

A SYMMETRY PROBLEM FROM PROBABILITY

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ABSTRACT. We examine solutions of two related boundary value problems for smooth domains in Euclidean space which arise from variational problems in probability. We show that the existence of solutions to each problem implies that the domain is a sphere.

1. INTRODUCTION

Let D be a connected open domain in \mathbf{R}^n with a smooth boundary S , where S is an oriented, compact, embedded hypersurface. In this note we prove the following result.

Theorem 1. *Let u be the solution of*

$$(1.1) \quad \begin{aligned} \Delta u &= -1 \text{ on } D, \\ u &= 0 \text{ on } S, \end{aligned}$$

and let U be the solution of

$$(1.2) \quad \begin{aligned} \Delta U &= -u \text{ on } D, \\ U &= 0 \text{ on } S. \end{aligned}$$

Suppose that

$$(1.3) \quad \frac{\partial u}{\partial n} \frac{\partial U}{\partial n} = k$$

where k is a constant and $\partial/\partial n$ is the inward-pointing normal derivative along S . Then S is a sphere and u and U are radial.

This symmetry result is closely related to the fundamental result of Serrin [S], and our proof follows Serrin's.

We were led to Theorem 1 by a study of variational problems for functionals defined on the space of embedded, compact, oriented, unparametrized hypersurfaces in \mathbf{R}^n which bound a connected open domain [KM2]. We denote this space by $\mathcal{SH}(\mathbf{R}^n)$. The functionals considered were defined in terms of exit times for Brownian motion.

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Specifically, given $S \in SH(\mathbf{R}^n)$, let $D = D(S)$ be the domain bounded by S and let $|D|$ be the volume of D . Let τ_S be the first exit time of Brownian motion from $D(S)$; that is,

$$\tau_S = \tau_S(\omega) = \inf\{t \geq 0; W_t(\omega) \in S\}$$

where ω belongs to a probability space and W_t is n -dimensional Brownian motion. Define a functional $\mathcal{V} : SH(\mathbf{R}^n) \rightarrow \mathbf{R}$ by

$$\mathcal{V}(S) = \frac{1}{|D|} \int_D E_x(\tau_S^2) - (E_x(\tau_S))^2 dx$$

where $E_x(\tau_S)$ is the expected value of the random variable τ_S , given that Brownian motion starts at x . The functional \mathcal{V} denotes the average variance of the first exit time for Brownian motion for the domain bounded by S . For v_0 an arbitrary constant, define $SH_{v_0}(\mathbf{R}^n)$ by

$$SH_{v_0}(\mathbf{R}^n) = \{S \in SH(\mathbf{R}^n) : |D(S)| = v_0\}.$$

Restrict \mathcal{V} to the subspace $SH_{v_0}(\mathbf{R}^n)$. By computing the first variation of the functional under volume-preserving infinitesimal variations of the domain $D(S)$, it follows that at critical points of \mathcal{V} , the overdetermined boundary value problem in Theorem 1 must admit a solution [KM2]. Thus, by Theorem 1, a critical point of \mathcal{V} (restricted to $SH_{v_0}(\mathbf{R}^n)$) must be a sphere.

Fix $x_0 \in \mathbf{R}^n$. If

$$SH_{v_0}(\mathbf{R}^n, x_0) = \{S \in SH_{v_0}(\mathbf{R}^n) : x_0 \in D(S)\},$$

a similar symmetry problem is associated with the functional $\mathcal{E} : SH_{v_0}(\mathbf{R}^n, x_0) \rightarrow \mathbf{R}$ defined by

$$\mathcal{E}(S) = E_{x_0}(\tau_S).$$

At critical points of \mathcal{E} , solutions of (1.1) satisfy

$$(1.4) \quad \frac{\partial u}{\partial n} \frac{\partial G}{\partial n}(x_0, \cdot) = k,$$

where $G(x_0, \cdot)$ is the Green's function for the domain $D(S)$ with pole at x_0 and k is a constant [KM1]. For this symmetry problem, we have

Theorem 2. *If u satisfies (1.1) and (1.4), then S must be a sphere centered at x_0 .*

2. PROOF OF THEOREM 1

We follow most of the notation of Serrin's result in [S]; for the reader's convenience we recap the relevant parts of the proof there. Fix a hyperplane $T_0 \subset \mathbf{R}^n$ not intersecting D . Move the hyperplane parallel to itself, toward D ; it will eventually cut off from D an open cap, $\Sigma(T)$, where T is parallel to T_0 and intersects D . The reflection of $\Sigma(T)$ in T , $\Sigma'(T)$, will initially lie in D . This will remain true as T continues to move into D , until one of the following two conditions occurs:

- (i) $\Sigma'(T)$ is internally tangent to S at a point P not on T , or
- (ii) T is orthogonal to S at some point Q .

Below, we will assume T satisfies either (i) or (ii) above. Define a function V in $\Sigma' = \Sigma'(T)$ by $V(x) = U(x')$, where x' is the reflection of x in T . Let $v(x) = u(x')$ for $x \in \Sigma'$; note that $\Delta V(x) = -v(x)$ and $\Delta v = -1$.

Moreover, V and v satisfy the following boundary conditions in Σ' : on $T \cap \partial\Sigma'$, $V = U$ and $v = u$; on the remainder, $T^c \cap \partial\Sigma'$, $V = 0$ and $v = 0$. Also, on this latter

set, $U \geq 0$ and $u \geq 0$. This follows because $\Sigma' \subset D$ and the maximum principle for the laplacian shows that U and u are positive in D .

As pointed out in [S], we will have shown S to be a sphere if, for arbitrary initial T_0 , D is symmetric about the hyperplane T satisfying (i) or (ii) above. The symmetry of D about T follows if either $U \equiv V$ or $u \equiv v$ in Σ' . We assume neither condition holds and obtain a contradiction.

If $u \not\equiv v$ in Σ' , then by the maximum principle applied to $u - v$ it follows that $u - v > 0$ in the interior of Σ' . Since $\Delta(U - V) = v - u < 0$ in Σ' , it similarly follows that $U > V$ in Σ' .

First suppose T satisfies condition (i) above. As in [S], the boundary point maximum principle shows that

$$\frac{\partial}{\partial n}(u - v) > 0 \text{ and } \frac{\partial}{\partial n}(U - V) > 0 \text{ at } P.$$

This contradicts the fact that at P ,

$$\frac{\partial u}{\partial n} \frac{\partial U}{\partial n} = k \text{ and } \frac{\partial v}{\partial n} \frac{\partial V}{\partial n} = k.$$

Second, suppose T satisfies condition (ii) above. Choose coordinates in \mathbf{R}^n so that the origin is at Q , the positive x_n -axis has the same direction as the inward normal to S at Q , and the positive x_1 -axis is normal to T , pointing away from Σ' .

Let \vec{s} be a vector at Q which is nontangential with respect to $\partial\Sigma'$, and let $\partial/\partial s$ be the corresponding directional derivative. We need the following result, essentially Lemma 1 of [S]:

Lemma. *Suppose $w \in C^2(\overline{\Sigma'})$, $\Delta w \leq 0$ and $w \geq 0$ in Σ' , and $w = 0$ at Q . Then if $w \not\equiv 0$, either*

$$\frac{\partial w}{\partial s} > 0 \text{ or } \frac{\partial^2 w}{\partial s^2} > 0 \text{ at } Q.$$

We apply the lemma to both $w = u - v$ and $w = U - V$, with $\partial/\partial s = -\partial/\partial x_1 + \partial/\partial x_n$. Near Q ,

$$(2.1) \quad v(x_1, x_2, \dots, x_n) = u(-x_1, x_2, \dots, x_n).$$

Denoting $\partial/\partial x_i$ with a subscript i , it follows that $v_n = u_n$ at Q . Because $u = 0$ on S , it follows easily that $\partial u/\partial x_i = 0$ at Q for $1 \leq i \leq n - 1$. Hence $u_1 = 0$ and $v_1 = 0$ at Q , and thus

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial s}$$

at Q . Similarly, $\partial U/\partial s = \partial V/\partial s$ at Q . By the Lemma and the assumptions that $U \not\equiv V$ and $u \not\equiv v$, it follows that

$$(2.2) \quad \frac{\partial^2}{\partial s^2}(u - v) > 0 \text{ and } \frac{\partial^2}{\partial s^2}(U - V) > 0 \text{ at } Q.$$

Now consider

$$\frac{\partial^2}{\partial s^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_n^2} - 2 \frac{\partial^2}{\partial x_1 \partial x_n}.$$

By (2.1), it follows that

$$(2.3) \quad v_{11} = u_{11} \text{ and } v_{nn} = u_{nn}.$$

Similar reasoning shows that

$$(2.4) \quad V_{11} = U_{11} \quad \text{and} \quad V_{nn} = U_{nn}.$$

Since U and u vanish on S , we may rewrite (1.3) as $\nabla U \cdot \nabla u = k$ on S , i.e.,

$$\sum_{i=1}^n U_i u_i = k \quad \text{on } S.$$

Differentiating this equality at $Q \in S$ in the direction $\partial/\partial x_1$ tangent to S at Q , it follows that

$$U_n u_{n1} + u_n U_{n1} = 0,$$

because $U_i = 0$ and $u_i = 0$ at Q for $1 \leq i \leq n-1$. Similarly, $V_n v_{n1} + v_n V_{n1} = 0$ at Q . Because $U_n = V_n$ and $u_n = v_n$ at Q , we may subtract to obtain

$$(2.5) \quad U_n(v_{n1} - u_{n1}) + u_n(V_{n1} - U_{n1}) = 0.$$

Combining (2.3), (2.4), and (2.5), we find that

$$U_n \frac{\partial^2}{\partial s^2}(u - v) + u_n \frac{\partial^2}{\partial s^2}(U - V) = 0 \quad \text{at } Q.$$

But this contradicts (2.2), because both U_n and u_n are positive at Q .

Remark. Notice that $\Delta^2 U = 1$. Bennett [B] proves the following symmetry result for this fourth-order equation: if U and its normal derivative vanish on S , and ΔU is constant on S , then D is a ball. His proof follows the method of [W], rather than that of [S]. The authors, however, were unable to prove the results here by that method.

3. PROOF OF THEOREM 2

For convenience, write $G(x) = G(x_0, x)$ and choose the sign of G so that $G > 0$ in D . With that choice, $G(x)$ is determined by

$$\begin{aligned} \Delta G(x) &= -\delta_{x_0}(x) \quad \text{in } D, \\ G &= 0 \quad \text{on } S \end{aligned}$$

where δ_{x_0} is the delta mass at x_0 .

As above, the method of moving planes yields a T satisfying (i) or (ii). Again, if x' is the reflection of x in T , define $v(x) = u(x')$ for $x \in \Sigma' = \Sigma'(T)$. Also define $\tilde{G}(x) = G(x')$. To prove Theorem 2, it suffices to show that D must be symmetric about T , as in the proof of Theorem 1, and in addition that $x_0 \in T$.

To begin with, assume x_0 indeed lies on T . The function $G - \tilde{G}$ is harmonic in Σ' , vanishing on $T \cap \partial\Sigma'$ and nonnegative on $T^c \cap \partial\Sigma'$. The proof of Theorem 1, with $G - \tilde{G}$ replacing $U - V$, shows that $v \equiv u$ and $\tilde{G} \equiv G$ in Σ' . Hence D is symmetric about T .

We next show that $x_0 \notin T$ inevitably leads to a contradiction.

Case one: $x_0 \in \Sigma'$. Because $\Delta(G - \tilde{G}) = -\delta_{x_0}$, $G - \tilde{G}$ is still superharmonic in Σ' , hence $G - \tilde{G} > 0$ there. Following the techniques of Theorem 1, we are led to a contradiction.

Case two: $x_0 \notin \Sigma' \cup T \cup \Sigma$. Because $G - \tilde{G}$ is harmonic in Σ' in this case, $G \geq \tilde{G}$ by the maximum principle. (If $x_0 \in \partial\Sigma'$, the maximum principle may still be used, since $\tilde{G}(x) \rightarrow 0$ and $G(x) \rightarrow +\infty$ as $x \rightarrow x_0$.) By the methods of Theorem 1, it follows that D is symmetric about T , and $D = \Sigma' \cup (T \cap D) \cup \Sigma$. But this contradicts $x_0 \in D$ and $x_0 \notin \Sigma' \cup T \cup \Sigma$.

Case three: $x_0 \in \Sigma$. In Σ' , $\Delta(G - \tilde{G}) = \delta_{x_0'}$, so $G - \tilde{G}$ is now *subharmonic*, preventing direct treatment of $u - v$ and $G - \tilde{G}$. A contradiction is obtained by considering the family $\{T_1\}$ of moving planes which are parallel to T but which start at an initial position on the opposite side of D and move in the direction opposite to that of the family which led to T . Again, it is required that the reflected cap $\Sigma'(T_1)$ remain in D . The family $\{T_1\}$ will reach a plane T_1^* satisfying (i) or (ii) above. In doing so, however, $\{T_1\}$ will never move past T because of the conditions $\Sigma'(T_1) \subset D$ and $\Sigma'(T) \subset D$. Since $x_0 \in \Sigma \equiv \Sigma(T)$, it follows that $x_0 \notin \Sigma(T_1^*)$ and $x_0 \notin T_1^*$, so that either case one or case two above holds for T_1^* replacing T . Both those cases, however, resulted in contradictions.

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