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HOPF SUBALGEBRAS OF POINTED HOPF ALGEBRAS AND APPLICATIONS

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Abstract. In this paper we construct certain Hopf subalgebras of a pointed Hopf algebra over a field of characteristic 0. Some applications are given in the case of Hopf algebras of dimension 6, p^2 and pq, where p and q are different prime numbers.

1. Preliminaries

Throughout this paper k will be an algebraically closed field of characteristic 0. In the first part of this note we shall prove that for any finite dimensional pointed Hopf algebra over k there is a Hopf subalgebra generated as an algebra by two elements g and x, where g is a group-like element and x is a $g, 1$-primitive element (Theorem 2). This result is then used for describing the isomorphism classes of pointed Hopf algebras of dimension p^2 and for proving that a pointed Hopf algebra of dimension pq is semisimple (p and q are different prime numbers). In the second part of the paper we shall prove that any Hopf algebra of dimension 6 is semisimple, so by [1], it is a group algebra or the dual of the group algebra of the symmetric group S_3.

Let H be a finite dimensional Hopf algebra over an algebraically closed field k, with $\mathrm{char}(k) = 0$. We recall that an element $g \neq 0$ is called a *group-like element* if $\Delta(g) = g \otimes g$. By definition, $x \in H$ is a *g, h-primitive element* if $\Delta(x) = x \otimes g + h \otimes x$, where g, h are two group-like elements. In the particular case when $g = h = 1$ we say that x is a *primitive element.* We denote by $G(H)$, $P(H)$ and $P_{g,h}(H)$, respectively, the sets of group-like elements, of primitive elements and of g, h-primitive elements of H. A Hopf algebra H is called *pointed* if all its simple subcoalgebras are of dimension one. The results of the following proposition are "folklore", so their proofs will be omitted.

Proposition 1. *Let H be a finite dimensional Hopf algebra over k.*

(a) *If H' is a pointed commutative Hopf subalgebra of H, then $H' = k[G']$, where G' is a certain subgroup of $G(H)$.*

(b) $P(H) = 0$.

(c) *Let H be a pointed Hopf algebra. Then $G(H) = \{1\}$ if and only if* $\dim(H) = 1$. *Moreover, if H is not cosemisimple, then there is $g \in G(H)$ such that $P_{g,1}(H)$ is not contained in the coradical of H.*

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Theorem 2. *Let H be a pointed Hopf algebra. If H is not semisimple, then there exist two natural numbers m, n, with $m \neq 1$ and m divides n, an mth primitive root of 1 (denoted by ω) and two elements $g, x \in H$ such that*

(a) *$gx = \omega xg$;*

(b) *g is a group-like element of order n;*

(c) *$x \in P_{g,1}(H)$ and x^m is either 0 or $g^m - 1$.*

Proof. Let $g \neq 1$ be a group-like element as in the third part of Proposition 1. Let ϕ_g be the inner automorphism of H afforded by g. Let n be the order of g. Obviously ϕ_g is semisimple, so its restriction to $P_{g,1}(H)$ has an eigenvalue $\omega \neq 1$; otherwise there is x in $P_{g,1}(H)$ which is not in $k[G(H)]$, such that $gx = xg$. The subalgebra generated by x and g is a group algebra (it is pointed and commutative), thus $x \in k[G(H)]$, a contradiction. We choose an eigenvalue $\omega \neq 1$ and a corresponding eigenvector x of ϕ_g. Hence $gx = \omega xg$ and x is in $P_{g,1}(H)$ by construction. Let m be the order of ω. Of course, m divides n, so we have only to prove that x^m equals either 0 or $g^m - 1$. Indeed, by [3, Proposition 1] we obtain $\Delta(x^m) = x^m \otimes g^m + 1 \otimes x^m$; therefore the subalgebra H' generated by g and x^m is a group algebra (being a commutative Hopf subalgebra of H). We end the proof by remarking that x^m is a g^m, 1-primitive element in H'. ☐

Let n be a natural number and let ω be a primitive nth-root of 1. We recall that, by definition, $H_{n^2,\omega}$ is the Hopf algebra generated as an algebra by two elements g and x satisfying the relations $g^n = 1$, $x^n = 0$, $gx = \omega xg$. The coalgebra structure is defined such that g is a group-like element and x is $g, 1$-primitive.

Corollary 3 (Andruskiewitsch, Chin). *If p is a prime natural number and H is a pointed Hopf algebra of dimension p^2, then $H \simeq k[G]$ or $H \simeq H_{p^2,\omega}$, where G is a group with p^2 elements and ω is a certain primitive nth-root of 1.*

Corollary 4. *Let p and q be two different prime numbers. If H is a pointed Hopf algebra of dimension pq, then H is semisimple.*

2. Hopf algebras of dimension 6

In this section we shall obtain the complete classification of Hopf algebras of dimension 6, as an application of Corollary 4. Namely, we shall prove the following

Theorem 5. *Let H be a Hopf algebra of dimension 6. Then H is isomorphic to $k[C_6]$, $k[S_3]$ or $k[S_3]^*$, where C_6 and S_3 are respectively the cyclic group with 6 elements and the symmetric group with 6 elements.*

Proof. We have to show that any Hopf algebra of dimension 6 is semisimple, as such a Hopf algebra is isomorphic to $k[C_6]$, $k[S_3]$ or $k[S_3]^*$ (see [1]). Let us suppose that H is a 6-dimensional Hopf algebra which is not semisimple. By the preceding corollary, H is neither pointed nor cosemisimple (any finite dimensional cosemisimple Hopf algebra over a field of characteristic 0 is semisimple). Then the coradical of H is isomorphic to $M_2(k)^*$ or $M_2(k)^* \oplus k$. The first case is not possible, as ε_{H^*} would induce an algebra map from $M_2(k) \simeq H^*/J(H^*)$ to k. Thus the coradical of H must be $M_2(k)^* \oplus k$ and, by [2, Thm. 5.4.2], there exists a coideal I of dimension 1 such that $H = \operatorname{corad}(H) \oplus I$. Let x be an element of I which is not 0. Then $\Delta(x) = x \otimes a + b \otimes x$, where a and b are in H. Writing explicitly the equality $(\Delta \otimes I_H)\Delta(x) = (I_H \otimes \Delta)\Delta(x)$ we can see easily that

$$\Delta(a) = a \otimes a + c \otimes x, \quad \Delta(b) = b \otimes b + x \otimes c, \quad \Delta(c) = a \otimes c + c \otimes b,$$

where $c \in H$. Therefore the vector space generated by a, b, c and x is a subcoalgebra C of H. The coalgebra $M_2(k)^*$ is simple, hence $M_2(k)^* \cap C = M_2(k)^*$ or $M_2(k)^* \cap C = 0$. In the first case it follows that $M_2(k)^* = C$ and then $x \in M_2(k)^*$, which contradicts the choice of x. In conclusion $M_2(k)^* \cap C = 0$, which implies $\dim(C) \leq 2$. Actually, one gets $\dim(C) = 2$ and $M_2(k)^* \oplus C = H$. C cannot be cosemisimple, otherwise H is semisimple, so $\operatorname{corad}(C) = k1$ and $H_1 = C$. But $C_1 = \operatorname{corad}(C) \oplus P(C)$, by [2, Lemma 5.3.2], thus $0 \neq P(C) \subseteq P(H)$, a contradiction with the second part of Proposition 1. ☐

Remark 6. The referee informed us that the results of the preceding theorem were already obtained by R. Williams [4].

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