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TRIEBEL-LIZORKIN SPACES ASSOCIATED WITH LAGUERRE AND HERMITE EXPANSIONS

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Abstract. It is proved that Triebel-Lizorkin spaces for some Laguerre and Hermite expansions are well-defined.

1. Introduction

Let D be a self-adjoint positive operator acting on $L^2(\mathfrak{M})$, and let dE be its spectral resolution, that is,

$$Df = \int_0^\infty \lambda dE(\lambda) f. \tag{1.1}$$

For $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and a C^∞ function φ satisfying

$$\operatorname{supp}\varphi \subset [1/2, 2], \quad |\varphi(\lambda)| > c > 0 \quad \text{for} \quad \lambda \in [3/4,\ 7/4], \tag{1.2}$$

we define the Triebel-Lizorkin norm associated with D (and with φ) by

$$\|f\|_{D_p^{\alpha}\ ^{q}(\varphi)} = \|[\sum_{\mu\in\mathbb{Z}} (2^{\mu\alpha}|Q_\mu f|)^q]^{1/q}\|_{L^p(X)}, \tag{1.3}$$

where

$$Q_\mu f = \varphi(2^{-\mu}D)f = \int_0^\infty \varphi(2^{-\mu}\lambda)\, dE(\lambda) f. \tag{1.4}$$

Note that if $D = \Delta = -\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^d$, then the norm $\|f\|_{\Delta_p^{\alpha}\ ^{q}(\varphi)}$ is equivalent to the classical Triebel-Lizorkin norm $\|f\|_{F_p^{2\alpha}\ ^{q}}$.

Triebel-Lizorkin spaces associated with the one-dimensional Hermite operator

$$\mathcal{H} = -\frac{\partial^2}{\partial x^2} + x^2$$

were studied by J. Epperson in [E1] and [E2]. It was proved there, using Mehler's formula, that the definition of the corresponding space $F_{\mathcal{H}_p^{\alpha,q}}$ is independent of the particular choice of the function φ.

The present paper continues these studies. We consider Triebel-Lizorkin spaces associated with some Laguerre expansions and multidimensional Hermite expansions. We use some ideas from [E1] combined with Heisenberg group methods (cf. [HJ]). Symbolic calculus for sublaplacians on Heisenberg groups (cf. Theorem 2.3) plays an essential role in our paper.

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2. Symbolic calculus on Heisenberg groups

Let $\mathbb{H}_d$ be the $2d+1$ dimensional Heisenberg group, that is, $\mathbb{H}_d = \mathbb{C}^d \times \mathbb{R}$ with the multiplication $hh' = (z,t)(z',t') = (z+z', t+t' + \frac{1}{2}\Im(z\bar{z'}))$. Let X_j, Y_j be the elements of the Lie algebra of $\mathbb{H}_d$ which we identify with the left-invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}. \tag{2.1}$$

The corresponding right-invariant vector fields are:

$$\tilde{X}_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \qquad \tilde{Y}_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}. \tag{2.2}$$

The sublaplacian L on $\mathbb{H}$ defined by

$$L = -\sum_{j=1}^{d} X_j^2 + Y_j^2$$

is a positive, homogeneous of degree 2, left-invariant subelliptic differential operator. Let dE be the spectral resolution for L, that is, $Lf = \int_0^\infty \lambda dE(\lambda) f$. If m is a bounded function on $(0,\infty)$, then the operator

$$m(L)f = \int_0^\infty m(\lambda) dE(\lambda) f$$

is left-invariant and bounded on $L^2(\mathbb{H}_d)$.

The following theorem due to Hulanicki (cf. [H]) is the basic tool in our paper.

Theorem 2.3. *If $m \in \mathcal{S}(\mathbb{R})$, then*

$$m(L)f = f * M, \tag{2.4}$$

with M in the Schwartz space $\mathcal{S}(\mathbb{H}_d)$ of functions on $\mathbb{H}_d$.

Moreover, if for $s > 0$ we set $m^s(\lambda) = m(s\lambda)$, then

$$m^s(L) = f * M_s, \tag{2.5}$$

where

$$M_s(h) = M_s(z,t) = s^{-Q/2} M(\frac{z}{\sqrt{s}}, \frac{t}{s}). \tag{2.6}$$

Here $Q = 2d+2$ is the homogeneous dimension of $\mathbb{H}_d$.

3. Laguerre functions

Let

$$\mathcal{L}_k^m(w) = (2\pi)^{-1/2} \left(\frac{k!}{(k+m)!} \right)^{1/2} w^{m/2} L_k^m(w) e^{-w/2}, \quad w > 0, \tag{3.1}$$

be the Laguerre function of type m, $m = 0,1,2,...$, where

$$L_k^m(w) = \sum_{j=0}^{k} \binom{k+m}{k-j} \frac{(-w)^j}{j!} \tag{3.2}$$

is the corresponding Laguerre polynomial of type m, $m = 0,1,2,....$

Let $\mathbb{H}/\Gamma$ denote the reduced Heisenberg group, where $\Gamma = \{(0, 2\pi n) \ : n \in \mathbb{Z}\}$ is a normal discrete central subgroup of $\mathbb{H} = \mathbb{H}_1$. For $p > 0$ and nonnegative integer

m we consider the space $L^p_m(\mathbb{H}/\Gamma)$ which consists of L^p functions f which have the form

$$f(z,t) = e^{it}e^{-im\theta}f_0(r)\ , \quad z = re^{i\theta}. \tag{3.3}$$

It is well known (cf. [T]) that if a C^2 function f on $\mathbb{H}/\Gamma$ has the form (3.3), then Lf is of the same form, where $L = -X^2 - Y^2$ is the sublaplacian on $\mathbb{H}/\Gamma$. Moreover, the functions

$$\phi^m_k(z,u) = e^{iu}e^{-im\theta}\mathcal{L}^m_k(|z|^2/2) \tag{3.4}$$

form an orthonormal basis of $L^2_m(\mathbb{H}/\Gamma)$, and

$$L\phi^m_k = d_k\phi^m_k, \quad \text{where} \ \ d_k = 2k+1. \tag{3.5}$$

Note that the map $W : L^p(\mathbb{R}^+) \to L^p_m(\mathbb{H}/\Gamma)$ given by

$$f(z,u) = (Wg)(z,u) = e^{iu}e^{-im\theta}g(|z|^2/2), \quad \text{where} \ \ z = e^{i\theta}|z|, \tag{3.6}$$

is an isometry from $L^p(\mathbb{R}^+)$ onto $L^p_m(\mathbb{H}/\Gamma)$. If, moreover, f and g are related by (3.6), then

$$\langle g, \mathcal{L}^m_k\rangle = \langle f, \phi^m_k\rangle. \tag{3.7}$$

Consequently, if $g = \sum_k \langle g, \mathcal{L}^m_k\rangle\mathcal{L}^m_k$, then

$$Wg = \sum_k \langle Wg, \phi^m_k\rangle\phi^m_k. \tag{3.8}$$

4. Triebel-Lizorkin spaces for Laguerre expansions

Let φ be a C^∞ function satisfying (1.2). We define the linear operators Q_μ on $L^2(\mathbb{R}^+)$ by

$$Q_\mu\mathcal{L}^m_k = \varphi(2^{-\mu}d_k)\mathcal{L}^m_k. \tag{4.1}$$

For $g = \sum_k a_k\mathcal{L}^m_k, 0 < p, q < \infty$, and $\alpha \in \mathbb{R}$, the Triebel-Lizorkin norm $\|g\|_{\mathcal{L}^{\alpha}_p{}^q(\varphi)}$ is defined by

$$\|g\|_{\mathcal{L}^{\alpha}_p{}^q(\varphi)} = \|[\sum_{\mu\in\mathbb{Z}}(2^{\mu\alpha}|Q_\mu g|)^q]^{1/q}\|_{L^p(\mathbb{R}^+)}. \tag{4.2}$$

On the space $L^2_m(\mathbb{H}/\Gamma)$ we define the corresponding operators $\tilde{Q}_\mu$ by setting

$$\tilde{Q}_\mu\phi^m_k = \varphi(2^{-\mu}d_k)\phi^m_k. \tag{4.3}$$

Obviously for f and g related by (3.6),

$$\|g\|_{\mathcal{L}^{\alpha}_p{}^q(\varphi)} = \|[\sum_{\mu\in\mathbb{Z}}(2^{\mu\alpha}|\tilde{Q}_\mu f|)^q]^{1/q}\|_{L^p(\mathbb{H}/\Gamma)}. \tag{4.4}$$

Our goal in this section is the following

Theorem A. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $0 < q < \infty$. If $\varphi^{(1)}$ and $\varphi^{(2)}$ are two C^∞ functions satisfying (1.2), then there exists a constant Csuch that*

$$C^{-1}\|g\|_{\mathcal{L}^{\alpha}_p{}^q(\varphi^{(1)})} \le \|g\|_{\mathcal{L}^{\alpha}_p{}^q(\varphi^{(2)})} \le C\|g\|_{\mathcal{L}^{\alpha}_p{}^q(\varphi^{(1)})}.$$

On the reduced Heisenberg group $\mathbb{H}/\Gamma$ let $d((z,t),(z',t'))$ be a distance function given by

$$d((z,t),(z',t')) = \inf_{n\in\mathbb{Z}}\{|(z,t)^{-1}(z',t')(0,2\pi n)|\}, \tag{4.5}$$

where $|(z,t)| = |z| + |t|^{1/2}$ is a homogeneous norm on $\mathbb{H}$.

For $a > 0$ and f of the form (3.3) we define an analogue of the Peetre maximal operator:

$$\tilde{A}_\mu f(z,t) = \sup_{(z',t')\in\mathbb{H}/\Gamma} \frac{|\tilde{Q}_\mu f(z',t')|}{(1+2^{\mu/2}d((z,t),(z',t')))^a}. \tag{4.6}$$

Note that if

$$\bar{A}_\mu f(z) = \sup_{z'\in\mathbb{R}^2} \frac{|\tilde{Q}_\mu f(z',0)|}{(1+2^{\mu/2}|z-z'|)^a}, \tag{4.7}$$

then

$$\bar{A}_\mu f(z) = \tilde{A}_\mu f(z,t). \tag{4.8}$$

Let

$$\tilde{B}_\mu f(z,t) = \sup_{(z',t')\in\mathbb{H}/\Gamma} \frac{|\nabla\tilde{Q}_\mu f(z',t')|}{(1+2^{\mu/2}d((z,t),(z',t')))^a}, \tag{4.9}$$

where $|\nabla\tilde{Q}_\mu f(z',t')| = |X\tilde{Q}_\mu f(z',t')| + |Y\tilde{Q}_\mu f(z',t')|$.

Lemma 4.10. *For every $a > 0$ there is a constant $C > 0$ such that*

$$\tilde{B}_\mu f(z,t) \le C2^{\mu/2}\tilde{A}_\mu f(z,t). \tag{4.11}$$

Proof. Let ψ be a C^∞ function satisfying (1.2) such that

$$\sum_{\mu\in\mathbb{Z}} \psi(2^{-\mu}\lambda)\varphi(2^{-\mu}\lambda) = 1 \quad \text{for} \quad \lambda > 0. \tag{4.12}$$

For the function $\zeta(\lambda) = \sum_{j=-1}^{j=1}\varphi(2^j\lambda)\psi(2^j\lambda)$ we denote by $M_{2^{-\mu}}(z,t)$ the convolution kernel on $\mathbb{H}$ that corresponds to the operator $\zeta(2^{-\mu}L)$, where L is the sublaplacian on $\mathbb{H}$. By Theorem 2.3

$$\begin{aligned}
|X\tilde{Q}_\mu f(z,t)| &= |X\int_{\mathbb{H}/\Gamma}\sum_{n\in\mathbb{Z}}\tilde{Q}_\mu f(z',t')M_{2^{-\mu}}((z',t')^{-1}(z,t)(0,2\pi n))dz'dt'| \\
&= |2^{5\mu/2}\int_{\mathbb{R}^2}\int_0^{2\pi}\sum_{n\in\mathbb{Z}}(XM)(2^{\mu/2}(z-z'),2^\mu(t-t'-\frac{1}{2}\Im(z'\bar{z})+2\pi n)) \\
&\qquad\qquad \times \tilde{Q}_\mu f(z',t')dz'dt'| \\
&\le 2^{5\mu/2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}|(XM)(2^{\mu/2}(z-z'),2^\mu(-t'))\tilde{Q}_\mu f(z',0)|dz'dt' \\
&\le C_b 2^{3\mu/2}\int_{\mathbb{R}^2}(1+2^{\mu/2}|z-z'|)^{-b}|\tilde{Q}_\mu f(z',0)|\,dz' \\
&\le C_b 2^{\mu/2}\bar{A}_\mu f(z'')(1+2^{\mu/2}|z''-z|)^a.
\end{aligned}$$

Similarly,

$$|Y\tilde{Q}_\mu f(z,t)| \le C_b 2^{\mu/2}\bar{A}_\mu f(z'')(1+2^{\mu/2}|z''-z|)^a.$$

Now, applying (4.8), we get (4.11). □

For $h = (z, t) \in \mathbb{H}/\Gamma$ and $\delta > 0$, let $\mathcal{B}_h(\delta)$ be the ball in $\mathbb{H}/\Gamma$ centered at h and radius δ, that is, $\mathcal{B}_h(\delta) = \{h_1 = (z_1, t_1) : d(h_1, h) < \delta\}$. We shall denote by $|\mathcal{B}_h(\delta)|$ the volume of this ball. Note that $|\mathcal{B}_h(\delta)|$ is comparable with δ^4 for $\delta < 1$ and with δ^2 for $\delta \geq 1$. Since $|\tilde{Q}_\mu f(z,t)|$ does not depend on t, we set $|\bar{Q}_\mu f(z)| = |\tilde{Q}_\mu f(z,t)|$.

Lemma 4.13. *$\tilde{A}_\mu f(z,t) \leq C[\mathcal{M}(|\bar{Q}_\mu f|^r)(z)]^{1/r}$, where $\mathcal{M}$ is the classical Hardy-Littlewood maximal operator on $\mathbb{R}^2$ and $r = 2/a$.*

Proof. We conclude from the mean value theorem for stratified groups (cf. [FS], Theorem 1.41) that there is a constant C such that for $h_3 \in \mathcal{B}_0(2^{-\mu/2}\delta)$

$$|\tilde{Q}_\mu f(h_1h_2)| \leq C|\tilde{Q}_\mu f(h_1h_2h_3)| + C2^{-\mu/2}\delta \sup_{h_4\in\mathcal{B}_0(C2^{-\mu/2}\delta)} |\nabla\tilde{Q}_\mu f(h_1h_2h_4)|.$$

This gives

$$\begin{aligned}
|\tilde{Q}_\mu f(h_1h_2)| &\leq C\left(|\mathcal{B}_0(2^{-\mu/2}\delta)|^{-1}\int_{\mathcal{B}_0(2^{-\mu/2}\delta)} |\tilde{Q}_\mu f(h_1h_2h_3)|^r dh_3\right)^{1/r} \\
&\quad + C2^{-\mu/2}\delta \sup_{h_4\in\mathcal{B}_0(C2^{-\mu/2}\delta)}\left(\frac{|\nabla\tilde{Q}_\mu f(h_1h_2h_4)|}{(1+2^{\mu/2}d(0,h_2h_4))^a}(1+2^{\mu/2}d(0,h_2h_4))^a\right) \\
&\leq C\left(|\mathcal{B}_0(2^{-\mu/2}\delta)|^{-1}\int_0^{\min(2\pi,2^{-\mu}\delta^2)}\int_{|z_3|<2^{-\mu/2}\delta}|\tilde{Q}_\mu f(h_1h_2h_3)|^r dh_3\right)^{1/r} \\
&\quad + C2^{-\mu/2}\delta\tilde{B}_\mu f(h_1)\sup_{h_4\in\mathcal{B}_0(C2^{-\mu/2}\delta)}(1+2^{\mu/2}d(0,h_2h_4))^a \\
&\leq C\left((2^{-\mu/2}\delta)^{-2}\int_{|z_3|<2^{-\mu/2}\delta}|\bar{Q}_\mu f(z_1+z_2+z_3)|^r dz_3\right)^{1/r} \\
&\quad + C2^{-\mu/2}\delta(1+\delta+2^{\mu/2}d(0,h_2))^a\tilde{B}_\mu f(h_1).
\end{aligned}$$

Using Lemma 4.10, we get

$$\begin{aligned}
|\tilde{Q}_\mu f(h_1h_2)| &\leq C\left(\frac{(2^{-\mu/2}\delta+|z_2|)^2}{2^{-\mu}\delta^2}\right)^{1/r} \\
&\qquad \times\left(\frac{1}{(2^{-\mu/2}\delta+|z_2|)^2}\int_{|z_3|<2^{-\mu/2}\delta+|z_2|}|\bar{Q}_\mu f(z_1+z_3)|^r dz_3\right)^{1/r} \\
&\qquad + C\delta(1+\delta+2^{\mu/2}d(0,h_2))^a\tilde{A}_\mu f(h_1).
\end{aligned}$$

Finally there is a constant C such that for any $\delta \in (0,1)$

$$\begin{aligned}
|\tilde{Q}_\mu f(h_1h_2)| &\leq C\delta^{-2/r}(1+2^{\mu/2}d(0,h_2))^{2/r}(\mathcal{M}(|\bar{Q}_\mu f(z_1)|^r))^{1/r} \\
&\quad + C\delta(1+2^{\mu/2}d(0,h_2))^a\tilde{A}_\mu f(h_1),
\end{aligned}$$

which completes the proof of the lemma. □

Proof of Theorem A. Let $0 < r < \min\{p,q\}$ and $a = 2/r$. For $\varphi^{(2)}$ let $\psi^{(2)}$ be a C^∞ function satisfying (1.2) such that (4.12) holds. If $\tilde{R}_\nu^{(2)}$ is the linear operator determined by $\tilde{R}_\nu^{(2)}\phi_k^m = \psi^{(2)}(2^{-\nu}d_k)\phi_k^m$, then

$$\tilde{Q}_\mu^{(1)} = \sum_{\nu=\mu-1}^{\mu+1}\tilde{Q}_\mu^{(1)}\tilde{R}_\nu^{(2)}\tilde{Q}_\nu^{(2)}. \tag{4.14}$$

By Theorem 2.3 the kernels $K_{\nu,\mu}((z,t),(z',t'))$ of the operators $\tilde{Q}_\mu^{(1)}\tilde{R}_\nu^{(2)}$, $|\nu-\mu|\leq 1$, are bounded by $C_b 2^{d\nu/2}(1+2^{\nu/2}|z'-z|)^{-b}$, thus

$$\begin{aligned}|\tilde{Q}_\mu^{(1)}f(z,t)| &\leq C_b\sum_{\nu=\mu-1}^{\mu+1}\int_{\mathbb{R}^2}2^{d\nu/2}(1+2^{\nu/2}|z'-z|)^{a-b}\bar{A}_\nu^{(2)}f(z)\,dz' \\ &\leq C_b\sum_{\nu=\mu-1}^{\mu+1}\bar{A}_\nu^{(2)}f(z).\end{aligned}$$

From Lemma 4.13 we conclude

$$|\tilde{Q}_\mu^{(1)}f(z,t)|\leq C\sum_{\nu=\mu-1}^{\mu+1}[\mathcal{M}(|\bar{Q}_\nu^{(2)}f|^r)(z)]^{1/r}.$$

Using the Fefferman-Stein vector-valued maximal inequality, we get

$$\begin{aligned}\|f\|_{H_p^{\alpha\ q}(\varphi^{(1)})} &\leq C\|(\sum_{\mu=-\infty}^{\infty}(2^{\mu\alpha}[\mathcal{M}(|\bar{Q}_\mu^{(2)}f|^r)(z)]^{1/r})^q)^{1/q}\|_{L^p} \\ &\leq C\|(\sum_{\mu=-\infty}^{\infty}(2^{\mu\alpha r}[|\bar{Q}_\mu^{(2)}f|^r(z)])^{q/r})^{r/q}\|_{L^{p/r}}^{1/r}\leq C\|f\|_{H_p^{\alpha\ q}(\varphi^{(2)})}. \quad \square\end{aligned}$$

5. Triebel-Lizorkin spaces associated with the Hermite operator

Let

$$H=-\Delta+|x|^2$$

be the Hermite operator on $\mathbb{R}^d$. Our main goal in the present section is to prove the following theorem which states that the definition of the Trieblel-Lizorkin space $H_p^{\alpha,q}(\varphi)$ does not depend on the particular choice of φ (cf. (1.3)).

Theorem B. *Let $\alpha\in\mathbb{R}$, $0<p<\infty$, and $0<q<\infty$. If $\varphi^{(1)}$ and $\varphi^{(2)}$ are two C^∞ functions satisfying (1.2), then there exists a constant Csuch that*

$$C^{-1}\|f\|_{H_p^{\alpha\ q}(\varphi^{(1)})}\leq\|f\|_{H_p^{\alpha\ q}(\varphi^{(2)})}\leq C\|f\|_{H_p^{\alpha\ q}(\varphi^{(1)})}.$$

Let $h_{\mathbf{m}}$ be normalized eigenfunctions of H with corresponding eigenvalues $a_{\mathbf{m}}$, that is, $Hh_{\mathbf{m}}=a_{\mathbf{m}}h_{\mathbf{m}}$.

For $m\in\mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz class of functions on $\mathbb{R}$, and $\mu\in\mathbb{Z}$ define

$$Q_\mu=m(2^{-\mu}H). \tag{5.1}$$

Then

$$Q_\mu h_{\mathbf{m}}=m(2^{-\mu}a_{\mathbf{m}})h_{\mathbf{m}}. \tag{5.2}$$

Obviously, for $f=\sum_{\mathbf{m}}\langle f,h_{\mathbf{m}}\rangle h_{\mathbf{m}}$, we have

$$Q_\mu f(x)=\sum_{\mathbf{m}}m(2^{-\mu}a_{\mathbf{m}})\langle f,h_{\mathbf{m}}\rangle h_{\mathbf{m}}(x)=\int_{\mathbb{R}^d}f(y)K_\mu(x,y)dy,$$

where

$$K_\mu(x,y)=\sum_{\mathbf{m}}m(2^{-\mu}a_{\mathbf{m}})h_{\mathbf{m}}(x)h_{\mathbf{m}}(y).$$

Let π be the Schrödinger representation of $\mathbb{H}_d$ defined by

$$\pi_{(z,t)}f(u) = e^{i(x\cdot u+\frac{1}{2}x\cdot y+t)}f(y+u). \tag{5.3}$$

Then $\pi_{Y_j} = \frac{\partial}{\partial x_j}$, $\pi_{X_j} = ix_j$, and consequently $\pi_L = -\Delta + |x|^2$ is the Hermite operator on $\mathbb{R}^d$. For $m \in \mathcal{S}(\mathbb{R})$, let $M(x,y,t)$ be the convolution kernel for the operator $m(L)$ on $\mathbb{H}_d$ (cf. Theorem 2.3). Then

$$m(H)f = \pi_M f = \int_{\mathbb{H}_d} M(x,y,t)\pi_{(x,y,t)}f\,dxdydt. \tag{5.4}$$

Applying (5.4) and Theorem 2.3, we get

$$K_\mu(x,y) = \int_{\mathbb{R}^d}\int_{-\infty}^{\infty} 2^{d\mu/2}M(w,2^{\mu/2}(y-x),t)e^{i(\frac{1}{2}2^{-\mu/2}w\cdot(x+y)+2^{-\mu}t)}dtdw. \tag{5.5}$$

Lemma 5.6. *For every $b>0$ there is a constant C_b such that*

$$|K_\mu(x,y)| \le C_b 2^{d\mu/2}(1+2^{\mu/2}|x-y|)^{-b}, \tag{5.7}$$

$$|\frac{\partial}{\partial x_j}K_\mu(x,y)| \le C_b 2^{(d+1)\mu/2}(1+2^{\mu/2}|x-y|)^{-b} \quad \textit{for}\ \ j=1,2,...,d. \tag{5.8}$$

Proof. The estimate (5.7) is a consequence of (5.5) and Theorem 2.3.

In order to obtain (5.8) we use the fact that $\frac{\partial}{\partial x_j}K_\mu(x,y)$ is the kernel which corresponds to $\pi_{Y_j}\pi_{M_{2^{-\mu}}}$. As $\pi_{Y_j}\pi_{M_{2^{-\mu}}} = -\pi_{\tilde{Y}_j M_{2^{-\mu}}}$, we have

$$\frac{\partial}{\partial x_j}K_\mu(x,y) = -\int_{\mathbb{R}^d}\int_{\infty}^{\infty} 2^{\mu/2}(\tilde{Y}_j M)_{2^{-\mu}}(w,y-x,t)e^{i(\frac{1}{2}w\cdot(x+y)+t)}dtdw.$$

This gives

$$|\frac{\partial}{\partial x_j}K_\mu(x,y)| \le 2^{\mu/2}2^{d\mu/2}\int_{\mathbb{R}^d}\int_{-\infty}^{\infty}|\tilde{Y}_j M(w,2^{\mu/2}(y-x),t)|dtdw. \tag{5.9}$$

The function $\tilde{Y}_j M$ is in the Schwartz class on $\mathbb{H}_d$, therefore (5.8) follows from (5.9). □

Using Lemma 5.6, Theorem B can be proved in the same way as Theorem 1.1 in [E1].

Remark. Standard arguments show that the norms of the Triebel-Lizorkin spaces for Laguerre and Hermite expansions for parameters $\alpha = 0$, $1 < p < \infty$, and $q = 2$ are equivalent to the L^p norms.

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