

## COMMUTATOR IDEALS AND SEMICOMMUTATOR IDEALS OF TOEPLITZ OPERATORS ASSOCIATED WITH FLOWS II

PAUL S. MUHLY AND JINGBO XIA

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ABSTRACT. We prove that for a flow with at most one fixed point, the commutator ideal and the semicommutator ideal of the associated Toeplitz algebra coincide. We further show that the situation becomes much more complicated for flows with at least two fixed points.

### INTRODUCTION

Let  $X$  be a compact Hausdorff space on which  $\mathbf{R}$  acts continuously as a group of homeomorphisms. That is, there is a continuous map  $(x, t) \mapsto x + t$  from  $X \times \mathbf{R}$  onto  $X$  such that  $(x + t) + s = x + (t + s)$ ,  $x + 0 = x$ , and such that, for each  $t$ ,  $x \mapsto x + t$  is a homeomorphism on  $X$ . Such an action of  $\mathbf{R}$  on  $X$  is usually referred to as a flow. Given  $f \in C(X)$  and  $x \in X$ , we denote the function  $t \mapsto f(x + t)$  on  $\mathbf{R}$  by  $f_x$ . Let  $A(X)$  denote the collection of  $\varphi \in C(X)$  such that  $\varphi_x \in H^\infty(\mathbf{R})$  for every  $x \in X$ .

Recall from [9] (also see [8]) that an *analytic representation* of a given flow  $(X, \mathbf{R})$  is a pair  $(\pi, P)$  where  $\pi$  is a unital  $C^*$ -algebra representation of  $C(X)$  on some Hilbert space  $K$  and  $P$  is an orthogonal projection on  $K$  such that  $PK$  is invariant under  $\pi(A(X))$ . That is,  $\pi(\varphi)P = P\pi(\varphi)P$  for every  $\varphi \in A(X)$ . A canonical example of analytic representation is  $K = L^2(\mathbf{R})$ ,  $\pi(f) = \pi_x(f)$ , the multiplication by  $f_x$  for some fixed  $x$ , and  $P = P_{\text{Hardy}}$ , the orthogonal projection from  $L^2(\mathbf{R})$  to the  $H^2(\mathbf{R})$ , the Hardy space associated with the upper half-plane. In fact the class  $\{(\pi_x, P_{\text{Hardy}}) : x \in X\}$  of analytic representations are the ones that we are most interested in.

Each analytic representation  $(\pi, P)$  of the flow gives rise to a Toeplitz algebra  $\mathcal{T} = \mathcal{T}(\pi, P)$ , i.e., the  $C^*$ -algebra on  $PK$  generated by  $\{P\pi(f)P : f \in C(X)\}$ . We write  $\mathcal{C} = \mathcal{C}(\pi, P)$  for the commutator ideal of  $\mathcal{T}$ . We define the semicommutator ideal  $\mathcal{C}_{1/2} = \mathcal{C}_{1/2}(\pi, P)$  of  $\mathcal{T}$  to be the ideal generated by  $\{P\pi(fg)P - P\pi(f)P\pi(g)P : f, g \in C(X)\}$ . Obviously  $\mathcal{C}_{1/2} \supset \mathcal{C}$ . In this note we continue the investigation [2, 8, 9] of the relation between  $\mathcal{C}$  and  $\mathcal{C}_{1/2}$ . The reader is referred to the Introduction of [9] for an extended discussion of what motivates our investigation. However, it might be worthwhile to point out that both ideals,  $\mathcal{C}$  and  $\mathcal{C}_{1/2}$ , play important roles in the  $K$ -theory of  $\mathcal{T}(\pi, P)$ , and in the spectral analysis of operators in this algebra. The structure of  $\mathcal{T}(\pi, P)$  strikes us as quite mysterious,

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in general. In some situations, it seems to reflect the orbit structure of the flow, much like what happens in the setting of transformation group  $C^*$ -algebras, while in others, it seems much more refractory. We hope that the present investigation will help to shed more light on this intriguing question.

It was shown in [9] that if the flow  $(X, \mathbf{R})$  has no fixed points, then  $\mathcal{C} = \mathcal{C}_{1/2}$  for every analytic representation  $(\pi, P)$  of the flow. While this result covers a large class of flows, it left the following classical example unexplained: Consider the  $C^*$ -algebra  $C_0^\dagger(\mathbf{R})$  obtained from  $C_0(\mathbf{R})$  by adjoining the constant functions on  $\mathbf{R}$ . There is a natural  $\mathbf{R}$ -translation on  $C_0^\dagger(\mathbf{R})$ , which induces a flow  $(\mathcal{X}, \mathbf{R})$ , where the maximal ideal space  $\mathcal{X}$  of  $C_0^\dagger(\mathbf{R})$  is just the one-point compactification of  $\mathbf{R}$ . Obviously the “infinity” in  $\mathcal{X}$  is a fixed point of the flow; in fact it is the only fixed point. Yet for any analytic representation of  $(\mathcal{X}, \mathbf{R})$  we always have  $\mathcal{C} = \mathcal{C}_{1/2}$ . Note that this example is derived from the classical Toeplitz operators on  $\mathbf{R}$  with continuous symbol functions which possess the limit  $\lim_{|t| \rightarrow \infty} f(t)$ .

Using techniques completely different from those in [9], we will show in this note that for any flow  $(X, \mathbf{R})$  which has at most one fixed point, we have  $\mathcal{C} = \mathcal{C}_{1/2}$  for every analytic representation  $(\pi, P)$ . A well-known example of Gohberg and Krupnik [6] which we mentioned in [9] tells us that  $\mathcal{C}$  and  $\mathcal{C}_{1/2}$  can differ for flows with two fixed points. While the above-mentioned result for  $\mathcal{C} = \mathcal{C}_{1/2}$  seems to be optimal in light of the Gohberg-Krupnik example, we will further show that the relation between the equality  $\mathcal{C} = \mathcal{C}_{1/2}$  and the set of fixed points of the flow is actually quite complicated.

We will produce a flow  $(X_{\mathcal{F}}, \mathbf{R})$  which has at least two fixed points and, yet, which has the property that  $\mathcal{C} = \mathcal{C}_{1/2}$  for every analytic representation  $(\pi, P)$ . In this example the space  $X_{\mathcal{F}}$  is the closure of a single orbit. This flow is obtained from the flow in the Gohberg-Krupnik example by adjoining certain oscillatory functions on  $\mathbf{R}$ . The point is that if the oscillation is slow enough, then the flow will have multiple fixed points.

But just in case the reader gets the wrong impression that one always has  $\mathcal{C} = \mathcal{C}_{1/2}$  when functions  $\{f_x : f \in C(X)\}$  oscillate on  $\mathbf{R}$ , we will further show that this need not be the case. We will show that if oscillation is added into the Gohberg-Krupnik example in certain way, we still have  $\mathcal{C} \neq \mathcal{C}_{1/2}$  for representations on the Hardy space.

## 1. THE COINCIDENCE OF $\mathcal{C}$ AND $\mathcal{C}_{1/2}$

Given a flow  $(X, \mathbf{R})$ , let  $\mathcal{B}$  denote the  $\|\cdot\|_\infty$ -closure of  $\{\sum_{j=1}^n \varphi_j \bar{\psi}_j : n \in \mathbf{N}, \varphi_j, \psi_j \in A(X)\}$ , which is a  $C^*$ -subalgebra of  $C(X)$ .

**Proposition 1.** *If the flow  $(X, \mathbf{R})$  has the property that  $\mathcal{B} = C(X)$ , then  $\mathcal{C} = \mathcal{C}_{1/2}$  for every analytic representation  $(\pi, P)$ .*

*Proof.* Given a pair  $(\pi, P)$ , let us write  $T_f = P\pi(f)P$  for  $f \in C(X)$ . If  $\varphi_1, \varphi_2, \psi_1, \psi_2 \in A(X)$ , then

$$\begin{aligned} T_{(\varphi_1 \bar{\psi}_1)(\varphi_2 \bar{\psi}_2)} &= T_{(\bar{\psi}_1 \bar{\psi}_2)(\varphi_1 \varphi_2)} = T_{\bar{\psi}_1} T_{\bar{\psi}_2} T_{\varphi_1} T_{\varphi_2} \\ &= T_{\bar{\psi}_1 \varphi_1} T_{\bar{\psi}_2 \varphi_2} + T_{\bar{\psi}_1} (T_{\bar{\psi}_2} T_{\varphi_1} - T_{\varphi_1} T_{\bar{\psi}_2}) T_{\varphi_2} \in T_{\varphi_1 \bar{\psi}_1} T_{\varphi_2 \bar{\psi}_2} + \mathcal{C}. \end{aligned}$$

Hence the proposition follows from the density of  $\mathcal{B}$  and the identity

$$T_{(u_1+\dots+u_m)(v_1+\dots+v_n)} - T_{(u_1+\dots+u_m)}T_{(v_1+\dots+v_n)} = \sum_{i=1}^m \sum_{j=1}^n (T_{u_i v_j} - T_{u_i} T_{v_j}).$$

□

For each  $f \in C(X)$ , let  $\text{sp}(f)$  denote the spectrum, in the sense of spectral synthesis (see [1, 4, 7, 10] and [9]), of the function  $f$  with respect to the flow  $(X, \mathbf{R})$ . It is well known that  $\text{sp}(\bar{f}) = -\text{sp}(f)$  and  $A(X) = \{\varphi \in C(X) : \text{sp}(\varphi) \subset [0, \infty)\}$ . Also,

$$(1.1) \quad \text{sp}(fg) \subset [a + b, \infty) \text{ if } \text{sp}(f) \subset [a, \infty) \text{ and } \text{sp}(g) \subset [b, \infty).$$

Furthermore, the set  $\{f \in C(X) : \text{sp}(f) \text{ is bounded}\}$  is dense in  $C(X)$ .

For each  $d > 0$  and each  $s \in \mathbf{R}$ , let

$$C_{s,d}(X) = \{u_1 + u_2 : u_1, u_2 \in C(X), \text{sp}(u_1) \subset [-d + s, \infty), \text{sp}(u_2) \subset (-\infty, d + s]\}.$$

Then  $C_{s,d}(X)$  is dense in  $C(X)$ . To verify this claim, consider  $f \in C(X)$  such that  $\text{sp}(f) \subset [-R, R]$  for some  $R > d + |s|$ . Let  $0 \leq \eta \leq 1$  be a  $C^\infty$ -function on  $\mathbf{R}$  such that  $\eta = 1$  on  $[-R - 1, R + 1]$  and  $\eta = 0$  on  $[-R - 2, R + 2]$ . If  $\hat{\eta}(t) = \int_{\mathbf{R}} \eta(\lambda) e^{-it\lambda} d\lambda / 2\pi$ , then  $f = \int \hat{\eta}(t) f_t dt$ , where  $f_t(x) = f(x + t)$ . Now there are  $C^\infty$ -functions  $0 \leq \eta_1 \leq 1$  and  $0 \leq \eta_2 \leq 1$  on  $\mathbf{R}$  such that  $\eta_1 = 0$  on  $(-\infty, (-d/2) + s]$ ,  $\eta_2 = 0$  on  $[(d/2) + s, \infty)$ , and  $\eta = \eta_1 + \eta_2$ . Let

$$\hat{\eta}_i(t) = \int_{\mathbf{R}} \eta_i(\lambda) e^{-it\lambda} d\lambda / 2\pi \quad \text{and} \quad u_i = \int \hat{\eta}_i(t) f_t dt, \quad i = 1, 2.$$

Then  $f = u_1 + u_2$  with  $\text{sp}(u_1) \subset [(-d/2) + s, \infty)$  and  $\text{sp}(u_2) \subset (-\infty, (d/2) + s]$ .

Define  $\mathcal{B}_0$  to be the linear span of  $\{\varphi_1 \bar{\varphi}_2 + \varphi_3 + \bar{\varphi}_4 : \varphi_i \in C(X), \text{sp}(\varphi_i) \subset [\delta, \infty)$  for some  $\delta = \delta(\varphi_1, \dots, \varphi_4) > 0, i = 1, \dots, 4\}$ . Obviously  $\mathcal{B}_0$  is closed under the linear operations, the multiplication, and the complex conjugation. Let  $\mathcal{B}_0$  denote the  $\|\cdot\|_\infty$ -closure of  $\mathcal{B}_0$  in  $C(X)$ . By a simple approximation argument,  $\mathcal{B}_0$  in particular contains every  $f \in C(X)$  with the property that  $\text{sp}(f) \subset \mathbf{R} \setminus (-\delta, \delta)$  for some  $\delta > 0$ . Let  $F$  denote the collection of the fixed points of the flow  $(X, \mathbf{R})$ .

**Proposition 2.**  $\mathcal{B}_0$  is a (not necessarily proper) ideal in  $C(X)$ . Moreover, if  $F_0 = \{x \in X : f(x) = 0 \text{ for every } f \in \mathcal{B}_0\}$ , then  $F_0 \subset F$ .

*Proof.* Suppose  $\varphi_1, \varphi_2 \in C(X)$  are such that  $\text{sp}(\varphi_i) \subset [\delta, \infty)$  for some  $\delta > 0, i = 1, 2$ . We claim that  $f\varphi_1 \bar{\varphi}_2 \in \mathcal{B}_0$  for every  $f \in C(X)$ . By the density of  $C_{0,\delta/2}(X)$  in  $C(X)$ , to verify this claim, it suffices to consider the case where  $f \in C_{0,\delta/2}(X)$ . Thus  $f = u_1 + \bar{u}_2$  with  $\text{sp}(u_i) \subset [-\delta/2, \infty), i = 1, 2$ . Now, by (1.1),  $\text{sp}(u_i \varphi_j) \subset [\delta/2, \infty), i, j = 1, 2$ . Hence  $f\varphi_1 \bar{\varphi}_2 = (u_1 \varphi_1) \bar{\varphi}_2 + \varphi_1 (\bar{u}_2 \bar{\varphi}_2) \in \mathcal{B}_0$ . We also claim that  $(\varphi_1 + \bar{\varphi}_2)f \in \mathcal{B}_0$  for every  $f \in C(X)$ . Since  $\mathcal{B}_0$  is closed under complex conjugation, it suffices to show that  $f\varphi_1 \in \mathcal{B}_0$ . For this purpose we may assume  $f \in C_{-\delta/2,\delta/4}(X)$ . That is, we may assume  $f = v_1 + \bar{v}_2$  where  $\text{sp}(v_1) \subset [(-\delta/2) + (-\delta/4), \infty) = [-3\delta/4, \infty)$  and  $\text{sp}(\bar{v}_2) \subset (-\infty, (-\delta/2) + (\delta/4)] = (-\infty, -\delta/4]$ . By (1.1),  $\text{sp}(v_1 \varphi_1) \subset [\delta/4, \infty)$ , which implies  $v_1 \varphi_1 \in \mathcal{B}_0$ . On the other hand, since  $\text{sp}(v_2) \subset [\delta/4, \infty)$ ,  $\bar{v}_2 \varphi_1 \in \mathcal{B}_0$  by definition. Hence  $\mathcal{B}_0$  is an ideal in  $C(X)$ .

We know that  $\text{sp}(f_t) = \text{sp}(f)$  for any  $f \in C(X)$  and  $t \in \mathbf{R}$ . Hence  $\mathcal{B}_0$  is invariant under the map  $f \mapsto f_t$ . This implies that  $F_0$  is an invariant set of the flow. Consequently we have the subflow  $(F_0, \mathbf{R})$ . To complete the proof, we must show that the action of  $\mathbf{R}$  on  $F_0$  is trivial. Equivalently, it suffices to show that every  $\xi \in C(F_0)$  is invariant with respect to  $(F_0, \mathbf{R})$ .

For this purpose we extend  $\xi$  to a continuous function on  $X$ . For each  $\epsilon > 0$ , let  $0 \leq \eta \leq 1$  be a  $C^\infty$ -function on  $\mathbf{R}$  such that  $\eta = 1$  on  $[-\epsilon/3, \epsilon/3]$  and  $\eta = 0$  on  $\mathbf{R} \setminus (-\epsilon/2, \epsilon/2)$ . Let  $\hat{\eta}(t) = \int_{\mathbf{R}} \eta(\lambda) e^{-it\lambda} d\lambda / 2\pi$  and  $\zeta = \int_{\mathbf{R}} \hat{\eta}(t) \xi_t dt$ . It is easy to see that if  $h$  is a  $C^\infty$ -function supported in  $(-\epsilon/4, \epsilon/4)$  and  $\hat{h}(t) = \int_{\mathbf{R}} h(\lambda) e^{-it\lambda} d\lambda$ , then  $\int_{\mathbf{R}} \hat{h}(t) (\xi - \zeta)_t dt = 0$ . This implies that  $\text{sp}(\xi - \zeta) \subset \mathbf{R} \setminus (-\epsilon/4, \epsilon/4)$  and, therefore, that  $\xi - \zeta \in \mathcal{B}_0$ . In other words,  $\xi = \zeta$  on  $F_0$ . On the other hand, it follows from the definition of  $\eta$  that  $\text{sp}(\zeta) \subset [-\epsilon, \epsilon]$ . But the spectrum of  $\xi$  with respect to  $(F_0, \mathbf{R})$  is a subset of  $\text{sp}(\zeta) \subset [-\epsilon, \epsilon]$  because  $\zeta = \xi$  on  $F_0$ . Since  $\epsilon$  is arbitrary,  $\xi$  is invariant on  $F_0$ . This completes the proof.  $\square$

**Theorem 3.** *If the flow  $(X, \mathbf{R})$  has at most one fixed point, then  $\mathcal{C} = \mathcal{C}_{1/2}$  for every analytic representation  $(\pi, P)$ .*

*Proof.* If  $F$  is at most a singleton set, then by Proposition 2  $\mathcal{B}_0$  is either a maximal ideal or equals  $C(X)$  itself. In either case we have  $\mathcal{B} = C(X)$ . The theorem follows from Proposition 1.  $\square$

As we will see in the next section, it is still possible to have  $\mathcal{C} = \mathcal{C}_{1/2}$  for every  $(\pi, P)$  for some flows with at least two fixed points.

## 2. A FLOW WITH AT LEAST TWO FIXED POINTS

Let  $f$  be a real-valued  $C^\infty$ -function on  $\mathbf{R}$  such that  $f = 0$  on  $(-\infty, -1]$  and  $f = 1$  on  $[0, \infty)$ . Let  $g$  denote the Hilbert transform of  $f$ . That is,

$$g(t) = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{|t-s| \geq \epsilon} \left( \frac{1}{t-s} + \frac{s}{1+s^2} \right) f(s) ds.$$

By the properties of  $f$ , it is straightforward to verify that  $g$  is uniformly continuous on  $\mathbf{R}$ . Therefore so is the function

$$u = \exp(f + ig).$$

We have  $u \in H^\infty(\mathbf{R})$ . To see that this is so let  $f_n$  be a smooth truncation of  $f$  so that  $f_n = 0$  outside  $(-n, n)$ . Then obviously  $f_n + ig_n \in H^\infty(\mathbf{R})$ , where  $g_n$  is the Hilbert transform of  $f_n$ . Since  $\exp(f_n + ig_n) \rightarrow u$  pointwise and since  $|\exp(f_n + ig_n)| \leq e^f \leq e$ , it follows from the dominated convergence theorem that  $\int_{\mathbf{R}} u(t) h(t) dt = 0$  for every  $h \in H^1(\mathbf{R})$ . Hence  $u \in H^\infty(\mathbf{R})$ .

Let  $\mathcal{F}$  be the  $C^*$ -subalgebra of  $C_{\text{bu}}(\mathbf{R})$  generated by the constant function 1 and the translations of  $u$  and  $\bar{u}$ . Let  $X_{\mathcal{F}}$  be the maximal ideal space of  $\mathcal{F}$  and let  $\rho: \mathcal{F} \rightarrow C(X_{\mathcal{F}})$  be the Gelfand transform. The natural translation  $\varphi(\cdot) \mapsto \varphi(\cdot + t)$  on  $\mathcal{F}$  induces a flow  $(X_{\mathcal{F}}, \mathbf{R})$ . For each  $\xi \in \mathcal{F}$ ,  $\rho(\xi) \in A(X_{\mathcal{F}})$  if and only if  $\xi \in H^\infty(\mathbf{R})$ . Hence in this case  $\mathcal{B} = C(X_{\mathcal{F}})$ . By Proposition 1, for any analytic representation  $(\pi, P)$  of this flow, we have  $\mathcal{C} = \mathcal{C}_{1/2}$ .

Each  $t \in \mathbf{R}$  corresponds to a  $[t] \in X_{\mathcal{F}}$  and  $\{[t] : t \in \mathbf{R}\}$  is dense in  $X_{\mathcal{F}}$ . That is,  $X_{\mathcal{F}}$  is the closure of a single orbit. We will next show that the flow  $(X_{\mathcal{F}}, \mathbf{R})$  has at least two fixed points.

Since  $\mathcal{F}$  is separable,  $X_{\mathcal{F}}$  is metrizable. Hence, by the compactness of  $X_{\mathcal{F}}$ , there is a sequence of natural numbers  $n_1 < \dots < n_k < \dots$  and  $x_+, x_- \in X_{\mathcal{F}}$  such that  $[n_k] \rightarrow x_+$  and  $[-n_k] \rightarrow x_-$  as  $k \rightarrow \infty$ . We claim that  $x_+$  and  $x_-$  are fixed points of the flow. Since  $[\pm n_k + r] \rightarrow x_{\pm} + r$ , to verify this claim, it suffices to show that  $[\pm n_k + r] \rightarrow x_{\pm}$  for any  $r \in \mathbf{R}$ . For this purpose we use the fact that  $\{u_{\alpha} : \alpha \in \mathbf{R}\}$  is a set of generators for the  $C^*$ -algebra  $\mathcal{F}$ . Here  $u_{\alpha}(t) = u(t + \alpha)$ . Thus it suffices to prove that

$$\lim_{k \rightarrow \infty} u_{\alpha}(\pm n_k + r) = \lim_{k \rightarrow \infty} u_{\alpha}(\pm n_k)$$

for any  $r, \alpha \in \mathbf{R}$ . This will follow if we can show that

$$\lim_{|t| \rightarrow \infty} (g(t + \gamma) - g(t)) = 0$$

for every  $\gamma \in \mathbf{R}$ . But this is quite obvious. For direct computation of the Hilbert transform yields that, when  $|t| > 1$ , we have

$$g(t) = \int_{-1}^0 \left( \frac{1}{t-s} + \frac{s}{1+s^2} \right) f(s) ds + \frac{t}{|t|} \log |t|.$$

This verifies that both  $x_+$  and  $x_-$  are fixed points of the flow.

Since  $f = 0$  on  $(-\infty, -1]$  and  $f = 1$  on  $[0, \infty)$ , we have

$$\rho(|u|^2)(x_+) = \lim_{k \rightarrow \infty} |u(n_k)|^2 = e^2 \quad \text{and} \quad \rho(|u|^2)(x_-) = \lim_{k \rightarrow \infty} |u(-n_k)|^2 = 1.$$

Therefore  $x_+ \neq x_-$ .

Note that  $\mathcal{F}$  properly contains  $C_{*,*}(\mathbf{R})$ , the collection of continuous functions  $\varphi$  on  $\mathbf{R}$  such that the limits  $\lim_{t \rightarrow \infty} \varphi(t)$  and  $\lim_{t \rightarrow -\infty} \varphi(t)$  exist but do not necessarily coincide. Indeed  $C_{*,*}(\mathbf{R})$  is the  $C^*$ -subalgebra of  $\mathcal{F}$  generated by 1 and the translations of  $|u|^2$ . As we mentioned in [9], the natural flow induced by the translation on  $C_{*,*}(\mathbf{R})$  provides the Gohberg-Krupnik example where  $\mathcal{C} \neq \mathcal{C}_{1/2}$  on the space  $H^2(\mathbf{R})$ . One might say that  $\mathcal{F}$  is obtained from  $C_{*,*}(\mathbf{R})$  by adjoining certain oscillating functions on  $\mathbf{R}$ . But because of the nature of the oscillation,  $\mathcal{C} = \mathcal{C}_{1/2}$  for every analytic representation of  $(X_{\mathcal{F}}, \mathbf{R})$ .

### 3. $\mathcal{C} \neq \mathcal{C}_{1/2}$ WITH OSCILLATING SYMBOLS

We will next show that if oscillation is added to  $C_{*,*}(\mathbf{R})$  in ways different from the preceding section, then we may still have  $\mathcal{C} \neq \mathcal{C}_{1/2}$ .

Let  $\tilde{w}$  be a real-valued continuous function on  $\mathbf{R}$  such that

$$\lim_{t \rightarrow +\infty} \tilde{w}(t) = 1 \quad \text{and} \quad \lim_{t \rightarrow -\infty} \tilde{w}(t) = 0.$$

Suppose that  $w$  is a real-valued, bounded, uniformly continuous function on  $\mathbf{R}$  with the property that for each  $k \in \mathbf{N}$ , there is a  $\lambda_k \in \mathbf{R}$  such that

$$w(t + \lambda_k) = \tilde{w}(t) \quad \text{if} \quad |t| \leq k.$$

Let  $\mathcal{W}$  denote the  $C^*$ -algebra generated by 1 and the translations of  $w$ . Again, if  $X_{\mathcal{W}}$  is the maximal ideal space of  $\mathcal{W}$ , then the natural translation on  $\mathcal{W}$  induces a flow  $(X_{\mathcal{W}}, \mathbf{R})$ . Consider the natural Hardy representation  $(\pi, P)$  of this flow. That is, we take  $K = L^2(\mathbf{R})$ ,  $P$  the orthogonal projection from  $L^2(\mathbf{R})$  onto  $H^2(\mathbf{R})$ , and

$$\pi(\rho(g)) = M_g, \quad g \in \mathcal{W},$$

where  $\rho : \mathcal{W} \rightarrow C(X_{\mathcal{W}})$  is the Gelfand transform. Let us denote the usual Toeplitz operator with symbol  $f$  on  $H^2(\mathbf{R})$  by  $T_f$ . That is,  $T_f = PM_f|H^2(\mathbf{R})$ .

Let  $\mathcal{T}(\mathcal{W})$  be the  $C^*$ -algebra on  $H^2(\mathbf{R})$  generated by  $\{T_g : g \in \mathcal{W}\}$  and let  $\mathcal{C}(\mathcal{W})$  be its commutator ideal. Also, let  $\mathcal{C}_{1/2}(\mathcal{W})$  denote the ideal of  $\mathcal{T}(\mathcal{W})$  generated by  $\{T_{fg} - T_f T_g : f, g \in \mathcal{W}\}$ .

**Theorem 4.**  $\mathcal{C}(\mathcal{W}) \neq \mathcal{C}_{1/2}(\mathcal{W})$ .

*Proof.* Define

$$u = \exp(-2\pi iw).$$

Note that the invertible operator  $\exp(-2\pi iT_w)$  is contained in  $T_{\exp(-2\pi iw)} + \mathcal{C}_{1/2}(\mathcal{W}) = T_u + \mathcal{C}_{1/2}(\mathcal{W})$ . Thus, to prove the theorem, it suffices to show that the set  $T_u + \mathcal{C}(\mathcal{W})$  does not contain any invertible operator.

For this purpose let us write  $\mathcal{W}_0$  for the collection of finite sums of finite products of translations of  $w$  and complex numbers. If there were an invertible operator in  $T_u + \mathcal{C}(\mathcal{W})$ , then there would be an operator of the form

$$(3.1) \quad A = \sum_{i=1}^n T_{\varphi_{i1}^1} \dots T_{\varphi_{im}^1} [T_{\varphi_i^2}, T_{\varphi_i^3}] T_{\varphi_{i1}^4} \dots T_{\varphi_{im}^4}$$

with  $\varphi_{i(j)}^\nu \in \mathcal{W}_0$  such that  $T_u + A$  is invertible. We will show that this leads to a contradiction.

Each symbol function  $\varphi_{i(j)}^\nu \in \mathcal{W}_0$  which appears in (3.1) can be expressed as a finite product-sum

$$(3.2) \quad \varphi_{i(j)}^\nu = d + \sum_p c_p w_{\gamma_{p1}} \dots w_{\gamma_{p\ell(p)}},$$

where  $c_p$  and  $d$  are complex numbers. (Recall  $g_\gamma(t) = g(t + \gamma)$ .) Define

$$(3.3) \quad \tilde{\varphi}_{i(j)}^\nu = d + \sum_p c_p \tilde{w}_{\gamma_{p1}} \dots \tilde{w}_{\gamma_{p\ell(p)}}.$$

Since  $w_{\lambda_k} = \tilde{w}$  on  $[-k, k]$ , we have  $(w_\gamma)_{\lambda_k} = (w_{\lambda_k})_\gamma = \tilde{w}_\gamma$  on  $[-k - \gamma, k - \gamma] \supset [-k + |\gamma|, k - |\gamma|]$ . Thus if  $c \geq |\gamma_{pq}|$  for every pair  $(p, q)$  in the above, then

$$(\varphi_{i(j)}^\nu)_{\lambda_k} = \tilde{\varphi}_{i(j)}^\nu \quad \text{on } [-k + c, k - c]$$

if  $k$  is large enough. This implies that, for every  $h \in L^2(\mathbf{R})$ ,

$$(3.4) \quad \lim_{k \rightarrow \infty} \|((\varphi_{i(j)}^\nu)_{\lambda_k} - \tilde{\varphi}_{i(j)}^\nu)h\| = 0.$$

Using the relation between (3.2) and (3.3) for each set of indices  $(\nu, i(j))$ , we obtain an operator

$$\tilde{A} = \sum_{i=1}^n T_{\tilde{\varphi}_{i1}^1} \dots T_{\tilde{\varphi}_{im}^1} [T_{\tilde{\varphi}_i^2}, T_{\tilde{\varphi}_i^3}] T_{\tilde{\varphi}_{i1}^4} \dots T_{\tilde{\varphi}_{im}^4}.$$

Since each  $\tilde{\varphi}_i^\nu$  is continuous on  $\mathbf{R}$  and possesses limits at  $-\infty$  and  $\infty$ ,  $[T_{\tilde{\varphi}_i^2}, T_{\tilde{\varphi}_i^3}]$  is a compact operator. Thus  $\tilde{A}$  is compact. Define

$$v = \exp(-2\pi i\tilde{w}).$$

Since  $\tilde{w}(-t) \rightarrow 0$  and  $\tilde{w}(t) \rightarrow 1$  as  $t \rightarrow \infty$ , the winding number of  $v$  about 0 is  $-1$ . By the conformal equivalence between the upper half-plane and the disc, it is easy to see that the Toeplitz operator  $T_v$  is a Fredholm operator of index 1 [3, 4]. Thus

$$\ker(T_v + \tilde{A}) \neq \{0\}.$$

Let  $g_0$  be a unit vector in  $\ker(T_v + \tilde{A})$ . For each  $k \in \mathbf{N}$ , define the unitary operator  $(U_k f)(t) = f(t + \lambda_k)$  on  $L^2(\mathbf{R})$ .  $H^2(\mathbf{R})$  is invariant under each  $U_k$ . (3.4) implies that  $\lim_{k \rightarrow \infty} \|(U_k A U_k^* - \tilde{A})g_0\| = 0$ . Similarly  $\lim_{k \rightarrow \infty} \|(U_k T_u U_k^* - T_v)g_0\| = 0$ . Therefore

$$(3.5) \quad \lim_{k \rightarrow \infty} \|U_k(T_u + A)U_k^*g_0\| = \lim_{k \rightarrow \infty} \|(U_k(T_u + A)U_k^* - (T_v + \tilde{A}))g_0\| = 0.$$

On the other hand, by the alleged invertibility of  $T_u + A$ , there should be a  $\delta > 0$  such that  $\|U_k(T_u + A)U_k^*g_0\| = \|(T_u + A)U_k^*g_0\| \geq \delta\|U_k^*g_0\| = \delta$  for all  $k$ . This and (3.5) result in a contradiction.  $\square$

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242

*E-mail address:* [muhly@math.uiowa.edu](mailto:muhly@math.uiowa.edu)

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, BUFFALO, NEW YORK 14214

*E-mail address:* [JXMTH@ubvms.cc.buffalo.edu](mailto:JXMTH@ubvms.cc.buffalo.edu)