Let $\mathcal{K} = \partial B_2 \times \mathbb{C}$ and let $\mathcal{N} = (\mathbb{D} \times \mathbb{C}) \setminus \overline{\mathcal{K}}$ be a singularity set projecting onto $\Delta = \{z \in \mathbb{D} \mid |z| = 1\}$ such that $\mathcal{N}$ is contained in the tube $\{((\lambda, w) \mid |w - a(\lambda)| < r\}$. Then there exists an analytic function $\lambda \rightarrow f(\lambda)$ such that $|f(\lambda) - a(\lambda)| \leq 4r$ for each $\lambda$ in the unit disk.
We shall make use of Alexander’s and Wermer’s technique in the proof of Theorem 2.

If \( h : \mathbb{C}^2 \rightarrow \mathbb{C} \) is a smooth function then \( |\nabla h| \) shall denote the Euclidean length of the vector

\[
\left( \frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \frac{\partial h}{\partial y_1}, \frac{\partial h}{\partial y_2} \right),
\]

where \( (z_1, z_2) = (x_1 + x_2i, y_1 + y_2i) \). For \( K \) a singularity set over the ball we shall prove the following.

**Theorem 2.** Let \( h \) be a \( C^1 \) function in a neighborhood of the closed ball in \( \mathbb{C}^2 \), \(|\nabla h| \leq 1 \) on \( \overline{B}_2 \). Suppose \( K \) is a singularity set projecting onto \( \overline{B}_2 \) such that \( K \) is contained in the tube

\[
T = \{ (z, w) \in \mathbb{B}_2 \times \mathbb{C} \mid |w - h(z)| < \epsilon \}.
\]

Then there exists an analytic polynomial \( F \) in \( \mathbb{C}^2 \) such that

\[
|F(z) - h(z)| < 26\sqrt{\epsilon} \text{ on } \overline{B}_2.
\]

Suppose that \( L \) is a complex affine subspace of \( \mathbb{C}^2 \) of complex dimension 1 which meets \( B_2 \) and let \( p \) be the point on \( L \) nearest to \((0, 0)\). Then the points on \( L \) which meet \( S \) form a circle in \( L \) with center \( p \) and \( L \cap \overline{B}_2 \) is a disk embedded complex affinely in \( \mathbb{C}^2 \). We shall write \( \Delta = L \cap \overline{B}_2 \) and refer to \( \Delta \) as a complex affine slice of \( \overline{B}_2 \). Our technique shall be to use the fact that \( h \) is uniformly near an analytic function on every complex affine slice of \( \overline{B}_2 \) (from the results over the disk) and try to prove that \( h \) is uniformly near an analytic function on \( B_2 \) as a function of two variables. We let \( C(\overline{B}_2) \) denote the space of continuous complex-valued functions on \( \overline{B}_2 \) with supremum norm and \( A(\overline{B}_2) \) the subspace of functions which are analytic on \( B_2 \).

**Theorem 1.** Suppose \( h \) is \( C^1 \) in a neighborhood of the closed ball in \( \mathbb{C}^2 \), \(|\nabla h| \leq 1 \) on \( \overline{B}_2 \) and that given any complex affine slice \( \Delta \) (a disk), there exists a polynomial \( g_\Delta \) on \( \Delta \) such that

\[
|h - g_\Delta| < \epsilon \text{ on } \Delta.
\]

Then in \( C(\overline{B}_2) \),

\[
\text{dist}(h, A(\overline{B}_2)) < 13\sqrt{\epsilon}.
\]

**Proof.** First we note that the theorem is trivial if \( \epsilon \geq 1 \), because then the fact that \(|\nabla h| \leq 1 \) means that \( h \) is uniformly within 1 of the function which is constantly \( h(0) \), and \( 1 < 13\sqrt{\epsilon} \). Thus we assume that \( \epsilon < 1 \).

We shall show that \( C[h] \), the Cauchy integral of \( h \), satisfies

\[
\text{dist}(C[h]_{\Delta}, h|_{\Delta}) < \sqrt{5} \epsilon \text{ in } L^2(\Delta)
\]

where \( \Delta = \text{a diametrical complex linear slice of } \overline{B}_2 \), \( \nu_1 = \text{normalized area measure on } \Delta \) and \( L^2(\Delta) = \{ f \text{ analytic in } \text{int } \Delta \mid f \in L^2(\Delta) \} \). We shall also define \( A^2(\Delta) = \{ f \text{ analytic in } \text{int } \Delta \mid f \in L^2(\Delta) \} \). Suppose first that \( \Delta = \overline{B}_2 \cap \{(z_1, z_2) \mid z_2 = 0 \} \). Let \( \sigma = \text{normalized volume measure} \)
on $S$. Using a technique used by Rudin in [6],

$$C[h](z_1,0) = \int_{\partial B_2} \frac{h(\zeta_1, \zeta_2)}{(1-\langle z_1, \zeta_1 \rangle)^2} d\sigma(\zeta_1, \zeta_2)$$

$$= \int_{B_1} \frac{1}{(1-\langle z_1, \zeta_1 \rangle)^2} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\zeta_1, \zeta_2 e^{i\theta}) d\theta \right) d\nu_1(\zeta_1).$$

(See [6], pp. 15, 39.) For fixed $\zeta_1$ define $k(\zeta_1, \zeta_2)$ to be the harmonic extension of $h(\zeta_1, x)$, $|x| = \sqrt{1-|\zeta_1|^2}$, to the region where $|x| \leq \sqrt{1-|\zeta_1|^2}$. Then the above equation becomes

$$C[h](z_1,0) = \int_{B_1} \frac{k(\zeta_1,0)}{(1-\langle z_1, \zeta_1 \rangle)^2} d\nu_1(\zeta_1).$$

We check that $k$ is continuous when restricted to $\{z_2 = 0\}$. We have $k(\zeta_1,0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(z_1, e^{i\theta} \sqrt{1-|z_1|^2}) d\theta$; for $z'_1$ near $z_1$, $h(z'_1, e^{i\theta} \sqrt{1-|z'_1|^2})$ is uniformly close to $h(z_1, e^{i\theta} \sqrt{1-|z_1|^2})$ so $k(z'_1,0)$ is close to $k(z_1,0)$.

**Claim.** For $\zeta_1 \in \Delta$,

$$|h(\zeta_1,0) - k(\zeta_1,0)| < 2\epsilon.$$

**Proof of Claim.** From (1), we get that for fixed $\zeta_1$, there exists a polynomial $f_{\zeta_1}$ in $\zeta_2$ such that

$$|h(\zeta_1, \zeta_2) - f_{\zeta_1}(\zeta_2)| < \epsilon \quad \text{for} \quad |\zeta_2| \leq \sqrt{1-|\zeta_1|^2}.$$ 

Thus

$$|h(\zeta_1,0) - k(\zeta_1,0)| \leq |h(\zeta_1,0) - f_{\zeta_1}(0)| + |f_{\zeta_1}(0) - k(\zeta_1,0)|$$

$$< \epsilon + \sup_{|\zeta_2| = \sqrt{1-|\zeta_1|^2}} |f_{\zeta_1}(\zeta_2) - k(\zeta_1, \zeta_2)|$$

(since $f_{\zeta_1}$ and $k$ are harmonic in $\zeta_2$)

$$= \epsilon + \sup_{|\zeta_2| = \sqrt{1-|\zeta_1|^2}} |f_{\zeta_1}(\zeta_2) - h(\zeta_1, \zeta_2)|$$

$$< 2\epsilon,$$

as claimed.

Let $\Pi : L^2(\Delta) \rightarrow A^2(\Delta)$ be orthogonal projection. From what we know about the Bergman kernel, (4) is the orthogonal projection of $k$ to $A^2(\Delta)$, as a function of $z_1$. This means that $\Pi(k|_{\Delta}) = C[h]|_{\Delta}$. Then in $L^2(\Delta)$,

$$\text{dist}(h|_{\Delta}, C[h]|_{\Delta})^2 = \text{dist}(h|_{\Delta}, \Pi(h|_{\Delta}))^2 + \text{dist}(\Pi(h|_{\Delta}), C[h]|_{\Delta})^2$$

by the Pythagorean theorem.

Since $\text{dist}(h|_{\Delta}, A(\Delta)) < \epsilon$ in $C(\Delta)$, $\text{dist}(h|_{\Delta}, A^2(\Delta)) < \epsilon$ in $L^2(\Delta)$, so we have $\text{dist}(h|_{\Delta}, \Pi(h|_{\Delta})) < \epsilon$ in $L^2(\Delta)$. Thus (6) is now

$$< \epsilon^2 + \text{dist}(\Pi(h|_{\Delta}), \Pi(k|_{\Delta}))^2$$

$$< \epsilon^2 + \text{dist}(h|_{\Delta}, k|_{\Delta})^2$$

$$\leq \epsilon^2 + (2\epsilon)^2$$

from (5)

$$= 5\epsilon^2.$$
Thus (3) holds for \( \Delta = \overline{B}_2 \cap \{(z_1, z_2)|z_2 = 0\} \).

Now if \( \Delta \) is an arbitrary complex linear diametrical slice of \( \overline{B}_2 \), let \( U \) be a unitary transformation such that \( U((z_2 = 0) \cap \overline{B}_2) = \Delta \). Then \( h \circ U \) satisfies the original hypotheses of the theorem, so

\[
\text{dist}(h \circ U|\{z_2=0\}, C[h \circ U]|\{z_2=0\}) < \sqrt{5}\epsilon \text{ in } L^2(B_2 \cap \{z_2 = 0\})
\]

\[
\Rightarrow \text{dist}(h \circ U|\{z_2=0\}, C[h] \circ U|\{z_2=0\}) < \sqrt{5}\epsilon \text{ in } L^2(B_2 \cap \{z_2 = 0\})
\]

(since the Cauchy integral operator commutes with unitary transformations).

Since \( U((z_2 = 0) \cap \overline{B}_2) = \Delta \) and \( U \) induces a natural isometry from \( L^2(\Delta) \) to \( L^2(B_2 \cap \{z_2 = 0\}) \), we have (3).

Now let \( f = C[h] \). Fix any diametrical slice \( \Delta \). By (1) we can choose a polynomial \( g_\Delta \in A(\Delta) \) such that \( |h - g_\Delta| < \epsilon \) on \( \Delta \). Then \( \text{dist}(h|\Delta, g_\Delta) < \epsilon \) in \( L^2(\Delta) \) \( \Rightarrow \text{dist}(f|\Delta, g_\Delta) \leq \text{dist}(f, h|\Delta) + \text{dist}(h|\Delta, g_\Delta) \) in \( L^2(\Delta) \) \( < \sqrt{5}\epsilon + \epsilon \) from (3) \( = (1 + \sqrt{5})\epsilon \equiv a\epsilon \) so that

\[
(7) \quad \text{dist}(f|\Delta, g_\Delta) < a\epsilon \text{ in } A^2(\Delta)
\]

Define

\[
\begin{align*}
 f^r(z e^{i\theta}) &= f(z e^{i\theta}), z \in \text{int } \Delta, \\
g_\Delta^r(z e^{i\theta}) &= g_\Delta(z e^{i\theta}), z \in \Delta, \\
h^r(z e^{i\theta}) &= h(z e^{i\theta}), z \in \Delta.
\end{align*}
\]

Claim. For \( r \leq 1 - \sqrt{\epsilon} \),

\[
(8) \quad \| f^r - g_\Delta^r \|_2 \leq a\epsilon \frac{r}{\sqrt{2}} \text{ in } H^2(\Delta \cap S).
\]

Proof of claim. Suppose not. Note that \( \| f^r - g_\Delta^r \|_2 \) is increasing in \( r \) since \( f^r \) and \( g_\Delta^r \) are analytic. Thus

\[
\| f^r - g_\Delta^r \|_{H^2} \geq a\epsilon \frac{r}{\sqrt{2}} \text{ for } r > 1 - \sqrt{\epsilon}.
\]

Using polar coordinates,

\[
\int_\Delta |f - g_\Delta|^2 d\nu_1 = 2 \int_0^1 r dr \frac{1}{2\pi} \int_{-\pi}^\pi |f^r(e^{i\theta}) - g_\Delta^r(e^{i\theta})|^2 d\theta
\]

\[
= 2 \int_0^1 r \| f^r - g_\Delta^r \|_{H^2}^2 dr
\]

\[
> 2 \int_{1 - \sqrt{\epsilon}}^1 r \| f^r - g_\Delta^r \|_{H^2}^2 dr
\]

\[
\geq 2 \int_{1 - \sqrt{\epsilon}}^1 r a^2 e^{\frac{3}{2}} dr, \text{ from (8)}
\]

\[
= 2a^2 e^{\frac{3}{2}} \int_{1 - \sqrt{\epsilon}}^1 r dr \geq a^2 e^{\frac{3}{2}} (2\sqrt{\epsilon} - \epsilon)
\]

\[
\geq a^2 e^{\frac{3}{2}} (2\sqrt{\epsilon} - \sqrt{\epsilon}) = a^2 e^{\frac{3}{2}} \sqrt{\epsilon} = a^2 \epsilon^2
\]

(recall \( \epsilon < 1 \)) so \( \| f - g_\Delta \|_2 > a\epsilon \) in \( A^2(\Delta) \), a contradiction of (7). Thus the claim (8) holds. This means that for \( r \leq 1 - \sqrt{\epsilon} \), \( \| f^r - h^r \|_{L^2(\Delta \cap S)} \leq \| f^r - g_\Delta^r \|_{L^2(\Delta \cap S)} \)
\[ g'' - h'' \|_{L^2(\Delta \cap S)} < e \varepsilon^2 + \varepsilon \] (from (1) and (8)) \leq (2 + \sqrt{5})\varepsilon^2 and we conclude

\[ f'' - h'' \|_{L^2(\Delta \cap S)} \leq (2 + \sqrt{5})\varepsilon^2 \] for \( r \leq 1 - \sqrt{\varepsilon} \).

Note that the conclusion holds for all diametrical slices \( \Delta \).

Now define \( F : B_2 \rightarrow \mathbb{C} \) by \( F(z) = \int_R f(ze^{-it})\phi(t)dt \) where \( \phi \in C^\infty(\mathbb{R}) \), \( \phi \geq 0 \), \( spt\phi \subset [-\sqrt{\varepsilon}, \sqrt{\varepsilon}] \), \( \int \phi dx = 1 \), and \( \phi \leq \frac{1}{\sqrt{2\pi}} \). Also define \( H(z) = \int_R h(ze^{-it})\phi(t)dt \). Since \( f \) and \( \phi \) are smooth in \( B_2 \), \( f \) analytic, \( F \) is analytic in \( B_2 \) by differentiation under the integral sign. Then for \( |z| \leq 1 - \sqrt{\varepsilon} \),

\[ |F(z) - H(z)| \leq \left| \int_R (f(ze^{-it}) - h(ze^{-it}))\phi(t)dt \right| \]

\[ \leq \sqrt{\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} |f(ze^{-it}) - h(ze^{-it})|^2 dt} \sqrt{\int \phi(t)^2 dt} \]

\[ \leq (\sqrt{2\pi})(2 + \sqrt{5})\varepsilon^2 \sqrt{2\sqrt{\varepsilon}} \sup_{t \in R} |\phi(t)|^2 , \text{ from (9)} \]

\[ \leq (\sqrt{2\pi})(2 + \sqrt{5})\varepsilon^2 \sqrt{2\sqrt{\varepsilon}} \frac{1}{2\varepsilon} \]

\[ \leq (\sqrt{2\pi})(2 + \sqrt{5})\sqrt{\varepsilon} \]

and for \( z \in B_2 \),

\[ |H(z) - h(z)| = \left| \int_R (h(ze^{-it}) - h(z))\phi(t)dt \right| \]

\[ \leq \sup_{t \in spt\phi} |h(ze^{-it}) - h(z)| \| \phi \|_1 \]

\[ \leq \sqrt{\varepsilon} \text{ (since } |\nabla h| \leq 1 \). \]

Thus for \( |z| \leq 1 - \sqrt{\varepsilon} \),

\[ |F(z) - h(z)| \leq |F(z) - H(z)| + |H(z) - h(z)| \]

\[ \leq (\sqrt{2\pi})(2 + \sqrt{5})\sqrt{\varepsilon} + \sqrt{\varepsilon} , \text{ from (10) and (11)} \]

\[ < 12\sqrt{\varepsilon} . \]

Lastly define \( J : \overline{B}_2 \rightarrow \mathbb{C} \) by \( J(z) = F(z(1 - \sqrt{\varepsilon})) \). Then \( J \in A(B_2) \). For \( z \in \overline{B}_2 \),

\[ |J(z) - h(z)| = |F(z(1 - \sqrt{\varepsilon})) - h(z)| \]

\[ \leq |F(z(1 - \sqrt{\varepsilon})) - h(z(1 - \sqrt{\varepsilon}))| + |h(z(1 - \sqrt{\varepsilon})) - h(z)| \]

\[ < 12\sqrt{\varepsilon} + \sqrt{\varepsilon} , \text{ from (12) and since } |\nabla h| \leq 1 \]

\[ \leq 13\sqrt{\varepsilon} . \]

Thus \( \text{dist}(h, A(B_2)) < 13\sqrt{\varepsilon} \) in \( C(\overline{B}_2) \), which is (2).

We now prove Theorem 2, which we state again for the reader’s convenience.

**Theorem 2.** Let \( h \) be a \( C^1 \) function in a neighborhood of the closed ball in \( \mathbb{C}_2 \) with \( |\nabla h| \leq 1 \) on \( \overline{B}_2 \). Suppose \( K \) is a singularity set projecting onto \( \overline{B}_2 \) such that \( K \) is contained in the tube

\[ T = \{(z, w) \in \overline{B}_2 \times \mathbb{C} \mid |w - h(z)| < \epsilon \}. \]
Then there exists an analytic polynomial $F$ in $\mathbb{C}^2$ such that
\begin{equation}
|F(z) - h(z)| < 26\sqrt{\epsilon} \text{ on } \overline{B}_2.
\end{equation}

\textbf{Proof.} Our main goal is to show that on every complex affine slice $\Delta$ of $\overline{B}_2$, there exists a polynomial $g_\Delta$ such that
\begin{equation}
|h - g_\Delta| < 4\epsilon.
\end{equation}
We can then apply Theorem 1 with (1) replaced by (15) and conclude (14) since $13\sqrt{4} = 26$. Let us now fix such a $\Delta$.

To prove (15), we can follow a method used by Alexander and Wermer in [2] to show that $\text{dist}(h|_{\Delta \cap S}, A(\Delta)) < 2\epsilon$. Let $P : \overline{B}_2 \times \mathbb{C} \rightarrow \overline{B}_2$ be projection and let $K'$ be the set $\{(z, w) \mid |w - h(z)| \leq \epsilon, z \in \overline{B}_2\}$. We claim that: some element of the polynomial convex hull of $K' \cap P^{-1}(\partial \Delta)$ lies over $\text{int } \Delta$. To see this, we first note that $K \cap P^{-1}(\text{int } \Delta)$ is a singularity set in $P^{-1}(\Delta)$ since the intersection of a pseudoconvex set with an affine subspace is pseudoconvex in that affine subspace. Then by the Proposition, the elements of $K \cap P^{-1}(\text{int } \Delta)$ (which is nonempty by assumption) are in the polynomial hull of $K \cap P^{-1}(\partial \Delta)$. Since $K' \supseteq K$, the claim holds. Then by Theorem 1 of [2], there exists $\phi \in H^\infty(\Delta)$ such that $(z, \phi(z)) \in K'$ for a.e. $z \in \Delta \cap S$. This means that $\|h - \phi\|_\infty \leq \epsilon < 2\epsilon$. Then we conclude that since $h$ is continuous, $\text{dist}(h|_{\Delta \cap S}, A(\Delta)) < 2\epsilon$. Choose a polynomial $g_\Delta$ on $\Delta$ such that
\begin{equation}
|h - g_\Delta| < 2\epsilon \text{ on } \Delta \cap S.
\end{equation}
Now consider the polynomial $w - g_\Delta$ on the set $K \cap (\Delta \times \mathbb{C})$. Let $(z_0, w_0) \in K \cap (\Delta \times \mathbb{C})$. Since $(z_0, w_0)$ is in the polynomial hull of $K \cap P^{-1}(\partial \Delta)$,
\begin{equation}
|w_0 - g_\Delta(z_0, w_0)| \leq \sup_{(z, w) \in K, z \in \Delta \cap S} |w - g_\Delta(z, w)|.
\end{equation}
Then
\begin{align*}
\sup_{z \in \Delta} |h(z) - g_\Delta(z)| &\leq \sup_{(z, w) \in (K \cap (\Delta \times \mathbb{C}))} |h(z) - w| + \sup_{(z, w) \in (K' \cap (\Delta \times \mathbb{C}))} |w - g_\Delta(z)| \\
&\leq \epsilon + \sup_{(z, w) \in K, z \in \Delta \cap S} |w - g_\Delta(z)|, \text{ from (13) and (17)} \\
&\leq \epsilon + \sup_{(z, w) \in K, z \in \Delta \cap S} |w - h(z)| + \sup_{(z, w) \in K, z \in \Delta \cap S} |h(z) - g_\Delta(z)| \\
&< \epsilon + \epsilon + 2\epsilon, \text{ from (13) and (16)} \\
&\leq 4\epsilon.
\end{align*}
This shows that (15) holds, so the theorem is proven. \hfill \Box

\textbf{References}

Department of Mathematics, Brown University, Providence, Rhode Island 02912
E-mail address: mwhittle@math.brown.edu

Current address: Department of Mathematics, Texas A&M University, College Station, Texas 77843-3368