A NOTE ON \(p\)-HYPONORMAL OPERATORS

TADASI HURUYA

(Communicated by Palle E. T. Jorgensen)

Abstract. Let \(T\) be a \(p\)-hyponormal operator on a Hilbert space with polar decomposition \(T = U|T|\) and let \(\tilde{T} = |T|^rU|T|^{r-t}\) for \(r > 0\) and \(r \geq t \geq 0\). We study order and spectral properties of \(\tilde{T}\). In particular we refine recent Furuta’s result on \(p\)-hyponormal operators.

1. Introduction

An operator means a bounded linear transformation from a Hilbert space into itself. For an operator \(T\), let \(U|T|\) denote the polar decomposition of \(T\), where \(U\) is a partially isometric operator, \(|T|\) is a positive square root of \(T^*T\) and \(N(T) = N(|T|) = N(U)\), where \(N(S)\) denotes the kernel of an operator \(S\).

An operator \(T\) is said be \(p\)-hyponormal if \((T^*T)^p \geq (TT^*)^p\) for \(1 \geq p > 0\). If \(p = 1\), \(T\) is called hyponormal, and if \(p = \frac{1}{2}\), \(T\) is called semi-hyponormal. A \(p\)-hyponormal operator \(T = U|T|\) is \(q\)-hyponormal for \(p \geq q\) [11] and \(|T|^p\) is hyponormal. In [15], Xia introduced the class of semi-hyponormal operators and obtained results analogous to those of hyponormal operators. Aluthge [1] studied \(p\)-hyponormal operators for \(1 \geq p > 0\). In particular, he defined the operator \(\tilde{T} = |T|^\frac{1}{2}U|T|^{\frac{1}{2}}\) which is called the Aluthge transformation of \(T\). Aluthge transformations have significant applications (see, e.g., [4], [7], [9], [12]). Recently Furuta [9] extended order properties of Aluthge transformations to those of operators \(\tilde{T} = |T|^qU|T|^q\) with \(N(U) = N(U^*)\) (see also Addendum in [9]). In this paper, we refine Furuta’s result by dropping this kernel condition. Applying this result, we give a general version of Patel’s theorem [12, Theorem 1] on the normality for a \(p\)-hyponormal operator. We also study spectral properties of \(p\)-hyponormal operators.

Throughout this paper, let \(1 \geq p > 0\).

2. Generalized Aluthge Transformations

In this section, using Furuta inequality we study order properties of \(p\)-hyponormal operators. We also generalize Patel’s Theorem [12] on the normality of a \(p\)-hyponormal operator.

Received by the editors December 28, 1995 and, in revised form, July 12, 1996.
1991 Mathematics Subject Classification. Primary 47A63, 47B20; Secondary 47A10.
Key words and phrases. Furuta inequality, hyponormal operator, Weyl spectrum.

©1997 American Mathematical Society

3617
Lemma 1. Let \( T = U|T| \) be the polar decomposition of a \( p \)-hyponormal operator on a Hilbert space \( H \). Then there exists an isometric operator \( V \) satisfying \( V|T| = U|T| \) and \( |T|V = |T|U \).

Moreover \( \hat{U} = \begin{pmatrix} V & I - VV^* \\ 0 & -V^* \\ \end{pmatrix} \) is a unitary operator on \( H \oplus H \) such that \( \hat{U} \begin{pmatrix} |T| \\ 0 \\ \end{pmatrix} = \begin{pmatrix} |T| \\ 0 \\ \end{pmatrix} \).

Proof. Proof is based on an idea of [16, p. 41, Lemma 3.5]. Since \( N(U) = N(|T|) = N(T) \subseteq N(T^*) \), we define \( V \) by \( V\xi = U\xi \) for \( \xi \in H \ominus N(U) \) and \( V\xi = \xi \) for \( \xi \in N(U) \). It is easy to see that \( V \) has the desired properties.

Furuta established the following result as an extension of Löwner-Heinz inequality.

The Furuta inequality ([8, Theorem 1]). If \( A \geq B \geq 0 \), then for each \( r \geq 0 \),

(i) \((B^r A^p B^*)^{\frac{1}{q}} \geq (B^r B^p B^*)^{\frac{1}{q}}\)

and

(ii) \((A^r A^p A^*)^{\frac{1}{q}} \geq (A^r B^p A^*)^{\frac{1}{q}}\)

hold for \( p \geq 0 \) and \( q \geq 1 \) with \((1 + 2r)q \geq p + 2r\).

The domain surrounded by \( p, q \) and \( r \) in the figure is the best possible one for the Furuta inequality in [14].

Theorem 2. Let \( T = U|T| \) be the polar decomposition of a \( p \)-hyponormal operator. For \( r > 0 \) and \( r \geq t \geq 0 \), let \( q = \min\left\{ \frac{2 + t}{r}, \frac{r + (r - t)}{r}, 1 \right\} \) and \( \tilde{T} = |T|^t U|T|^{-t} \). Then \( \tilde{T} \) satisfies that \((\tilde{T}^*\tilde{T})^q \geq |T|^{2rq} \geq (\tilde{T}\tilde{T}^*)^q \). In particular, \( \tilde{T} \) is \( q \)-hyponormal.

Proof. We first prove that if \( W \) is a unitary operator such that \( T = W|T| \), then \( S = |T|^t W|T|^{-t} \) is \( q \)-hyponormal. This part is close to the proof of [1, Theorem 2] or [9, Theorem 1]. Put

\[ A = W^*|T|^{2p}W, \quad B = |T|^{2p} \quad \text{and} \quad C = W|T|^{2p}W^*. \]
Then we have that for any $s > 0$,

$$A^s = W|T|^{2sp}W$$

and $C^s = W|T|^{2sp}W^*$. Since $T$ is $p$-hyponormal, we have $(T^*T)^p \geq (TT^*)^p$, or equivalently

$$A \geq B \geq C.$$ 

Let $q' = \frac{1}{q}$. Then the Furuta inequality (i) gives

$$\left((S^*S)^q\right) = \left(|T|^{-t}W^*|T|^{2t}W|T|^{-t}\right)^{\frac{1}{q'}}$$

$$= (B \frac{p}{p+1} A \frac{p}{p+1} B \frac{p}{p+1})^{\frac{1}{q'}} \geq (B \frac{p}{p+1} B \frac{p}{p+1} B \frac{p}{p+1})^{\frac{1}{q'}} = B \frac{p}{p+1},$$

since $(1 + 2\frac{r-1}{2p})q' \geq \frac{p+t-r}{p} \frac{r}{p+r-t} = \frac{p}{p} = \frac{2}{2} + \frac{2}{2p}$ and $q' \geq 1$.

Similarly, the Furuta inequality (ii) gives

$$B \frac{p}{p+1} = (B \frac{p}{p+1} B \frac{p}{p+1} B \frac{p}{p+1})^{\frac{1}{q'}} \geq (B \frac{p}{p+1} C \frac{p}{p} B \frac{p}{p+1})^{\frac{1}{q'}}$$

$$= \left(|T|^{t}W|T|^{2(r-t)}W^*|T|^{t}\right)^{\frac{1}{q'}} = (SS^*)^q,$$

since $(1 + 2\frac{r-1}{2p})q' \geq \frac{p+t-r}{p} \frac{r}{p+r-t} = \frac{p}{p} = \frac{2}{2} + \frac{2}{2p}$ and $q' \geq 1$.

Hence $(S^*S)^q \geq |T|^{2q} \geq (SS^*)^q$ and $S$ is $q$-hyponormal.

Suppose that $U$ is not unitary. By Lemma 1, we choose an isometric operator $V$ such that $V|T| = U|T|$ and $|T|V = |T|U$. Put $\hat{U} = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{T} = \hat{U} \begin{pmatrix} |T| & 0 \\ 0 & 0 \end{pmatrix}$, $\hat{S} = |\hat{T}|^t\hat{U}|\hat{T}|^{-t}$ and $\hat{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Applying the above argument to $\hat{T}$, we have

$$\left(|\hat{T}|^{-t}\hat{U}^*|\hat{T}|^{2t}\hat{U}|\hat{T}|^{-t}\right)^q = (\hat{S}^*\hat{S})^q \geq |\hat{T}|^{2rq} \geq (\hat{S}\hat{S}^*)^q = (|\hat{T}|^t\hat{U}|\hat{T}|^{2(r-t)}\hat{U}^*|\hat{T}|^t)^q.$$

Then

$$\begin{pmatrix} |T|^{2rq} \\ 0 \end{pmatrix}, (\hat{S}\hat{S}^*)^q = \begin{pmatrix} (|T|^tU|T|^{2(r-t)}U^*|T|^{t})^q \\ 0 \end{pmatrix}.$$ 

By Hansen’s inequality [10],

$$\begin{pmatrix} (|T|^{-t}U^*|T|^{2t}U|T|^{-t})^q \\ 0 \end{pmatrix} = (\hat{E}(\hat{S}\hat{S})\hat{E})^q \geq \hat{E}(\hat{S}\hat{S})^q\hat{E}.$$

Since $\hat{E}(\hat{S}\hat{S})^q\hat{E} \geq \hat{E}|\hat{T}|^{2rq}\hat{E} = |\hat{T}|^{2rq}$, we obtain the desired inequalities.

**Remark.** With the notation of Theorem 2, Aluthge [1] and Furuta [9] considered $\hat{T}$ in the greatest $q$ for $r = 1$ and $r \geq p$.

**Theorem 3.** Let $T = U|T|$ be the polar decomposition of a $p$-hyponormal operator on a Hilbert space $H$. For $r > 0$ and $r \geq t \geq 0$, put $\tilde{T} = |T|^tU|T|^{-t}$. If $\tilde{T}$ is normal, then $T$ is normal.
Proof. There exists $q > 0$ such that $\bar{T}$ is $q$-hyponormal by Theorem 2. Moreover we have that
\[
(|T|^{-t}U^*|T|^{-t})^q = (\bar{T}^*\bar{T})^q 
\geq |T|^{2r}
\geq (\bar{T}\bar{T})^q = (|T|U|T|^{-t}T^{-t}U^*|T|^{-t})^q.
\]
Since $\bar{T}$ is normal,
\[
(|T|^{-t}U^*|T|^{-t})^q = |T|^{2r} = (|T|U|T|^{-t}U^*|T|^{-t})^q,
\]
that is,
\[
(*): \quad |T|^{-t}U^*|T|^{-t} = |T|^{2r} = |T|U|T|^{-t}U^*|T|^{-t}.
\]

For an operator $X$ on $H$, it is easy to see that $N(X) = X^*(H)^{\perp}$, so that $N(X) \perp = \bar{X}^*(H)$, the closure of $X^*(H)$. Let $s$ be any positive number. Since $N(|T|^{-t}) = N(|T|)$, it holds that
\[
N(|T|^{-t}) = |T|(H).
\]
Let $P$ denote the orthogonal projection having range $|T|(H)$. Since
\[
N(|T|^{-t}) = N(|T|) = N(T) \subseteq N(T^*) = N(U^*),
\]
we have that
\[
(**): \quad |T|^s P = |T|^s, \quad U^* P = U^*.
\]

(i) If $r - t > 0$, then the second equality of (*) implies that $|T|^{2r-t}|T|^t = |T|U|T|^{-t}U^*|T|^t$. It follows from (**) that
\[
|T|^{2r-t} = |T|^{2r-t} P = |T|U|T|^{-t}U^* P = |T|U^* |T|^{-t}U^*.
\]
Taking adjoints, we have that $|T|^{2r-t} = U|T|^{-t}U^* |T|^t$. Thus
\[
|T|^{2r-t} = U|T|^{-t}U^* = |T^*|^{2r-t},
\]
so that $|T| = |T^*|$. Therefore $T$ is normal.

(ii) If $r - t = 0$, then (*) implies that $U^*|T|^{2r} U = |T|^{2r} = |T|^tUU^*|T|^r$, and hence $|T|^{2r} = |T|^t P = |T|^tUU^* P = |T|^tUU^*$. Then we have that $N(U^*) \subseteq N(|T|) = N(U)$, so that $UU^* = U^*U$. We obtain that
\[
|T|^t^{2r} = U|T|^{2r} U^* = UU^* |T|^{2r} U^* = |T|^{2r}.
\]

Therefore $T$ is normal. This completes the proof.

3. Spectra

In this section, we show a property of the point spectrum of a $p$-hyponormal operator. Applying this property, we give a spectral mapping theorem for the Weyl spectrum of a $p$-hyponormal operator.

Throughout this section, let $r > 0$ and $r \geq t \geq 0$.

The following two lemmas are known. We include proofs for completeness.

Lemma 4. Let $A$ and $B$ be operators and suppose that $A = A^*$. Then $AB$ is invertible if and only if $BA$ is invertible.
Proof. (i) If $AB$ is invertible, there exists an operator $X$ such that $ABX = I$. Then $(BX)^*A = I$, so that $A$ is invertible. Since $AB$ and $A$ are invertible, so is $B$. Therefore $BA$ is invertible.

(ii) If $BA$ is invertible, a similar argument implies that $A$ and $B$ are invertible. Therefore $AB$ is invertible.

**Lemma 5.** If $T$ is an operator such that $T = V|T|$ with partial isometric operator $V$, then $\sigma(|T|^t|T|^{-t}) = \sigma(V|T|^t)$. 

**Proof.** It is well-known that $\sigma(|T|^t(V|T|^{-t})) - \{0\} = \sigma((V|T|^{-t})|T|^t) - \{0\}$ (see, for example, [3, Proposition 5.3 in Chapter I]). By Lemma 4 we have $\sigma(|T|^tV|T|^{-t}) = \sigma(V|T|^t)$.

The following is a general version of [5, Theorem 3].

**Theorem 6.** Let $T = U|T|$ be the polar decomposition of a $p$-hyponormal operator on a Hilbert space $H$. Then $\sigma(U|T|^r) = \{e^{i\theta} \rho^r : e^{i\theta} \rho \in \sigma(T)\}$.

**Proof.** If $U$ is unitary, the equality holds by [5, Theorem 3]. Suppose that $U$ is not unitary. From Lemma 1 there exists a unitary operator $W$ on $H \oplus H$ such that $W\begin{bmatrix} |T| & 0 \\ 0 & 0 \end{bmatrix} = W^* W = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$ is $p$-hyponormal. Put $A = \begin{bmatrix} |T| & 0 \\ 0 & 0 \end{bmatrix}$. Then we have that $\sigma(T) \cup \{0\} = \sigma(WA)$ and $\sigma(WA^*) = \sigma(U|T|^r) \cup \{0\}$. We choose $q > 0$ such that $WA$ and $WA^*$ are $q$-hyponormal.

Using [5, Theorem 3], we have that $\sigma(WA^*) = \{e^{i\theta} \rho^r : e^{i\theta} \rho \in \sigma(WA)\}$. Then $\sigma(U|T|^r) - \{0\} = \sigma(WA^*) - \{0\} = \{e^{i\theta} \rho^r : e^{i\theta} \rho \in \sigma(T) - \{0\}\}$, $0 \in \sigma(U|T|^r)$ and $0 \in \sigma(U|T|^r) = \sigma(T)$. Then $\sigma(U|T|^r) = \{e^{i\theta} \rho^r : e^{i\theta} \rho \in \sigma(T)\}$.

**Corollary 7.** Let $T = U|T|$ be the polar decomposition of a $p$-hyponormal operator. Then $\sigma(|T|^tU|T|^{-t}) = \{e^{i\theta} \rho^r : e^{i\theta} \rho \in \sigma(T)\}$.

**Proof.** By Lemma 5 we have that $\sigma(|T|^tU|T|^{-t}) = \sigma(U|T|^r)$. It follows from Theorem 6 that $\sigma(U|T|^r) = \{e^{i\theta} \rho^r : e^{i\theta} \rho \in \sigma(T)\}$.

**Theorem 8.** Let $T = U|T|$ be the polar decomposition of a $p$-hyponormal operator on a Hilbert space $H$. Then $|T|^tU|T|^{-t} \xi = e^{i\theta} \rho^r \xi$ if and only if $T \xi = e^{i\theta} \rho \xi$ for $e^{i\theta} \rho \in \mathbb{C}$ and $\xi \in H$.

**Proof.** (i) In the case of $\rho = 0$, let $\xi \in N(T)$. Since $\xi \in N(T) = N(U)$, $|T|^tU|T|^{-t} \xi = 0$. Conversely, if $\xi \in N(|T|^tU|T|^{-t})$, then $|T|^{r-t} \xi \in N(U)$. Since $|T|^{r-t} \xi = U|T|^{1+r-t} \xi = 0$ and $1 + r - t > 0$, we have that $\xi \in N(|T|^{1+r-t}) = N(|T|) = N(T)$.

(ii) In the case of $\rho \neq 0$, let $T \xi = e^{i\theta} \rho \xi$. Using [4, Theorem 4] (see also [12, Theorem 2]), we have $|T|^{r-t} \xi = e^{i\theta} \rho^r \xi$. Then we have $|T|^tU|T|^{-t} \xi = e^{i\theta} \rho^r \xi$.

Conversely, let $|T|^tU|T|^{-t} \xi = \beta \xi$, where $e^{i\theta} \rho^r = \beta$. We first assume that $t > 0$. Then there exists a unique vector $\eta$ in $\overline{T^*H}$, the closure of $T^*H$, such that $|T|^t \eta = \xi$.

Since $|T|^t\frac{1}{\beta}(U|T|^{-t} \xi) = \xi$ and $U|T|^{-t} \xi \in \overline{T^*H} \subseteq T^*H$, we have $\frac{1}{\beta} U|T|^{-t} \xi = \eta$. 


Then
\[ U|T|^r \eta = U|T|^{r-\epsilon}|T|^\epsilon \eta = U|T|^{r-\epsilon} \xi = \beta \eta. \]

It follows from [4, Theorem 4] that
\[ |T|^r \eta = |\beta| \eta \text{ and } U \eta = e^{i\theta} \eta. \]

Hence \( |T|^r |T|^\epsilon \eta = |T|^\epsilon |\beta| \eta \), that is, \( |T|^r \xi = |\beta| \xi = \rho^r \xi \). Also we have that
\[ U \xi = U|T|^\epsilon \eta = U|\beta| \xi = e^{i\theta} \eta = e^{i\theta}|T|^\epsilon \eta = e^{i\theta} \xi. \]

Therefore \( T \xi = e^{i\theta} \rho \xi \). If \( t = 0 \), then \( U|T|^r \xi = \beta \xi \). A similar argument implies that \( T \xi = e^{i\theta} \rho \xi \).

For an operator \( T \), let \( \pi_{00}(T) \) denote the set of isolated eigenvalues of finite multiplicity of \( T \) and let \( w(T) \) denote the Weyl spectrum, that is,
\[ w(T) = \cap \{ \sigma(T + K) : K \text{ compact} \}. \]

We need the following two conditions introduced by Baxley [2].

C-1: if \( \{ \lambda_n \} \) is an infinite sequence of distinct points of the set of eigenvalues of finite multiplicity of \( T \) and \( \{ x_n \} \) is any sequence of corresponding normalized eigenvectors, then the sequence \( \{ x_n \} \) does not converge.

C-2: if \( \lambda \in \pi_{00}(T) \), then \( T - \lambda I \) has closed range and index 0, that is,
\[ \dim N(T - \lambda I) = \dim (R(T - \lambda I))^{\perp} < \infty, \]
where \( R(T - \lambda I) \) denotes the range of \( T - \lambda I \).

**Proposition 9.** Let \( T = U|T| \) be the polar decomposition of a \( p \)-hyponormal operator. Then \( T \) satisfies C-2.

**Proof.** Let \( \lambda \in \pi_{00}(T) \). By [4, Theorem 4], \( N(T - \lambda I) \) is a reducing subspace for \( U \) and \( |T| \). Let \( T_1 \) and \( T_2 \) be the restrictions of \( T \) and \( U|T|^p \) to \( N(T - \lambda I)^{\perp} \), respectively. Then \( T_1 \) is \( p \)-hyponormal, \( \sigma(T_1) \subseteq \sigma(T) \) and \( \sigma(T_2) \subseteq \sigma(U|T|^p) \). By Corollary 7, \( \sigma(U|T|) \) and \( \sigma(T_1) \) are homeomorphic to \( \sigma(U|T|^p) \) and \( \sigma(T_2) \), respectively. Suppose that \( \lambda \in \sigma(T_1) \). Then \( \lambda = e^{i\theta} |\lambda| \) is an isolated point of \( \sigma(T_1) \). Since \( T_2 \) is hyponormal, it follows from [13, Theorem 2] that \( e^{i\theta} |\lambda|^p \) is an eigenvalue of \( T_2 \). By Theorem 8, \( \lambda \) is an eigenvalue of \( T_1 \). But this is a contradiction since \( N(T - \lambda I)^{\perp} \) contains no eigenvectors of \( T \) corresponding to \( \lambda \). Hence \( \lambda \notin \sigma(T_1) \) and we have that
\[ R(T - \lambda I) = R(T_1 - \lambda I) = N(T - \lambda I)^{\perp}. \]

Therefore, \( T - \lambda I \) has closed range and index 0.

Chô, Itoh and Ôshiro [6] showed that a \( p \)-hyponormal operator \( T \) holds Weyl’s theorem, that is, \( w(T) = \sigma(T) - \pi_{00}(T) \). Chô informed us that Patel gave a different proof by the property \( \pi_{00}(T) = \pi_{00}(|T|^2 U|T|^2) \). We give another proof.

**Theorem 10** ([6, Theorem]). If \( T \) is a \( p \)-hyponormal operator, then \( w(T) = \sigma(T) - \pi_{00}(T) \).

**Proof.** If \( Tx = zx \), then \( T^*x = \bar{z}x \) by [4, Theorem 4]. Then \( T \) satisfies C-1. By Proposition 9 \( T \) satisfies C-2. It follows from [2, Lemmas 3 and 4] that \( w(T) = \sigma(T) - \pi_{00}(T) \).

**Corollary 11.** Let \( T = U|T| \) be the polar decomposition of a \( p \)-hyponormal operator and put \( \bar{T} = |T|^r U|T|^{r-1} \). Then \( w(\bar{T}) = \{ e^{i\theta} \rho^r : e^{i\theta} \rho \in w(T) \} \).
Proof. Using Corollary 7 and Theorem 8, we have that
\[ \sigma(T) = \{ e^{i\theta} \rho : e^{i\theta} \rho \in \sigma(T) \} \text{ and } \pi_{00}(T) = \{ e^{i\theta} \rho : e^{i\theta} \rho \in \pi_{00}(T) \}. \]
It follows from Theorem 2 and Theorem 10 that
\[ w(T) = \sigma(T) - \pi_{00}(T). \]
Hence we obtain that \( w(T) = \{ e^{i\theta} \rho : e^{i\theta} \rho \in w(T) \} \).

We define \( \psi \) on \( C \) by \( \psi(\rho e^{i\theta}) = \rho e^{i\theta} \rho^{-t} \) and put \( \psi(T) = |T|^{t}|U|^{-t} \). Restating Corollary 7, Theorem 8 and Corollary 11, we have the following spectral mapping result:
\[ \sigma(\psi(T)) = \psi(\sigma(T)), \quad \pi_{00}(\psi(T)) = \psi(\pi_{00}(T)) \text{ and } w(\psi(T)) = \psi(w(T)). \]

ACKNOWLEDGMENT

The author would like to express his thanks to Professor Furuta for his advice and powerful encouragement and to Professor Chô for many valuable discussions during the preparation of this paper.

ADDENDUM

(1) We would like to cite the following result in Addendum of [9];

**Theorem 1’.** Let \( T = U|T| \) be the polar decomposition of \( p \)-hyponormal for \( 1 \geq p > 0 \) with \( N(T) = N(T^*) \). Then \( \tilde{T} = |T|^t|U|^{-t} \) is \( \frac{p+1}{s+t} \)-hyponormal for any \( s \geq 0 \) and \( t \geq \max\{p, s\} \).

(2) After this paper was written, the author have found Aluthge’s paper; “Some generalized theorems on \( p \)-hyponormal operators, Integral Equations and Operator Theory, 24 (1996), 497–501”, in which he proved a theorem (Theorem 1) closely related to our Theorem 2. In fact, our theorem implies his theorem.

REFERENCES

8. T. Furuta, *A \( \geq B \geq 0 \) assures \( (B^rAM^pB^r)^{1/q} \geq B^{(p+2r)/q} \) for \( r \geq 0, q \geq 0, q \geq 1 \) with \( 1 + 2r \) \( \geq p + 2r \)*, Proc. Amer. Math. Soc. 101 (1987), 85–88. MR 89b:47028

Faculty of Education, Niigata University, Niigata 950-21, Japan
E-mail address: huruya@ed.niigata-u.ac.jp