

TRIEBEL-LIZORKIN SPACES ASSOCIATED WITH LAGUERRE AND HERMITE EXPANSIONS

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ABSTRACT. It is proved that Triebel-Lizorkin spaces for some Laguerre and Hermite expansions are well-defined.

1. INTRODUCTION

Let D be a self-adjoint positive operator acting on $L^2(\mathfrak{M})$, and let dE be its spectral resolution, that is,

$$(1.1) \quad Df = \int_0^\infty \lambda dE(\lambda)f.$$

For $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and a C^∞ function φ satisfying

$$(1.2) \quad \text{supp } \varphi \subset [1/2, 2], \quad |\varphi(\lambda)| > c > 0 \quad \text{for } \lambda \in [3/4, 7/4],$$

we define the Triebel-Lizorkin norm associated with D (and with φ) by

$$(1.3) \quad \|f\|_{D_p^\alpha q(\varphi)} = \left\| \left[\sum_{\mu \in \mathbb{Z}} (2^{\mu\alpha} |Q_\mu f|)^q \right]^{1/q} \right\|_{L^p(X)},$$

where

$$(1.4) \quad Q_\mu f = \varphi(2^{-\mu} D)f = \int_0^\infty \varphi(2^{-\mu} \lambda) dE(\lambda)f.$$

Note that if $D = \Delta = -\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on \mathbb{R}^d , then the norm $\|f\|_{\Delta_p^\alpha q(\varphi)}$ is equivalent to the classical Triebel-Lizorkin norm $\|f\|_{F_p^{2\alpha} q}$.

Triebel-Lizorkin spaces associated with the one-dimensional Hermite operator

$$\mathcal{H} = -\frac{\partial^2}{\partial x^2} + x^2$$

were studied by J. Epperson in [E1] and [E2]. It was proved there, using Mehler's formula, that the definition of the corresponding space $F_{\mathcal{H}_p}^{\alpha, q}$ is independent of the particular choice of the function φ .

The present paper continues these studies. We consider Triebel-Lizorkin spaces associated with some Laguerre expansions and multidimensional Hermite expansions. We use some ideas from [E1] combined with Heisenberg group methods (cf. [HJ]). Symbolic calculus for sublaplacians on Heisenberg groups (cf. Theorem 2.3) plays an essential role in our paper.

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2. SYMBOLIC CALCULUS ON HEISENBERG GROUPS

Let \mathbb{H}_d be the $2d + 1$ dimensional Heisenberg group, that is, $\mathbb{H}_d = \mathbb{C}^d \times \mathbb{R}$ with the multiplication $hh' = (z, t)(z', t') = (z + z', t + t' + \frac{1}{2}\Im(z\bar{z}'))$. Let X_j, Y_j be the elements of the Lie algebra of \mathbb{H}_d which we identify with the left-invariant vector fields

$$(2.1) \quad X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}x_j \frac{\partial}{\partial t}.$$

The corresponding right-invariant vector fields are:

$$(2.2) \quad \tilde{X}_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j \frac{\partial}{\partial t}, \quad \tilde{Y}_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j \frac{\partial}{\partial t}.$$

The sublaplacian L on \mathbb{H} defined by

$$L = - \sum_{j=1}^d X_j^2 + Y_j^2$$

is a positive, homogeneous of degree 2, left-invariant subelliptic differential operator. Let dE be the spectral resolution for L , that is, $Lf = \int_0^\infty \lambda dE(\lambda)f$. If m is a bounded function on $(0, \infty)$, then the operator

$$m(L)f = \int_0^\infty m(\lambda)dE(\lambda)f$$

is left-invariant and bounded on $L^2(\mathbb{H}_d)$.

The following theorem due to Hulanicki (cf. [H]) is the basic tool in our paper.

Theorem 2.3. *If $m \in \mathcal{S}(\mathbb{R})$, then*

$$(2.4) \quad m(L)f = f * M,$$

with M in the Schwartz space $\mathcal{S}(\mathbb{H}_d)$ of functions on \mathbb{H}_d .

Moreover, if for $s > 0$ we set $m^s(\lambda) = m(s\lambda)$, then

$$(2.5) \quad m^s(L) = f * M_s,$$

where

$$(2.6) \quad M_s(h) = M_s(z, t) = s^{-Q/2} M\left(\frac{z}{\sqrt{s}}, \frac{t}{s}\right).$$

Here $Q = 2d + 2$ is the homogeneous dimension of \mathbb{H}_d .

3. LAGUERRE FUNCTIONS

Let

$$(3.1) \quad \mathcal{L}_k^m(w) = (2\pi)^{-1/2} \left(\frac{k!}{(k+m)!} \right)^{1/2} w^{m/2} L_k^m(w) e^{-w/2}, \quad w > 0,$$

be the Laguerre function of type m , $m = 0, 1, 2, \dots$, where

$$(3.2) \quad L_k^m(w) = \sum_{j=0}^k \binom{k+m}{k-j} \frac{(-w)^j}{j!}$$

is the corresponding Laguerre polynomial of type m , $m = 0, 1, 2, \dots$

Let \mathbb{H}/Γ denote the reduced Heisenberg group, where $\Gamma = \{(0, 2\pi n) : n \in \mathbb{Z}\}$ is a normal discrete central subgroup of $\mathbb{H} = \mathbb{H}_1$. For $p > 0$ and nonnegative integer

m we consider the space $L_m^p(\mathbb{H}/\Gamma)$ which consists of L^p functions f which have the form

$$(3.3) \quad f(z, t) = e^{it} e^{-im\theta} f_0(r), \quad z = re^{i\theta}.$$

It is well known (cf. [T]) that if a C^2 function f on \mathbb{H}/Γ has the form (3.3), then Lf is of the same form, where $L = -X^2 - Y^2$ is the sublaplacian on \mathbb{H}/Γ . Moreover, the functions

$$(3.4) \quad \phi_k^m(z, u) = e^{iu} e^{-im\theta} \mathcal{L}_k^m(|z|^2/2)$$

form an orthonormal basis of $L_m^2(\mathbb{H}/\Gamma)$, and

$$(3.5) \quad L\phi_k^m = d_k \phi_k^m, \quad \text{where } d_k = 2k + 1.$$

Note that the map $W : L^p(\mathbb{R}^+) \rightarrow L_m^p(\mathbb{H}/\Gamma)$ given by

$$(3.6) \quad f(z, u) = (Wg)(z, u) = e^{iu} e^{-im\theta} g(|z|^2/2), \quad \text{where } z = e^{i\theta}|z|,$$

is an isometry from $L^p(\mathbb{R}^+)$ onto $L_m^p(\mathbb{H}/\Gamma)$. If, moreover, f and g are related by (3.6), then

$$(3.7) \quad \langle g, \mathcal{L}_k^m \rangle = \langle f, \phi_k^m \rangle.$$

Consequently, if $g = \sum_k \langle g, \mathcal{L}_k^m \rangle \mathcal{L}_k^m$, then

$$(3.8) \quad Wg = \sum_k \langle Wg, \phi_k^m \rangle \phi_k^m.$$

4. TRIEBEL-LIZORKIN SPACES FOR LAGUERRE EXPANSIONS

Let φ be a C^∞ function satisfying (1.2). We define the linear operators Q_μ on $L^2(\mathbb{R}^+)$ by

$$(4.1) \quad Q_\mu \mathcal{L}_k^m = \varphi(2^{-\mu} d_k) \mathcal{L}_k^m.$$

For $g = \sum_k a_k \mathcal{L}_k^m$, $0 < p, q < \infty$, and $\alpha \in \mathbb{R}$, the Triebel-Lizorkin norm $\|g\|_{\mathcal{L}_p^\alpha{}^q(\varphi)}$ is defined by

$$(4.2) \quad \|g\|_{\mathcal{L}_p^\alpha{}^q(\varphi)} = \left\| \left[\sum_{\mu \in \mathbb{Z}} (2^{\mu\alpha} |Q_\mu g|)^q \right]^{1/q} \right\|_{L^p(\mathbb{R}^+)}.$$

On the space $L_m^2(\mathbb{H}/\Gamma)$ we define the corresponding operators \tilde{Q}_μ by setting

$$(4.3) \quad \tilde{Q}_\mu \phi_k^m = \varphi(2^{-\mu} d_k) \phi_k^m.$$

Obviously for f and g related by (3.6),

$$(4.4) \quad \|g\|_{\mathcal{L}_p^\alpha{}^q(\varphi)} = \left\| \left[\sum_{\mu \in \mathbb{Z}} (2^{\mu\alpha} |\tilde{Q}_\mu f|)^q \right]^{1/q} \right\|_{L^p(\mathbb{H}/\Gamma)}.$$

Our goal in this section is the following

Theorem A. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $0 < q < \infty$. If $\varphi^{(1)}$ and $\varphi^{(2)}$ are two C^∞ functions satisfying (1.2), then there exists a constant C such that*

$$C^{-1} \|g\|_{\mathcal{L}_p^\alpha{}^q(\varphi^{(1)})} \leq \|g\|_{\mathcal{L}_p^\alpha{}^q(\varphi^{(2)})} \leq C \|g\|_{\mathcal{L}_p^\alpha{}^q(\varphi^{(1)})}.$$

On the reduced Heisenberg group \mathbb{H}/Γ let $d((z, t), (z', t'))$ be a distance function given by

$$(4.5) \quad d((z, t), (z', t')) = \inf_{n \in \mathbb{Z}} \{|(z, t)^{-1}(z', t')(0, 2\pi n)|\},$$

where $|(z, t)| = |z| + |t|^{1/2}$ is a homogeneous norm on \mathbb{H} .

For $a > 0$ and f of the form (3.3) we define an analogue of the Peetre maximal operator:

$$(4.6) \quad \tilde{A}_\mu f(z, t) = \sup_{(z', t') \in \mathbb{H}/\Gamma} \frac{|\tilde{Q}_\mu f(z', t')|}{(1 + 2^{\mu/2}d((z, t), (z', t')))^a}.$$

Note that if

$$(4.7) \quad \bar{A}_\mu f(z) = \sup_{z' \in \mathbb{R}^2} \frac{|\tilde{Q}_\mu f(z', 0)|}{(1 + 2^{\mu/2}|z - z'|)^a},$$

then

$$(4.8) \quad \bar{A}_\mu f(z) = \tilde{A}_\mu f(z, t).$$

Let

$$(4.9) \quad \tilde{B}_\mu f(z, t) = \sup_{(z', t') \in \mathbb{H}/\Gamma} \frac{|\nabla \tilde{Q}_\mu f(z', t')|}{(1 + 2^{\mu/2}d((z, t), (z', t')))^a},$$

where $|\nabla \tilde{Q}_\mu f(z', t')| = |X\tilde{Q}_\mu f(z', t')| + |Y\tilde{Q}_\mu f(z', t')|$.

Lemma 4.10. *For every $a > 0$ there is a constant $C > 0$ such that*

$$(4.11) \quad \tilde{B}_\mu f(z, t) \leq C2^{\mu/2}\tilde{A}_\mu f(z, t).$$

Proof. Let ψ be a C^∞ function satisfying (1.2) such that

$$(4.12) \quad \sum_{\mu \in \mathbb{Z}} \psi(2^{-\mu}\lambda)\varphi(2^{-\mu}\lambda) = 1 \quad \text{for } \lambda > 0.$$

For the function $\zeta(\lambda) = \sum_{j=-1}^{j=1} \varphi(2^j\lambda)\psi(2^j\lambda)$ we denote by $M_{2^{-\mu}}(z, t)$ the convolution kernel on \mathbb{H} that corresponds to the operator $\zeta(2^{-\mu}L)$, where L is the sublaplacian on \mathbb{H} . By Theorem 2.3

$$\begin{aligned} |X\tilde{Q}_\mu f(z, t)| &= |X \int_{\mathbb{H}/\Gamma} \sum_{n \in \mathbb{Z}} \tilde{Q}_\mu f(z', t') M_{2^{-\mu}}((z', t')^{-1}(z, t)(0, 2\pi n)) dz' dt'| \\ &= |2^{5\mu/2} \int_{\mathbb{R}^2} \int_0^{2\pi} \sum_{n \in \mathbb{Z}} (XM)(2^{\mu/2}(z - z'), 2^\mu(t - t' - \frac{1}{2}\Im(z'\bar{z}) + 2\pi n)) \\ &\quad \times \tilde{Q}_\mu f(z', t') dz' dt'| \\ &\leq 2^{5\mu/2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} |(XM)(2^{\mu/2}(z - z'), 2^\mu(-t')) \tilde{Q}_\mu f(z', 0)| dz' dt' \\ &\leq C_b 2^{3\mu/2} \int_{\mathbb{R}^2} (1 + 2^{\mu/2}|z - z'|)^{-b} |\tilde{Q}_\mu f(z', 0)| dz' \\ &\leq C_b 2^{\mu/2} \bar{A}_\mu f(z'')(1 + 2^{\mu/2}|z'' - z|)^a. \end{aligned}$$

Similarly,

$$|Y\tilde{Q}_\mu f(z, t)| \leq C_b 2^{\mu/2} \bar{A}_\mu f(z'')(1 + 2^{\mu/2}|z'' - z|)^a.$$

Now, applying (4.8), we get (4.11). □

For $h = (z, t) \in \mathbb{H}/\Gamma$ and $\delta > 0$, let $\mathcal{B}_h(\delta)$ be the ball in \mathbb{H}/Γ centered at h and radius δ , that is, $\mathcal{B}_h(\delta) = \{h_1 = (z_1, t_1) : d(h_1, h) < \delta\}$. We shall denote by $|\mathcal{B}_h(\delta)|$ the volume of this ball. Note that $|\mathcal{B}_h(\delta)|$ is comparable with δ^4 for $\delta < 1$ and with δ^2 for $\delta \geq 1$. Since $|\tilde{Q}_\mu f(z, t)|$ does not depend on t , we set $|\tilde{Q}_\mu f(z)| = |\tilde{Q}_\mu f(z, t)|$.

Lemma 4.13. $\tilde{A}_\mu f(z, t) \leq C[\mathcal{M}(|\tilde{Q}_\mu f|^r)(z)]^{1/r}$, where \mathcal{M} is the classical Hardy-Littlewood maximal operator on \mathbb{R}^2 and $r = 2/a$.

Proof. We conclude from the mean value theorem for stratified groups (cf. [FS], Theorem 1.41) that there is a constant C such that for $h_3 \in \mathcal{B}_0(2^{-\mu/2}\delta)$

$$|\tilde{Q}_\mu f(h_1 h_2)| \leq C|\tilde{Q}_\mu f(h_1 h_2 h_3)| + C2^{-\mu/2}\delta \sup_{h_4 \in \mathcal{B}_0(C2^{-\mu/2}\delta)} |\nabla \tilde{Q}_\mu f(h_1 h_2 h_4)|.$$

This gives

$$\begin{aligned} |\tilde{Q}_\mu f(h_1 h_2)| &\leq C \left(|\mathcal{B}_0(2^{-\mu/2}\delta)|^{-1} \int_{\mathcal{B}_0(2^{-\mu/2}\delta)} |\tilde{Q}_\mu f(h_1 h_2 h_3)|^r dh_3 \right)^{1/r} \\ &\quad + C2^{-\mu/2}\delta \sup_{h_4 \in \mathcal{B}_0(C2^{-\mu/2}\delta)} \left(\frac{|\nabla \tilde{Q}_\mu f(h_1 h_2 h_4)|}{(1 + 2^{\mu/2}d(0, h_2 h_4))^a} (1 + 2^{\mu/2}d(0, h_2 h_4))^a \right) \\ &\leq C \left(|\mathcal{B}_0(2^{-\mu/2}\delta)|^{-1} \int_0^{\min(2\pi, 2^{-\mu}\delta^2)} \int_{|z_3| < 2^{-\mu/2}\delta} |\tilde{Q}_\mu f(h_1 h_2 h_3)|^r dh_3 \right)^{1/r} \\ &\quad + C2^{-\mu/2}\delta \tilde{B}_\mu f(h_1) \sup_{h_4 \in \mathcal{B}_0(C2^{-\mu/2}\delta)} (1 + 2^{\mu/2}d(0, h_2 h_4))^a \\ &\leq C \left((2^{-\mu/2}\delta)^{-2} \int_{|z_3| < 2^{-\mu/2}\delta} |\tilde{Q}_\mu f(z_1 + z_2 + z_3)|^r dz_3 \right)^{1/r} \\ &\quad + C2^{-\mu/2}\delta(1 + \delta + 2^{\mu/2}d(0, h_2))^a \tilde{B}_\mu f(h_1). \end{aligned}$$

Using Lemma 4.10, we get

$$\begin{aligned} |\tilde{Q}_\mu f(h_1 h_2)| &\leq C \left(\frac{(2^{-\mu/2}\delta + |z_2|)^2}{2^{-\mu}\delta^2} \right)^{1/r} \\ &\quad \times \left(\frac{1}{(2^{-\mu/2}\delta + |z_2|)^2} \int_{|z_3| < 2^{-\mu/2}\delta + |z_2|} |\tilde{Q}_\mu f(z_1 + z_3)|^r dz_3 \right)^{1/r} \\ &\quad + C\delta(1 + \delta + 2^{\mu/2}d(0, h_2))^a \tilde{A}_\mu f(h_1). \end{aligned}$$

Finally there is a constant C such that for any $\delta \in (0, 1)$

$$|\tilde{Q}_\mu f(h_1 h_2)| \leq C\delta^{-2/r}(1 + 2^{\mu/2}d(0, h_2))^{2/r}(\mathcal{M}(|\tilde{Q}_\mu f(z_1)|^r))^{1/r} + C\delta(1 + 2^{\mu/2}d(0, h_2))^a \tilde{A}_\mu f(h_1),$$

which completes the proof of the lemma. □

Proof of Theorem A. Let $0 < r < \min\{p, q\}$ and $a = 2/r$. For $\varphi^{(2)}$ let $\psi^{(2)}$ be a C^∞ function satisfying (1.2) such that (4.12) holds. If $\tilde{R}_\nu^{(2)}$ is the linear operator determined by $\tilde{R}_\nu^{(2)}\phi_k^m = \psi^{(2)}(2^{-\nu}d_k)\phi_k^m$, then

$$(4.14) \quad \tilde{Q}_\mu^{(1)} = \sum_{\nu=\mu-1}^{\mu+1} \tilde{Q}_\mu^{(1)} \tilde{R}_\nu^{(2)} \tilde{Q}_\nu^{(2)}.$$

By Theorem 2.3 the kernels $K_{\nu,\mu}((z, t), (z', t'))$ of the operators $\tilde{Q}_\mu^{(1)} \tilde{R}_\nu^{(2)}$, $|\nu - \mu| \leq 1$, are bounded by $C_b 2^{d\nu/2} (1 + 2^{\nu/2} |z' - z|)^{-b}$, thus

$$\begin{aligned} |\tilde{Q}_\mu^{(1)} f(z, t)| &\leq C_b \sum_{\nu=\mu-1}^{\mu+1} \int_{\mathbb{R}^2} 2^{d\nu/2} (1 + 2^{\nu/2} |z' - z|)^{a-b} \bar{A}_\nu^{(2)} f(z) dz' \\ &\leq C_b \sum_{\nu=\mu-1}^{\mu+1} \bar{A}_\nu^{(2)} f(z). \end{aligned}$$

From Lemma 4.13 we conclude

$$|\tilde{Q}_\mu^{(1)} f(z, t)| \leq C \sum_{\nu=\mu-1}^{\mu+1} [\mathcal{M}(|\bar{Q}_\nu^{(2)} f|^r)(z)]^{1/r}.$$

Using the Fefferman-Stein vector-valued maximal inequality, we get

$$\begin{aligned} \|f\|_{H_p^{\alpha, q}(\varphi^{(1)})} &\leq C \|(\sum_{\mu=-\infty}^{\infty} (2^{\mu\alpha} [\mathcal{M}(|\bar{Q}_\mu^{(2)} f|^r)(z)]^{1/r})^q)^{1/q}\|_{L^p} \\ &\leq C \|(\sum_{\mu=-\infty}^{\infty} (2^{\mu\alpha r} [|\bar{Q}_\mu^{(2)} f|^r(z)]^{q/r})^{r/q})^{1/r}\|_{L^{p/r}} \leq C \|f\|_{H_p^{\alpha, q}(\varphi^{(2)})}. \quad \square \end{aligned}$$

5. TRIEBEL-LIZORKIN SPACES ASSOCIATED WITH THE HERMITE OPERATOR

Let

$$H = -\Delta + |x|^2$$

be the Hermite operator on \mathbb{R}^d . Our main goal in the present section is to prove the following theorem which states that the definition of the Triebel-Lizorkin space $H_p^{\alpha, q}(\varphi)$ does not depend on the particular choice of φ (cf. (1.3)).

Theorem B. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $0 < q < \infty$. If $\varphi^{(1)}$ and $\varphi^{(2)}$ are two C^∞ functions satisfying (1.2), then there exists a constant C such that*

$$C^{-1} \|f\|_{H_p^{\alpha, q}(\varphi^{(1)})} \leq \|f\|_{H_p^{\alpha, q}(\varphi^{(2)})} \leq C \|f\|_{H_p^{\alpha, q}(\varphi^{(1)})}.$$

Let $h_{\mathbf{m}}$ be normalized eigenfunctions of H with corresponding eigenvalues $a_{\mathbf{m}}$, that is, $Hh_{\mathbf{m}} = a_{\mathbf{m}}h_{\mathbf{m}}$.

For $m \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz class of functions on \mathbb{R} , and $\mu \in \mathbb{Z}$ define

$$(5.1) \quad Q_\mu = m(2^{-\mu} H).$$

Then

$$(5.2) \quad Q_\mu h_{\mathbf{m}} = m(2^{-\mu} a_{\mathbf{m}}) h_{\mathbf{m}}.$$

Obviously, for $f = \sum_{\mathbf{m}} \langle f, h_{\mathbf{m}} \rangle h_{\mathbf{m}}$, we have

$$Q_\mu f(x) = \sum_{\mathbf{m}} m(2^{-\mu} a_{\mathbf{m}}) \langle f, h_{\mathbf{m}} \rangle h_{\mathbf{m}}(x) = \int_{\mathbb{R}^d} f(y) K_\mu(x, y) dy,$$

where

$$K_\mu(x, y) = \sum_{\mathbf{m}} m(2^{-\mu} a_{\mathbf{m}}) h_{\mathbf{m}}(x) h_{\mathbf{m}}(y).$$

Let π be the Schrödinger representation of \mathbb{H}_d defined by

$$(5.3) \quad \pi_{(z,t)}f(u) = e^{i(x \cdot u + \frac{1}{2}x \cdot y + t)}f(y + u).$$

Then $\pi_{Y_j} = \frac{\partial}{\partial x_j}$, $\pi_{X_j} = ix_j$, and consequently $\pi_L = -\Delta + |x|^2$ is the Hermite operator on \mathbb{R}^d . For $m \in \mathcal{S}(\mathbb{R})$, let $M(x, y, t)$ be the convolution kernel for the operator $m(L)$ on \mathbb{H}_d (cf. Theorem 2.3). Then

$$(5.4) \quad m(H)f = \pi_M f = \int_{\mathbb{H}_d} M(x, y, t)\pi_{(x,y,t)}f \, dx dy dt.$$

Applying (5.4) and Theorem 2.3, we get

$$(5.5) \quad K_\mu(x, y) = \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} 2^{d\mu/2}M(w, 2^{\mu/2}(y - x), t)e^{i(\frac{1}{2}2^{-\mu/2}w \cdot (x+y) + 2^{-\mu}t)} dt dw.$$

Lemma 5.6. *For every $b > 0$ there is a constant C_b such that*

$$(5.7) \quad |K_\mu(x, y)| \leq C_b 2^{d\mu/2}(1 + 2^{\mu/2}|x - y|)^{-b},$$

$$(5.8) \quad \left| \frac{\partial}{\partial x_j} K_\mu(x, y) \right| \leq C_b 2^{(d+1)\mu/2}(1 + 2^{\mu/2}|x - y|)^{-b} \quad \text{for } j = 1, 2, \dots, d.$$

Proof. The estimate (5.7) is a consequence of (5.5) and Theorem 2.3.

In order to obtain (5.8) we use the fact that $\frac{\partial}{\partial x_j} K_\mu(x, y)$ is the kernel which corresponds to $\pi_{Y_j} \pi_{M_{2^{-\mu}}}$. As $\pi_{Y_j} \pi_{M_{2^{-\mu}}} = -\pi_{\tilde{Y}_j M_{2^{-\mu}}}$, we have

$$\frac{\partial}{\partial x_j} K_\mu(x, y) = - \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} 2^{\mu/2}(\tilde{Y}_j M)_{2^{-\mu}}(w, y - x, t)e^{i(\frac{1}{2}w \cdot (x+y) + t)} dt dw.$$

This gives

$$(5.9) \quad \left| \frac{\partial}{\partial x_j} K_\mu(x, y) \right| \leq 2^{\mu/2} 2^{d\mu/2} \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} |\tilde{Y}_j M(w, 2^{\mu/2}(y - x), t)| dt dw.$$

The function $\tilde{Y}_j M$ is in the Schwartz class on \mathbb{H}_d , therefore (5.8) follows from (5.9). □

Using Lemma 5.6, Theorem B can be proved in the same way as Theorem 1.1 in [E1].

Remark. Standard arguments show that the norms of the Triebel-Lizorkin spaces for Laguerre and Hermite expansions for parameters $\alpha = 0$, $1 < p < \infty$, and $q = 2$ are equivalent to the L^p norms.

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REFERENCES

[D] J. Długosz, *Almost everywhere convergence of some summability methods for Laguerre series*, Studia Math. 82 (1985), 199–209. MR **87d**:42040
 [E1] J. Epperson, *Triebel-Lizorkin spaces for Hermite expansion*, Studia Math. 114 (1995), 87–103. MR **96c**:42059
 [E2] J. Epperson, *Hermite and Laguerre Wave Packet Expansions*, preprint.
 [FeS] C. Fefferman and E. Stein, *Some maximal inequalities*, Amer.J. Math. 93 (1972), 107–115.

- [FS] G. Folland and E. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, 1982. MR **87h**:43027
- [FJW] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, Conference Board of the Mathematical Sciences 79, AMS, (1991). MR **92m**:42041
- [H] A. Hulanicki, *A functional calculus for Rockland operators on nilpotent Lie groups*, *Studia Math.* 78 (1984), 253–266. MR **86g**:22009
- [HJ] A. Hulanicki and J.W. Jenkins, *Almost everywhere summability on nilmanifolds*, *Trans. Amer. Math. Soc.* 278 (1983), 703–715. MR **85f**:22011
- [P] J. Peetre, *On space of Triebel-Lizorkin type*, *Ark. Math.* 13 (1975), 123–130. MR **52**:1294
- [T] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, *Math. Notes* 42, Princeton Univ. Press, 1993. MR **94i**:42001
- [Tr] H. Triebel, *Theory of Function Spaces*, *Monographs Math.* 78. Birkhäuser, Basel 1983. MR **86j**:46026

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