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STATISTICAL LIMIT SUPERIOR AND LIMIT INFERIOR

J. A. FRIDY AND C. ORHAN

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ABSTRACT. Following the concept of statistical convergence and statistical cluster points of a sequence x, we give a definition of statistical limit superior and inferior which yields natural relationships among these ideas: e.g., x is statistically convergent if and only if st-liminf$x$ = st-limsup$x$. The statistical core of x is also introduced, for which an analogue of Knopp's Core Theorem is proved. Also, it is proved that a bounded sequence that is C_1-summable to its statistical limit superior is statistically convergent.

1. Introduction

The concepts of limit and cluster point of a sequence x have been extended ([7], [8] and [9]) to st-lim x and statistical cluster point using the natural density δ of a set K of positive integers:

$$\delta\{K\} := \lim_n \frac{1}{n}(\text{the number } k \leq n \text{ such that } k \in K).$$

(See [18], Chapter 11 for a basic theory of density.) The sequence x is statistically convergent to L, denoted st-lim$x = L$, if for every $\epsilon > 0$, $\delta\{k : |x_k - L| \geq \epsilon\} = 0$. Over the years, statistical convergence has been examined in number theory [6], trigonometric series [19] and summability theory [10]. It has also been considered in locally convex spaces [16]. Recently, generalizations of statistical convergence have been made in the study of strong integral summability [3] and the structure of ideals of bounded continuous functions on locally compact spaces [4]. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of the natural numbers [5]. A matrix characterization and a measure theoretical subsequence characterization of statistical convergence may be found in [11] and [17], respectively.

The number γ is called a statistical cluster point of x if for every $\epsilon > 0$ the set $\{k : |x_k - \gamma| < \epsilon\}$ does not have density zero. The purpose this paper is to present natural definitions of the concepts of statistical limit superior and inferior and to develop some statistical analogues of properties of the ordinary limit superior and inferior. The latter results include statistical analogues of Knopp's Core Theorem [13] and R.C. Buck's Theorem [1] on Cesàro summability of a sequence to its limit superior.

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2. Definitions and basic results

Throughout the paper k and n will always denote positive integers; x, y, and z will denote real number sequences; and $\mathbb{N}$ and $\mathbb{R}$ will denote the sets of positive integers and real numbers, respectively. If $K \subseteq \mathbb{N}$, then $K_n := \{k : k \leq n\}$, and $|K_n|$ denotes the cardinality of K_n.

For a real number sequence x let B_x denote the set:

$$B_x := \{b \in \mathbb{R} : \delta\{k : x_k > b\} \neq 0\};$$

similarly,

$$A_x := \{a \in \mathbb{R} : \delta\{k : x_k < a\} \neq 0\}.$$

Note that throughout this paper the statement $\delta\{K\} \neq 0$ means that *either* $\delta\{K\} > 0$ *or* K does not have natural density.

Definition 1. If x is a real number sequence, then the *statistical limit superior* of x is given by

$$\text{st-lim sup}x := \begin{cases} \sup B_x, & \text{if} \quad B_x \neq \emptyset, \\ -\infty & \text{if} \quad B_x = \emptyset. \end{cases}$$

Also, the *statistical limit inferior* of x is given by

$$\text{st-lim inf}x := \begin{cases} \inf A_x, & \text{if} \quad A_x \neq \emptyset, \\ +\infty, & \text{if} \quad A_x = \emptyset. \end{cases}$$

A simple example will help to illustrate the concepts just defined. Let the sequence x be given by

$$x_k := \begin{cases} k, & \text{if} \quad k \quad \text{is an odd square,} \\ 2, & \text{if} \quad k \quad \text{is an even square,} \\ 1, & \text{if} \quad k \quad \text{is an odd nonsquare,} \\ 0, & \text{if} \quad k \quad \text{is an even nonsquare.} \end{cases}$$

Note that although x is unbounded above, it is "statistically bounded" because the set of squares has density zero. Thus $B_x = (-\infty, 1)$ and st-lim sup$x = 1$. Also, x is not statistically convergent since it has two (disjoint) subsequences of positive density that converge to 0 and 1, respectively. (See Theorem 1 of [8].) Also note that the set of statistical cluster points of x is $\{0, 1\}$, and st-lim sup$x$ equals the greatest element while st-lim inf is the least element of this set. This observation suggests the main idea of the first theorem, which can be proved by a straightforward least upper bound argument.

Theorem 1. *If* $\beta = \text{st-}\limsup x$ *is finite, then for every positive number* ϵ

$$\delta\{k : x_k > \beta - \epsilon\} \neq 0 \quad \textit{and} \quad \delta\{k : x_k > \beta + \epsilon\} = 0. \tag{1}$$

Conversely, if (1) *holds for every positive* ϵ *then* $\beta = \text{st-}\limsup x$.

The dual statement for st-lim inf$x$ is as follows.

Theorem 1′. *If* $\alpha = \text{st-}\liminf x$ *is finite, then for every positive number* ϵ

$$\delta\{k : x_k < \alpha + \epsilon\} \neq 0 \quad \textit{and} \quad \delta\{k : x_k < \alpha - \epsilon\} = 0. \tag{2}$$

Conversely, if (2) *holds for every positive* ϵ *then* $\alpha = \text{st-}\liminf x$.

From the definition of statistical cluster point in [9] we see that Theorems 1 and $1'$ can be interpreted as saying that st-$\limsup x$ and st-$\liminf x$ are the greatest and least statistical cluster points of x. The next theorem reinforces that observation.

Theorem 2. *For any sequence* x, st-$\liminf x \ \leq\ $ st-$\limsup x$.

Proof. First consider the case in which st-$\limsup x = -\infty$. This implies that $B_x = \emptyset$, so for every b in $\mathbb{R}$, $\delta\{k : x_k > b\} = 0$. This implies that $\delta\{k : x_k \leq b\} = 1$, so for every a in $\mathbb{R}$, $\delta\{k : x_k < a\} \neq 0$. Hence, st-$\liminf x = -\infty$.

The case in which st-$\limsup\ x = +\infty$ needs no proof, so we next assume that $\beta =$ st-$\limsup\ x$ is finite, and let $\alpha :=$ st-$\liminf x$. Given $\epsilon > 0$ we show that $\beta + \epsilon \in A_x$, so that $\alpha \leq \beta + \epsilon$. By Theorem 1, $\delta\{k : x_k > \beta + \epsilon/2\} = 0$ because $\beta = \text{lub } B_x$. This implies that $\delta\{k : x_k \leq \beta + \frac{\epsilon}{2}\} = 1$ which, in turn, implies that $\delta\{k : x_k < \beta + \epsilon\} = 1$. Hence, $\beta + \epsilon \in A_x$. By definition $\alpha = \inf A_x$, so we conclude that $\alpha \leq \beta + \epsilon$; and since ϵ is arbitrary this gives us $\alpha \leq \beta$. ☐

From Theorem 2 and the above definition, it is clear that

$$\liminf x \ \leq\ \text{st-}\liminf x \ \leq\ \text{st-}\limsup x \ \leq\ \limsup x \tag{3}$$

for any sequence x.

A statistical limit point of a sequence x is defined in [9] as the limit of a subsequence of x whose indices do not have zero density. Since it was noted there that a bounded sequence might have *no* statistical limit point, one cannot say that st-$\limsup x$ is equal to the greatest such point. (Compare this to the remark following Theorem 1.) See, for example, Example 4 of [9]. This suggests the following question:

> If x does have a greatest statistical limit point μ, does it follow that $\mu =$ st-$\limsup x$?

The answer is "no," which is shown by the following sequence.

Example 1. Following Example 4 of [9] we let u be the uniformly distributed sequence $u = \{0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \dots\}$, and define

$$x_{2k-1} := 0 \text{ and } x_{2k} := u_k.$$

Then st-$\limsup x = 1$ because $\delta\{k : x_k > 1 - \epsilon\} = \epsilon/2$. Also, zero is the *only* statistical limit point of x because u has none (as shown in [9]). Hence, the greatest statistical limit point of x is zero, but st-$\limsup x = 1$.

The next result is another statistical analogue of a very basic property of convergent sequences. For clarity of presentation we first give a formal definition of another statistical concept.

Definition 2. The real number sequence x is said to be *statistically bounded* if there is a number B such that $\delta\{k : |x_k| > B\} = 0$.

Note that statistical boundedness implies that st-$\limsup$ and st-$\liminf$ are finite, so Properties (1) and (2) of Theorems 1 and $1'$ hold.

Theorem 3. *The statistically bounded sequence x is statistically convergent if and only if*

$$\text{st-}\liminf x \ =\ \text{st-}\limsup x.$$

Proof. Let $\alpha :=$ st-$\liminf x$ and $\beta :=$ st-$\limsup x$. First assume that st-$\lim x = L$ and $\epsilon > 0$. Then $\delta\{k : |x_k - L| \geq \epsilon\} = 0$, so $\delta\{k : x_k > L + \epsilon\} = 0$, which implies that $\beta \leq L$. We also have $\delta\{k : x_k < L - \epsilon\} = 0$ which implies that $L \leq \alpha$. Therefore $\beta \leq \alpha$, which we combine with Theorem 2 to conclude that $\alpha = \beta$.

Next assume $\alpha = \beta$ and define $L := \alpha$. If $\epsilon > 0$ then (1) and (2) of Theorems 1 and 1′ imply $\delta\{k : x_k > L + \frac{\epsilon}{2}\} = 0$ and $\delta\{k : x_k < L - \frac{\epsilon}{2}\} = 0$. Hence, st-$\lim x = L$. □

3. Summability theorems concerning st − lim sup

In [13] Knopp introduced the concept of the *core* of a sequence and proved the well-known Core Theorem. Since the core of a bounded sequence x is the closed convex hull of the set of limit points of x, we can replace limit points with statistical cluster points to produce a natural analogue of Knopp's core.

Definition 3. If x is a statistically bounded sequence, then the *statistical core* of x is the closed interval [st-$\liminf x$, st-$\limsup x$]. In case x is not statistically bounded, st-core$\{x\}$ is defined accordingly as either [st-$\liminf x, \infty$), $(-\infty, \infty)$, or ($-\infty$, st-$\limsup x$].

We shall denote the statistical core of x by st-core$\{x\}$, and K–core$\{x\}$ will denote the usual core. It is clear from (3) that for any real sequence x

$$\text{st-core}\{x\} \;\subseteq\; K\text{–core}\{x\}.$$

Recall that the Core Theorem asserts that $K-$core$\{Ax\} \;\subseteq\; K$–core$\{x\}$, whenever Ax exists for the nonnegative regular matrix A [12, p. 55]. In [15] Maddox proves a variant of the Core Theorem that $\limsup Ax \leq \limsup x$ for every bounded x if and only if A is regular and $\lim_n \sum_{k=0}^{\infty} |a_{nk}| = 1$. We shall prove a similar result for the st-core$\{x\}$. For this purpose let us recall some previous results and notations. In [2] Connor proved that the set of bounded statistically convergent sequences is equal to the set of bounded strongly p-Cesàro summable sequences ($S \sqcap \ell_\infty = \omega_p \sqcap \ell_\infty$). In [14] Maddox proved that a matrix A maps $\omega_p \sqcap \ell_\infty$ into c if and only if A is in the class $\mathcal{T}^*$, i.e., A is regular and $\lim_n \sum_{k\in E} |a_{nk}| = 0$ for every $E \subseteq \mathbb{N}$ such that $\delta\{E\} = 0$. Throughout the following we shall use the abbreviations

$$\alpha(x) := \text{st-}\liminf x \quad \text{and} \quad \beta(x) := \text{st-}\limsup x.$$

Lemma. *Suppose the matrix A satisfies $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$; then*

$$\limsup Ax \leq \text{st-}\limsup x \quad \textit{for every } x \in \ell_\infty \tag{4}$$

if and only if

$$A \in \mathcal{T}^* \quad \textit{and} \quad \lim_n \sum_{k=1}^{\infty} |a_{nk}| = 1. \tag{5}$$

Proof. Assume A satisfies (4) and $x \in \ell_\infty$. Then $\beta(x) \leq \limsup x$ and, since $\sup_n \sum_k |a_{nk}| < \infty$, $Ax \in \ell_\infty$. By (4) we have

$$-\beta(-x) \;\leq\; -\limsup(-Ax) \;\leq\; \limsup Ax \;\leq\; \beta(x),$$

or

$$\text{st-}\liminf x \;\le\; \liminf Ax \;\le\; \limsup Ax \;\le\; \beta(x). \tag{6}$$

If $x \in S \sqcap \ell_\infty$ we have $\alpha(x) = \beta(x) = \text{st-}\lim x$, so (6) implies that $\lim Ax = \text{st-}\lim x$. Hence, A maps $S \sqcap \ell_\infty$ into c, so by the theorems of Maddox and Connor, $A \in \mathcal{T}^*$. Also, since $\beta(x) \le \limsup x$, (4) implies that $\limsup Ax \le \limsup x$, and Maddox's variant of Knopp's Core Theorem yields

$$\lim_n \sum_{k=1}^{\infty} |a_{nk}| = 1.$$

Conversely, assume (5) and let x be bounded; then $Ax \in \ell_\infty$ and $\beta(x)$ is finite. Given $\epsilon > 0$ let $E := \{k : x_k > \beta(x) + \epsilon\}$. Thus $\delta\{E\} = 0$, and if $k \notin E$ then $x_k \le \beta(x) + \epsilon$. For any real number z we write

$$z^+ := \max\{z, 0\} \quad \text{and} \quad z^- := \max\{-z, 0\},$$

whence

$$|z| = z^+ + z^-, \quad z = z^+ - z^-, \quad \text{and} \quad |z| - z = 2z^-.$$

For a fixed positive integer m we write

$$\begin{aligned}
(Ax)_n &= \sum_{k<m} a_{nk}x_k + \sum_{k\ge m} a_{nk}x_k \\
&= \sum_{k<m} a_{nk}x_k + \sum_{k\ge m} a^+_{nk}x_k - \textstyle\sum_{k\ge m} a^-_{nk}x_k \\
&\le \|x\|_\infty \sum_{k<m} |a_{nk}| \quad + \sum_{\substack{k\ge m\\ k\notin E}} a^+_{nk}x_k + \sum_{\substack{k\ge m\\ k\in E}} a^+_{nk}x_k + \|x\|_\infty \sum_{k\ge m} (|a_{nk}| - a_{nk}) \\
&\le \|x\|_\infty \sum_{k<m} |a_{nk}| \quad + (\beta(x)+\epsilon) \sum_{\substack{k\ge m\\ k\notin E}} |a_{nk}| + \|x\|_\infty \sum_{\substack{k\ge m\\ k\in E}} |a_{nk}| \\
&\qquad\qquad + \|x\|_\infty \sum_{k\ge m} (|a_{nk}| - a_{nk}).
\end{aligned}$$

Taking the limit superior as $n \longrightarrow \infty$ and using (5) and the regularity of A, we get

$$\limsup (Ax)_n \le \beta(x) + \epsilon.$$

Since ϵ is arbitrary we conclude that (4) holds, and the proof is complete. □

It is clear that one can prove a similar result for $\alpha(x) \le \liminf Ax$, and therefore we have the following result.

Theorem 4 (Statistical Core Theorem)**.** *If the matrix A satisfies $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$, then*

$$K\text{–core}\{Ax\} \subseteq \text{st-core}\{x\} \textit{ for every } x \textit{ in } \ell_\infty$$

if and only if

$$A \in \mathcal{T}^* \quad \textit{and} \quad \lim_n \sum_{k=1}^{\infty} |a_{nk}| = 1.$$

In [1] Buck proved that a sequence that is C_1-summable to its limit superior is statistically convergent. The next theorem is a statistical analogue of that result.

Theorem 5. *If the sequence x is bounded above and C_1-summable to the number $\beta := \operatorname{st-lim\,sup} x$, then x is statistically convergent to β.*

Proof. Suppose that x is not statistically convergent to β. Then by Theorem 3, $\operatorname{st-lim\,inf} x < \beta$, so there is a number $\mu < \beta$ such that $\delta\{k : x_k < \mu\} \neq 0$. Let $K' := \{k : x_k < \mu\}$. By the definition of β, $\delta\{k : x_k > \beta + \epsilon\} = 0$ for every $\epsilon > 0$. Define

$$K'' := \{k : \mu \leq x_k \leq \beta + \epsilon\} \quad \text{and} \quad K''' := \{k : x_k > \beta + \epsilon\},$$

and let $B := \sup_k x_k < \infty$. Since $\delta\{K'\} \neq 0$, there are infinitely many n such that

$$\frac{1}{n}|K'_n| \ \geq \ d \ > \ 0, \tag{7}$$

and for each such n we have

$$\begin{aligned} (C_1 x)_n \quad &= \tfrac{1}{n} \sum_{k \in K'_n} x_k + \frac{1}{n} \sum_{k \in K''_n} x_k + \frac{1}{n} \sum_{k \in K'''_n} x_k \\ &< \frac{\mu}{n}|K'_n| + \frac{\beta + \epsilon}{n}|K''_n| + \frac{B}{n}|K'''_n| \\ &= \mu \frac{|K'_n|}{n} + (\beta + \epsilon)(1 - \frac{|K'_n|}{n}) + o(1) \\ &\leq \beta - d(\beta - \mu) + \epsilon(1 - d) + o(1). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary it follows that

$$\liminf C_1 x \leq \beta - d(\beta - \mu) < \beta.$$

Hence, x is not C_1-summable to β, which completes the proof. $\square$

By symmetry we have the dual result for lower bounds.

Corollary. *If the sequence x is bounded below and C_1-summable to the number $\alpha := \operatorname{st-lim\,inf} x$, then x is statistically convergent to α.*

Since Buck's Theorem, which was the motivation for Theorem 5, does not assume an upper bound of the sequence, it is natural to ask if that hypothesis could be eliminated from Theorem 5. The following example shows that the upper bound cannot be omitted or even replaced by the weaker assumption of a statistical upper bound.

Example 2. Let x be the sequence given by

$$x_k := \begin{cases} \sqrt{k}, & \text{if} \quad k \quad \text{is a square,} \\ 0, & \text{if} \quad k \quad \text{is an odd nonsquare,} \\ 1, & \text{if} \quad k \quad \text{is an even nonsquare.} \end{cases}$$

Since $\delta\{k : x_k = 0\} = 1/2 = \delta\{k : x_k = 1\}$, it is clear that $\operatorname{st-lim\,inf} x = 0$ and $\operatorname{st-lim\,sup} x = 1$. Therefore x is not statistically convergent. Also note that x is statistically bounded since $\delta\{k : |x_k| > 1\} = 0$. It remains to show that $C_1 x$ has limit $1 = \text{st-}\limsup x$. Let K^2 denote the set of squares, and let K^0 and K^1 denote, respectively, the sets of odd and even nonsquares. With $[t] := \max\{k : k \le t\}$, this yields

$$\begin{aligned}(C_1 x)_n &= \frac{1}{n}\sum_{k\in K_n^0} x_k + \frac{1}{n}\sum_{k\in K_n^1} x_k + \frac{1}{n}\sum_{k\in K_n^2} x_k \\ &= 0 + \frac{1}{n}\frac{[n-[\sqrt{n}]]}{2} + \frac{1}{n}\sum_{i\le\sqrt{n}} i \\ &= 1 + o(1).\end{aligned}$$

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Department of Mathematics and Computer Science, Kent State University, Kent, Ohio 44242-0001

E-mail address: `fridy@mcs.kent.edu`

Department of Mathematics, Faculty of Science, Ankara University, Ankara, 06100, Turkey

E-mail address: `orhan@science.ankara.edu.tr`