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A MONOTONEITY PROPERTY OF THE GAMMA FUNCTION

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Abstract. In this paper we obtain a monotoneity property for the gamma function that yields sharp asymptotic estimates for $\Gamma(x)$ as x tends to ∞, thus proving a conjecture about $\Gamma(x)$.

1. Introduction

For real and positive values of x the Euler gamma function Γ and its logarithmic derivative Ψ, the so-called digamma function, are defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,\ \ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.1}$$

For extensions of these functions to complex variables and for basic properties see [WW].

Over the past half century many authors have obtained inequalities for these important functions (see [A1], [A2] and bibliographies in those papers). In keeping with this tradition we here obtain a monotoneity property of the gamma function that yields a sharp asymptotic estimate for $\Gamma(x)$ as x tends to ∞.

In [AVV, Lemma 2.39] the following result was obtained.

1.2. Lemma.

$$\lim_{x\to\infty} \frac{\log\Gamma(1+\frac{x}{2})}{x\log x} = \frac{1}{2}, \tag{1.3}$$

and

$$f(x) \equiv \frac{1}{x}\log\Gamma(1+\frac{x}{2}) \tag{1.4}$$

is strictly increasing from $[2,\infty]$ *onto* $[0,\infty)$.

It was conjectured in [AVV, Remark 2.41] that the function in (1.3) is strictly increasing from $[2,\infty)$ onto $[0,1/2)$. In order to obtain an affirmative answer to the above conjecture we prove here the following result.

1.5. Theorem. *The function* $f(x) \equiv (\log\Gamma(x+1))/(x\log x)$ *is strictly increasing from* $(1,\infty)$ *onto* $(1-\gamma,1)$, *where* γ *is the Euler-Mascheroni constant. In particular, for* $x \in (1,\infty)$,

$$x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1} \tag{1.6}$$

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and

$$\lim_{x\to\infty} \frac{\log\Gamma(x)}{(x-1)\log(x-1)} = 1. \tag{1.7}$$

In [AVV] the the following result was also obtained.

1.8. Lemma. *Let $\Omega_n = \pi^{n/2}/\Gamma(1+n/2)$ denote the n-dimensional volume of the unit ball B^n in $\mathbb{R}^n$. Then*

$$\lim_{n\to\infty} \Omega_n^{1/(n\log n)} = e^{-\frac{1}{2}}, \tag{1.9}$$

$$\Omega_n^{1/n} \quad \text{decreases strictly to} \quad 0 \quad \text{as} \quad n\to\infty, \tag{1.10}$$

$$\sum_{n=2}^{\infty} \Omega_n^{1/\log n} \quad \text{is convergent.} \tag{1.11}$$

It was pointed out in [AVV, Remark 2.41] that if the function in (1.3) above has the conjectured property this would imply that $\Omega_n^{1/(n\log n)}$ is strictly decreasing for $n \geq 2$. Thus our Theorem 1.5 implies this monotoneity of $\Omega_n^{1/(n\log n)}$ (see Corollary 3.1). It should be observed that Ω_n itself is not monotone [BH, pp. 263, 264] (cf. [SV]).

In this paper we let $\mathbb{N}$ denote the set of positive integers and, for the real number x, let $[x]$ denote the integer satisfying $x-1 < [x] \leq x$.

2. Preliminary results

Before establishing the main theorem we need to prove some technical lemmas.

2.1. Lemma. *The function $f(x) \equiv \sum_{n=1}^{\infty} \frac{n-x}{(n+x)^3}$ is positive for $x \in [1,4)$.*

Proof. Let $u(t,x) = (t-x)/(t+x)^3$, for $t, x \in [1,\infty)$. Then u is strictly decreasing in t on $[2x,\infty)$ for any $x \in [1,\infty)$, since $\partial u/\partial t = 2(2x-t)/(x+t)^4$.

For $k \in \mathbb{N}$ and $x \geq 1$, let $v_k = u(k,x)$. Then $v_k(x)$ is strictly decreasing in k for $k \in \mathbb{N} \cap [2x,\infty)$ and

$$v_k(x) = u(k,x) > u(t,x) \quad \text{for} \quad t > k \geq 2x.$$

Hence, for $k \geq 2x \geq 2$,

$$v_k(x) = \frac{k-x}{(k+x)^3} = \int_k^{k+1} v_k(x)dt > \int_k^{k+1} u(t,x)dt,$$

so that

$$\begin{cases} \displaystyle\sum_{n=2[x]+2}^{\infty} v_n(x) > \sum_{n=2[x]+2}^{\infty} \int_n^{n+1} u(t,x)dt = \int_{2[x]+x}^{\infty} \frac{t-x}{(t+x)^3}dt \\ \displaystyle\qquad = \int_{2[x]+2}^{\infty} \left[\frac{1}{(t+x)^2} - \frac{2x}{(t+x)^3}\right] dt = \frac{2(1+[x])}{(2[x]+x+2)^2}. \end{cases} \tag{2.2}$$

It follows from (2.2) that

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{2[x]+1} v_n(x) + \sum_{n=2[x]+2}^{\infty} v_n(x) > \sum_{n=1}^{2[x]+1} v_n(x) + \frac{2(1+[x])}{(2[x]+x+2)^2} \\
&= \left[\frac{1-x}{(1+x)^3} + \frac{2-x}{(2+x)^3} + \cdots + \frac{[x]-x}{([x]+x)^3}\right] \\
&+ \left[\frac{[x]+1-x}{([x]+x+1)^3} + \frac{[x]+2-x}{([x]+x+2)^3} + \cdots + \frac{2[x]+1-x}{(2[x]+x+1)^3}\right] \\
&\qquad + \frac{2(1+[x])}{(2[x]+x+2)^2} \\
&\geq \frac{1}{(x+1)^3}\left[(1+x)+(2-x)+\cdots+([x]-x)\right] \\
&+ \frac{1}{(2[x]+x+1)^3}\left[([x]+1-x)+([x]+2-x)+\cdots+(2[x]+1-x)\right] \\
&\qquad + 2\frac{[x]+1}{(2[x]+x+2)^2} \\
&= \frac{[x]([x]+1-2x)}{2(x+1)^3} + \frac{([x]+1)(3[x]+2-2x)}{2(2[x]+x+1)^3} + \frac{2[x]+1}{(2[x]+x+2)^2} \\
&\equiv f_1(x).
\end{aligned}
$$

If $1 \leq x < 2$, then $[x] = 1$ and

$$
\begin{cases}
f_1(x) = \dfrac{1-x}{(x+1)^3} + \dfrac{5-2x}{(3+x)^3} + \dfrac{4}{(4+x)^2} \\[2ex]
\qquad = \dfrac{1}{(x+1)^3(4+x)^2}[3x^3+5x^2+4x+20] + \dfrac{5-2x}{(3+x)^3} > 0.
\end{cases} \tag{2.3}
$$

If $2 \leq x < 3$, then $[x] = 2$ and

$$
\begin{cases}
f_1(x) = \dfrac{3-2x}{(x+1)^3} + 3\dfrac{4-x}{(5+x)^3} + \dfrac{6}{(6+x)^2} \\[2ex]
\qquad = \dfrac{4x^3-3x^2-18x+114}{(x+1)^3(6+x)^2} + 3\dfrac{4-x}{(5+x)^3} \\[2ex]
\qquad > \dfrac{4x^3+33}{(x+1)^3(6+x)^2} + 3\dfrac{4-x}{(5+x)^3} > 0.
\end{cases} \tag{2.4}
$$

If $3 \le x < 4$, then $[x] = 3$ and

$$(2.5)\quad \begin{cases} f_1(x) = \dfrac{3(2-x)}{(x+1)^3} + \dfrac{2(11-2x)}{(7+x)^3} + \dfrac{8}{(8+x)^2} \\[2ex] \quad = \dfrac{5x^3 - 18x^2 - 72x + 392}{(x+1)^3(8+x)^2} + 2\dfrac{11-2x}{(7+x)^3} \\[2ex] \quad > \dfrac{1}{(8+x)^2}\left[\dfrac{5x^3 - 18x^2 - 72x + 392}{(x+1)^3} + \dfrac{22-4x}{7+x}\right] \\[2ex] \quad = \dfrac{x^4 + 27x^3 - 144x^2 - 50x + 2766}{(x+8)^2(x+7)(x+1)^3} \\[2ex] \quad > \dfrac{1072}{(x+8)^2(x+7)(x+1)^3} > 0. \end{cases}$$

The conclusion now follows from (2.2) – (2.5). □

2.6. Lemma. *The function* $g(x) \equiv x^2\Psi'(1+x) - x\Psi(1+x) + \log\Gamma(1+x)$ *is positive for all* $x \in [1, \infty)$.

Proof. From the well-known difference equation $\Gamma(x+1) = x\Gamma(x)$ [WW, p. 237] it follows easily that

$$(2.7)\quad \Psi(x+1) = \frac{1}{x} + \Psi(x),$$

from which we obtain

$$(2.8)\quad xg'(x) = x^2\Psi'(x) + x^3\Psi''(x) + 1 \equiv g_1(x).$$

Since

$$\Psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2},\ \Psi''(x) = -2\sum_{n=0}^{\infty} \frac{1}{(x+n)^3}$$

[Ah, p. 200, (31)], g_1 can be rewritten as

$$(2.9)\quad g_1(x) = x^2 \sum_{n=1}^{\infty} \frac{n-x}{(n+x)^3} = x^2 f(x),$$

where f is as in Lemma 2.1.

From Lemma 2.1, (2.8), and (2.9), we see that g is strictly increasing on $[1, 4]$, and hence, by [W, Exercise 2, p. 80] and [Ah, p. 199, (29)],

$$(2.10)\quad g(x) \ge g(1) = \Psi'(2) - \Psi(2) = \sum_{n=2}^{\infty} \frac{1}{n^2} - 1 + \gamma = \frac{\pi^2}{6} + \gamma - 2 = 0.2221\cdots > 0$$

for $x \in [1, 4]$.

Next, since

$$(2.11)\quad \begin{cases} \dfrac{1}{x} < \Psi'(x) < \dfrac{1}{x-1}, \\[2ex] \log x - \dfrac{1}{x} < \Psi(x) < \log x - \dfrac{1}{2x}, \end{cases}$$

for $x > 1$ (see [ABRVV, Theorem 3.1], [S, Lemma 4b], [A2, (2.2)]), it follows from (2.7) that

$$\begin{cases} g(x) = x^2\Psi'(x) - x\Psi(x) + \log x + \log\Gamma(x) - 2 \\ \qquad > x + (1-x)\log x + \log\Gamma(x) - \dfrac{3}{2} \equiv g_2(x). \end{cases} \tag{2.12}$$

Differentiation gives

$$g_2'(x) = \Psi(x) - (\log x - \frac{1}{x}),\ x > 1,$$

which is positive by (2.11). Hence g_2 is strictly increasing on $[1,\infty)$ so that, for $x \in [4,\infty)$,

$$g(x) > g_2(x) \geq g_2(4) = \frac{5}{2} - 3\log 4 + \log 6 = 0.1328\cdots > 0. \tag{2.13}$$

The result now follows from (2.10) and (2.13). □

2.14. Lemma. *The function* $h(x) \equiv x\Psi(1+x) - \log\Gamma(1+x)$ *is strictly increasing from* $[0,\infty)$ *onto* $[0,\infty)$. *Moreover,*

$$\lim_{x\to\infty} \frac{h(x)}{x} = 1 \quad \textit{and} \quad \frac{h(x)}{x} = 1 + \mathrm{O}(\frac{\log x}{x}) \tag{2.15}$$

as $x \to \infty$.

Proof. Differentiation gives

$$h'(x) = x\Psi'(x+1) > 0,$$

and the monotoneity of h follows. Clearly, $h(0) = 0$. Since

$$\begin{cases} \log\Gamma(x) = (x - \frac{1}{2})\log x - x + \frac{1}{2}\log(2\pi) + \mathrm{O}(\frac{1}{x}), \\ \Psi(x) = \log x - \frac{1}{2x} + \mathrm{O}(\frac{1}{x^2}) \end{cases} \tag{2.16}$$

as $x \to \infty$ by [S, Theorems 4, 5], we have

$$\lim_{x\to\infty} \frac{h(x)}{x} = \lim_{x\to\infty} \left[\log(1+x) - \frac{1}{2(x+1)} - \frac{1}{x}(x + \frac{1}{2})\log(x+1) + \frac{x+1}{x}\right]$$

$$= \lim_{x\to\infty} \left[1 - \frac{\log(1+x)}{2x}\right] = 1.$$

□

2.17. Lemma. *The function* $H(x) \equiv \log x - \frac{1}{h(x)}\log\Gamma(x+1)$ *is strictly increasing from* $[1,\infty)$ *onto* $[0,1)$. *Here* h *is as in Lemma* 2.14. *In particular, for all* $x \in (1,\infty)$,

$$1 - \frac{1}{\log x} < \frac{\log\Gamma(x+1)}{x\Psi(x+1)} < 1 - \frac{1}{1+\log x}. \tag{2.18}$$

Proof. Clearly, $H(1) = 0$. It follows from (2.15) and (2.16) that

$$\lim_{x\to\infty} H(x) = \lim_{x\to\infty}\left[\left(1-\frac{x}{h(x)}\right)\log x + \frac{x}{h(x)}\left(-\frac{\log x}{2x}+1-\frac{\log(2\pi)}{2x}+\mathrm{O}(\frac{1}{x^2})\right)\right]$$

$$= \lim_{x\to\infty}\left[\mathrm{O}\left(\frac{(\log x)^2}{h(x)}\right)+\frac{x}{h(x)}+\mathrm{O}(\frac{1}{x})\right]$$

$$= 1 + \lim_{x\to\infty}\mathrm{O}\left(\frac{1}{x}(\log x)^2\right) = 1.$$

Next, by differentiation, we get

$$\begin{aligned} H'(x) &= \frac{1}{x} - \frac{1}{(h(x))^2}\left[\Psi(x+1)h(x) - x\Psi'(x+1)\log\Gamma(x+1)\right] \\ &= \frac{1}{x} - \frac{1}{x(h(x))^2}\big[(x\Psi(x+1) - \log\Gamma(x+1) + \log\Gamma(x+1))h(x) \\ &\qquad - x^2\Psi'(x+1)\log\Gamma(x+1)\big] \\ &= \frac{\log\Gamma(x+1)}{x(h(x))^2}[x^2\Psi'(x+1) - h(x)] \\ &= \frac{g(x)}{x(h(x))^2}\log\Gamma(x+1), \end{aligned}$$

where g is as in Lemma 2.6. Hence the monotoneity of H follows from Lemma 2.6. The inequality (2.18) is clear. □

3. Proof of the Main Theorem

We now show how Theorem 1.5 follows from the lemmas in Section 2. By differentiation we get

$$(x\log x)^2 f'(x) = h(x)H(x),$$

where $f(x) = (\log\Gamma(x+1))/(x\log x)$ and where h and H are as in Lemmas 2.14 and 2.17, respectively. Hence, the monotoneity of f follows from Lemmas 2.14 and 2.17.

Next, by l'Hôpital's Rule, we have

$$f(1^+) = \lim_{x\to 1}\frac{\Psi(x+1)}{1+\log x} = \Psi(2) = -1-\gamma$$

[AS, 6.3.3] and, by (2.16),

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\frac{(x+\frac{1}{2})\log(x+1)-(x+1)+\frac{1}{2}\log(2\pi)}{x\log x} \\ &= \lim_{x\to\infty}\frac{\log(x+1)}{\log x} = 1. \end{aligned}$$

Inequality (1.6) and limit (1.7) are clear. □

3.1. Corollary. (1) *The function* $f(x) \equiv (\log \Gamma(1+\frac{x}{2}))/(x \log x)$ *is strictly increasing from* $[2,\infty)$ *onto* $[0,1/2)$ (*the conjecture in* [AVV, Remark 2.41] *is true*).

(2) *For* $n \in \mathbb{N}$, *let* $\Omega_n = \pi^{n/2}/\Gamma(1+n/2)$ *be the n-dimensional volume of the unit ball* B^n *in* $\mathbb{R}^n$. *Then the sequence* $G(n) \equiv \Omega_n^{1/(n \log n)}$ *is strictly decreasing for* $n \geq 2$, *with* $G(2) = \pi^{1/\log 4}$ *and* $\lim_{n\to\infty} G(n) = e^{-1/2}$.

Proof. (1) Let $t = x/2$. Then

$$f(x) = \frac{\log t}{2\log(2t)} \cdot \frac{\log \Gamma(t+1)}{t \log t},$$

and the conclusion follows from Theorem 1.5 since the function $(\log t)/\log(2t)$ is strictly increasing from $[1,\infty)$ onto $[0,1)$.

(2) Since

$$\log G(n) = \frac{1}{2}\frac{\log \pi}{\log n} - \frac{\log \Gamma(1+\frac{n}{2})}{n \log n},$$

the assertion follows from part (1). □

3.2. *Remark.* By methods similar to those used to prove Theorem 1.5 we can show that the function $f(x) \equiv (\log \Gamma(x+1))/((x-1)\log(2x))$ is strictly increasing from $[4.5,\infty)$ onto $[c,1)$, where $c = (\log(135\sqrt{\pi}/32))/(7\log 3)$. However, f is not monotone on $(1,\infty)$ since $f'(x) < 0$ when x is near 1.

3.3. Conjecture. $f(x) \equiv (\log \Gamma(x+1))/(x \log x)$ is concave on $(1,\infty)$.

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