HOPF SUBALGEBRAS OF POINTED HOPF ALGEBRAS
AND APPLICATIONS

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Abstract. In this paper we construct certain Hopf subalgebras of a pointed Hopf algebra over a field of characteristic 0. Some applications are given in the case of Hopf algebras of dimension 6, $p^2$ and $pq$, where $p$ and $q$ are different prime numbers.

1. Preliminaries

Throughout this paper $k$ will be an algebraically closed field of characteristic 0. In the first part of this note we shall prove that for any finite dimensional pointed Hopf algebra over $k$ there is a Hopf subalgebra generated as an algebra by two elements $g$ and $x$, where $g$ is a group-like element and $x$ is a $g$, 1-primitive element (Theorem 2). This result is then used for describing the isomorphism classes of pointed Hopf algebras of dimension $p^2$ and for proving that a pointed Hopf algebra of dimension $pq$ is semisimple ($p$ and $q$ are different prime numbers). In the second part of the paper we shall prove that any Hopf algebra of dimension 6 is semisimple, so by [1], it is a group algebra or the dual of the group algebra of the symmetric group $S_3$.

Let $H$ be a finite dimensional Hopf algebra over an algebraically closed field $k$, with char($k$) = 0. We recall that an element $g \neq 0$ is called a group-like element if $\Delta(g) = g \otimes g$. By definition, $x \in H$ is a $g$, $h$-primitive element if $\Delta(x) = x \otimes g + h \otimes x$, where $g$, $h$ are two group-like elements. In the particular case when $g = h = 1$ we say that $x$ is a primitive element. We denote by $G(H)$, $P(H)$ and $P_{g,h}(H)$, respectively, the sets of group-like elements, of primitive elements and of $g$, $h$-primitive elements of $H$. A Hopf algebra $H$ is called pointed if all its simple subcoalgebras are of dimension one. The results of the following proposition are “folklore”, so their proofs will be omitted.

**Proposition 1.** Let $H$ be a finite dimensional Hopf algebra over $k$.

(a) If $H'$ is a pointed commutative Hopf subalgebra of $H$, then $H' = k[G']$, where $G'$ is a certain subgroup of $G(H)$.

(b) $P(H) = 0$.

(c) Let $H$ be a pointed Hopf algebra. Then $G(H) = \{1\}$ if and only if dim($H$) = 1. Moreover, if $H$ is not cosemisimple, then there is a $g \in G(H)$ such that $P_{g,1}(H)$ is not contained in the coradical of $H$.

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Theorem 2. Let $H$ be a pointed Hopf algebra. If $H$ is not semisimple, then there exist two natural numbers $m, n$, with $m \neq 1$ and $m$ divides $n$, an $m$th primitive root of 1 (denoted by $\omega$) and two elements $g, x \in H$ such that
\begin{enumerate}[(a)]
\item $gx = \omega x g$;
\item $g$ is a group-like element of order $n$;
\item $x \in P_{g,1}(H)$ and $x^m$ is either 0 or $g^m - 1$.
\end{enumerate}

Proof. Let $g \neq 1$ be a group-like element as in the third part of Proposition 1. Let $\phi_g$ be the inner automorphism of $H$ afforded by $g$. Let $n$ be the order of $g$. Obviously $\phi_g$ is semisimple, so its restriction to $P_{g,1}(H)$ has an eigenvalue $\omega \neq 1$; otherwise there is $x \in P_{g,1}(H)$ which is not in $k[G(H)]$, such that $gx = x g$. The subalgebra generated by $x$ and $g$ is a group algebra (it is pointed and commutative), thus $x \in k[G(H)]$, a contradiction. We choose an eigenvalue $\omega \neq 1$ and a corresponding eigenvector $x$ of $\phi_g$. Hence $gx = \omega x g$ and $x$ is in $P_{g,1}(H)$ by construction. Let $m$ be the order of $\omega$. Of course, $m$ divides $n$, so we have only to prove that $x^m$ equals either 0 or $g^m - 1$. Indeed, by [3, Proposition 1] we obtain $\Delta(x^m) = x^m \otimes g^m + 1 \otimes x^m$; therefore the subalgebra $H'$ generated by $g$ and $x^m$ is a group algebra (being a commutative Hopf subalgebra of $H$). We end the proof by remarking that $x^m$ is a $g^m$, 1-primitive element in $H'$.

Let $n$ be a natural number and let $\omega$ be a primitive $n$th-root of 1. We recall that, by definition, $H_{n^2 \omega}$ is the Hopf algebra generated as an algebra by two elements $g$ and $x$ satisfying the relations $g^n = 1$, $x^n = 0$, $gx = \omega x g$. The coalgebra structure is defined such that $g$ is a group-like element and $x$ is $g$, 1-primitive.

Corollary 3 (Andruskiewitsch, Chin). If $p$ is a prime natural number and $H$ is a pointed Hopf algebra of dimension $p^2$, then $H \simeq k[G]$ or $H \simeq H_{p^2 \omega}$, where $G$ is a group with $p^2$ elements and $\omega$ is a certain primitive $n$th-root of 1.

Corollary 4. Let $p$ and $q$ be two different prime numbers. If $H$ is a pointed Hopf algebra of dimension $pq$, then $H$ is semisimple.

2. HOPF ALGEBRAS OF DIMENSION 6

In this section we shall obtain the complete classification of Hopf algebras of dimension 6, as an application of Corollary 4. Namely, we shall prove the following

Theorem 5. Let $H$ be a Hopf algebra of dimension 6. Then $H$ is isomorphic to $k[C_6]$, $k[S_3]$ or $k[S_3]^*$, where $C_6$ and $S_3$ are respectively the cyclic group with 6 elements and the symmetric group with 6 elements.

Proof. We have to show that any Hopf algebra of dimension 6 is semisimple, as such a Hopf algebra is isomorphic to $k[C_6]$, $k[S_3]$ or $k[S_3]^*$ (see [1]). Let us suppose that $H$ is a 6-dimensional Hopf algebra which is not semisimple. By the preceding corollary, $H$ is neither pointed nor cosemisimple (any finite dimensional cosemisimple Hopf algebra over a field of characteristic 0 is semisimple). Then the coradical of $H$ is isomorphic to $M_2(k)^*$ or $M_2(k)^* \oplus k$. The first case is not possible, as $\varepsilon_H$ would induce an algebra map from $M_2(k) \simeq H^*/J(H^*)$ to $k$. Thus the coradical of $H$ must be $M_2(k)^* \oplus k$ and, by [2, Thm. 5.4.2], there exists a coideal $I$ of dimension 1 such that $H = \text{corad}(H) \oplus I$. Let $x$ be an element of $I$ which is not 0. Then $\Delta(x) = x \otimes a + b \otimes x$, where $a$ and $b$ are in $H$. Writing explicitly the equality $(\Delta \otimes I_H)(\Delta(x)) = (I_H \otimes \Delta)(\Delta(x))$ we can see easily that
\begin{align*}
\Delta(a) &= a \otimes a + c \otimes x, \\
\Delta(b) &= b \otimes b + x \otimes c, \\
\Delta(c) &= a \otimes c + c \otimes b,
\end{align*}
where \( c \in H \). Therefore the vector space generated by \( a, b, c \) and \( x \) is a subcoalgebra \( C \) of \( H \). The coalgebra \( M_2(k)^* \) is simple, hence \( M_2(k)^* \cap C = M_2(k)^* \) or \( M_2(k)^* \cap C = 0 \). In the first case it follows that \( M_2(k)^* = C \) and then \( x \in M_2(k)^* \), which contradicts the choice of \( x \). In conclusion \( M_2(k)^* \cap C = 0 \), which implies \( \dim(C) \leq 2 \). Actually, one gets \( \dim(C) = 2 \) and \( M_2(k)^* \oplus C = H \). \( C \) cannot be cosemisimple, otherwise \( H \) is semisimple, so \( \corad(C) = k1 \) and \( H_1 = C \). But \( C_1 = \corad(C) \oplus P(C) \), by [2, Lemma 5.3.2], thus \( 0 \neq P(C) \subseteq P(H) \), a contradiction with the second part of Proposition 1.

**Remark 6.** The referee informed us that the results of the preceding theorem were already obtained by R. Williams [4].

**References**


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