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A GENERALIZATION OF THE DE BRANGES THEOREM

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Abstract. In this paper a generalization of de Branges' proof of the Bieberbach conjecture is given. The argument does not make use of the Askey-Gasper theorem.

1. Introduction

It is well known that in 1984 the American mathematician de Branges [1], [2] proved the famous Bieberbach conjecture, thus, solving completely a problem that had challenged many mathematicians over a period of seventy years.

In this paper, the author gives a generalization of de Branges' proof. The new methods used are quite elementary and yield de Branges' result without using the Askey-Gasper theorem [3].

2. Result

In this section we shall give the main theorem and its proof.

Theorem. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S \tag{1}$$

and

$$\log \frac{f(z)}{z} = \sum_{k=1}^{\infty} C_k z^k, \qquad |z| < 1. \tag{2}$$

If for any real number set $\{\lambda_k\}_{k=1}^{n}$, $\lambda_k \geq 0$ $(k = 1, 2, \dots, n)$, *the condition*

$$\lambda_k + 2 \sum_{p=k+1}^{n} (-1)^{p-k} \lambda_k \geq 0$$

is satisfied, then

$$\sum_{k=1}^{n} k\lambda_k |C_k|^2 \leq 4 \sum_{k=1}^{n} \frac{\lambda_k}{k}, \qquad n = 1, 2, \dots . \tag{3}$$

Equality holds if and only if $f(z)$ *is a rotation of the Koebe function.*

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Corollary. *In the theorem, set* $\lambda_k = n-k+1$ $(k=1,2,\ldots,n)$*, then it becomes de Branges theorem.*

Instead of the de Branges special function system, we consider the function system $S_k(t)$ $(k=1,2,\ldots,n)$ defined by the following system of differential equations:

$$\begin{cases} \dfrac{S'_{k+1}(t)}{k+1} + \dfrac{S'_k(t)}{k} = S_{k+1}(t) - S_k(t), \\ \qquad S_{n+1}(t) \equiv 0, & (4) \\ S_k(0) = \lambda_k, \lambda_k \geq 0 & (5) \\ \qquad (k=1,2,\ldots,n). \end{cases}$$

For a given set λ_k $(k=1,2,\ldots,n)$, (4) and (5) define a unique system of functions $\{S_k(t)\}_{k=1}^n$. Now we find expressions for the $S_k(t)$.

First, we can obtain from (4)

$$\frac{S'_k(t)}{k} = 2\sum_{m=k+1}^{n}(-1)^{m-k-1}S_m(t) - S_k(t) \qquad (k=1,2,\ldots,n). \tag{6}$$

The characteristic equation of this system of differential equations is

$$\Delta(\lambda) = \begin{vmatrix} 1+\lambda & -2\cdot 1 & 2\cdot 1 & \ldots & (-1)^{n-1}2\cdot 1 \\ 0 & 2+\lambda & -2\cdot 2 & & (-1)^{n-2}2\cdot 2 \\ 0 & 0 & 3+\lambda & (-1)^{n-3}2\cdot 3 & \\ \ldots & & \ldots & & \\ 0 & 0 & 0 & \ldots & n+\lambda \end{vmatrix} = 0. \tag{7}$$

So its characteristic roots are

$$\lambda_1 = -1, \lambda_2 = -2, \ldots, \lambda_n = -n. \tag{8}$$

Hence we can suppose

$$S_k(t) = \sum_{l=k}^{n} C_{k,l}e^{-lt} \qquad (k=1,2,\ldots,n), \tag{9}$$

where $C_{k,l}$ $(l=k,k+1,\ldots,n)$ are constants.

Lemma 1. *The functions* $S_k(t)$ $(k=1,2,\ldots,n)$ *in* (9) *have forms*

$$S_k(t) = C_{k,k}e^{-kt} + \sum_{p=k+1}^{n}\alpha_{k,p}C_{p,p}e^{-pt}, \tag{10}$$

where $\alpha_{k,p}$, $C_{p,p}$ $(p=k,k+1,\ldots,n)$ *are constants and*

$$\begin{cases} \alpha_{k,p} = \frac{(-1)^{p-k}2k(2p-1)(2p-2)\cdots(p+k+1)}{(p-k)!} \\ \qquad (p=k,k+1,k+2,\ldots,n), \\ \alpha_{k,k} = 1, \end{cases} \tag{11}$$

$$\sum_{m=k}^{p}(-1)^{m-k}\alpha_{m,p} = \frac{(-1)^{p-k}(2p-1)(2p-2)\cdots(p+k)}{(p-k)!}. \tag{12}$$

Proof. Substituting (9) in (6), we have

$$S_k'(t) + kS_k(t) = 2k \sum_{m=k+1}^{n} (-1)^{m-k-1} \left(\sum_{l=m}^{n} C_{m,l} e^{-lt} \right).$$

The solution of this differential equation is

$$S_k(t) = C_{k,k} e^{-kt} + 2k \sum_{p=k+1}^{n} \left[\sum_{q=k+1}^{p} (-1)^{q-k} \frac{C_{q,p}}{p-k} \right] e^{-pt}.$$

Therefore,

(13)

$$C_{k,p} = \frac{2k}{p-k} \sum_{q=k+1}^{p} (-1)^{q-k} C_{q,p} \qquad (k = 1, 2, \ldots, n-1;\ p = k+1, k+2, \ldots, n).$$

From (13) it may be obtained that

$$C_{k,p} = \alpha_{k,p} C_{p,p}, C_{k+1,p} = \alpha_{k+1,p} C_{p,p}, \ldots, C_{p-1,p} = \alpha_{p-1,p} C_{p,p}$$
$$(k = 1, 2, \ldots, n-1;\ p = k+1, k+2, \ldots, n).$$

When $k = p-1$, (13) is

$$C_{p-1,p} = -2(p-1) C_{p,p},$$

and $C_{p-1,p} - C_{p,p} = -(2p-1)C_{p,p}$. Namely (11) and (12) are valid when $k = p-1$.

Now suppose (11) and (12) are valid if $k = m+1, m+2, \ldots, p-1$. Then by (13)

$$\begin{aligned} C_{m,p} &= -\frac{2m}{p-m}(C_{m+1,p} - C_{m+2,p} + \cdots + (-1)^{p-m-1} C_{p,p}) \\ &= -\frac{2m}{p-m} \cdot \frac{(-1)^{p-m-1}(2p-1)(2p-2)\cdots(p+m+1)}{(p-m-1)!} C_{p,p} \\ &= \frac{(-1)^{p-m} 2m (2p-1)(2p-2)\cdots(p+m+1)}{(p-m)!} C_{p,p}. \end{aligned}$$

Hence (11) is true. Also,

$$\begin{aligned} &C_{m,p} - C_{m+1,p} + \cdots + (-1)^{p-m} C_{p,p} \\ &\quad = \left[\frac{(-1)^{p-m} 2m (2p-1)(2p-2)\cdots(p+m+1)}{(p-m)!} \right. \\ &\qquad \left. - \frac{(-1)^{p-m-1}(2p-1)(2p-2)\cdots(p+m+1)}{(p-m-1)!} \right] C_{p,p} \\ &\quad = \frac{(-1)^{p-m}(2p-1)(2p-2)\cdots(p+m)}{(p-m)!} C_{p,p}, \end{aligned}$$

and (12) is true as well. This completes the proof of Lemma 1. □

Lemma 2. *For any positive integer $n = 1, 2, \ldots,$ the identity*

(14)
$$\sum_{m=0}^{n} (-1)^m \frac{m^k}{m!(n-m)!} = 0$$

holds, where $k = 0, 1, 2, \ldots, n-1$.

Proof. By the binomial theorem

$$(1-x)^n = n! \sum_{m=0}^{n} (-1)^m \frac{x^m}{m!(n-m)!}.$$

Apply the operator $x\frac{d}{dx}$ to the $(1-x)^n$ k times, then let $x = 1$. This gives (14). □

Lemma 3. *In* (10) *the coefficients* $C_{k,k}$ *have the forms*

$$C_{k,k} = \sum_{p=k}^{n} (-1)^{p-k} \delta_{k,p} \lambda_p \qquad (k = 1, 2, \ldots, n), \tag{15}$$

where

$$\begin{aligned} \delta_{k,p} &= \begin{vmatrix} \alpha_{k,k+1} & \alpha_{k,k+2} & \ldots & \alpha_{k,p-1} & \alpha_{k,p} \\ 1 & \alpha_{k+1,k+2} & \ldots & \alpha_{k+1,p-1} & \alpha_{k+1,p} \\ \ldots & & \ldots & & \\ 0 & 0 & \ldots & 1 & \alpha_{p-1,p} \end{vmatrix} \\ &= \frac{(-1)^{p-k} 2k(2k+1)(2k+2)\cdots(p+k-1)}{(p-k)!} \qquad (p > k), \\ \delta_{k,k} &\equiv 1. \end{aligned} \tag{16}$$

Proof. It follows by (5) and (10) that

$$\begin{cases} C_{1,1} + \alpha_{1,2}\ C_{2,2} + \alpha_{1,3}\ C_{3,3} + \cdots + \alpha_{1,n}\ C_{n,n} = \lambda_1, \\ \qquad\qquad C_{2,2} + \alpha_{2,3}\ C_{3,3} + \cdots + \alpha_{2,n}\ C_{n,n} = \lambda_2, \\ \quad \ldots \qquad \ldots \qquad \ldots \\ \qquad\qquad\qquad\qquad\qquad\qquad C_{n,n} = \lambda_n. \end{cases} \tag{17}$$

We then obtain from (17)

$$\begin{aligned} C_{k,k} &= \begin{vmatrix} \lambda_k & \alpha_{k,k+1} & \ldots & \alpha_{k,n-1} & \alpha_{k,n} \\ \lambda_{k+1} & 1 & \ldots & \alpha_{k+1,n-1} & \alpha_{k+1,n} \\ \ldots & & \ldots & & \\ \lambda_{n-1} & 0 & \ldots & 1 & \alpha_{n-1,n} \\ \lambda_n & 0 & \ldots & 0 & 1 \end{vmatrix} \\ &= \sum_{p=k}^{n} (-1)^{p-k} \delta_{k,p} \lambda_p \qquad (k = 1, 2, \ldots, n). \end{aligned}$$

Calculating gives

$$\begin{aligned} \delta_{p-1,p} &= -\frac{2p-2}{1!}, \\ \delta_{p-2,p-1} &= -\frac{2p-4}{1!}, \delta_{p-2,p} = \frac{(2p-4)(2p-3)}{2!}. \end{aligned}$$

Suppose

$$\delta_{m,p} = \frac{(-1)^{p-m} 2m(2m+1)(2m+2)\cdots(p+m-1)}{(p-m)!}$$
$$(m = k+1, k+2, \ldots, p-1; m < p \le n).$$

Then by (16)

$$\begin{aligned}\delta_{k,p} &= \sum_{m=k+1}^{p} (-1)^{m-k-1} \alpha_{k,m} \delta_{m,p} \\ &= \sum_{m=k+1}^{p} (-1)^{p-m-1} \frac{2k(2m-1)(2m-2)\cdots(m+k+1)}{(m-k)!} \\ &\qquad \cdot \frac{2m(2m+1)(2m+2)\cdots(p+m-1)}{(p-m)!} \\ &= \sum_{m=k+1}^{p} (-1)^{p-m-1} \frac{2k(p+m-1)(p+m-2)\cdots(m+k+1)}{(m-k)!(p-m)!}.\end{aligned} \tag{18}$$

Let $m-k=\mu$; then (18) becomes

$$\begin{aligned}\delta_{k,p} &= \sum_{\mu=1}^{p-k} (-1)^{p-k-\mu-1} \frac{2k(p+k+\mu-1)(p+k+\mu-2)\cdots(2k+\mu+1)}{\mu!(p-k-\mu)!} \\ &= (-1)^{p-k} \frac{2k(2k+1)(2k+2)\cdots(p+k-1)}{(p-k)!} \\ &\quad + \sum_{\mu=0}^{p-k} (-1)^{p-k-\mu-1} \frac{2k(2k+\mu+1)(2k+\mu+2)\cdots(p+k+\mu-1)}{\mu!(p-k-\mu)!}.\end{aligned}$$

For fixed p and k, the numerator of second term in the last formula is a $(p-k-1)$th degree polynomial of μ. It must be equal to zero by Lemma 2. Hence

$$\delta_{k,p} = (-1)^{p-k} \frac{2k(2k+1)(2k+2)\cdots(p+k-1)}{(p-k)!}. \qquad \square$$

Lemma 4. *Let the functions $S_k(t)$ $(k=1,2,\ldots,n)$ be defined by the system of differential equations* (4), (5). *Then*

$$-\frac{1}{k} S_k'(0) = \lambda_k + 2 \sum_{p=k+1}^{n} (-1)^{p-k} \lambda_p. \tag{19}$$

Proof. It follows by the definition of $S_k(t)$ and (5) that

$$-\frac{1}{n} S_n'(0) = \lambda_n, \qquad -\frac{1}{n-1} S_{n-1}'(0) = \lambda_{n-1} - 2\lambda_n.$$

Now suppose

$$-\frac{1}{m} S_m'(0) = \lambda_m + 2 \sum_{p=m+1}^{n} (-1)^{p-m} \lambda_p \tag{20}$$

when $m = k+1, k+2, \dots, n$. However, by (10)

$$\begin{aligned} -\frac{1}{k+1}S'_{k+1}(0) &= C_{k+1,k+1} + \sum_{p=k+2}^{n} \frac{p}{k+1}\alpha_{k+1,p}C_{p,p} \\ &= C_{k+1,k+1} + \sum_{p=k+2}^{n} \alpha_{k+1,p}C_{p,p} + \sum_{p=k+2}^{n} \frac{p-k-1}{k+1}\alpha_{k+1,p}C_{p,p} \\ &= \lambda_{k+1} + \sum_{p=k+2}^{n} \frac{p-k-1}{k+1}\alpha_{k+1,p}C_{p,p}. \end{aligned}$$

Therefore

$$\sum_{p=k+2}^{n} \frac{p-k-1}{k+1}\alpha_{k+1,p}C_{p,p} = 2\sum_{p=k+2}^{n}(-1)^{p-k-1}\lambda_p. \tag{21}$$

On the other hand, by (10)

$$\begin{aligned} -\frac{1}{k}S'_k(0) &= C_{k,k} + \sum_{p=k+1}^{n} \frac{p}{k}\alpha_{k,p}C_{p,p} \\ &= C_{k,k} + \sum_{p=k+1}^{n} \alpha_{k,p}C_{p,p} + \sum_{p=k+1}^{n} \frac{p-k}{k}\alpha_{k,p}C_{p,p} \\ &= \lambda_k + \sum_{p=k+1}^{n} \frac{p-k}{k}\alpha_{k,p}C_{p,p} \\ &= \lambda_k - 2C_{k+1,k+1} + \sum_{p=k+2}^{n} \frac{p-k}{k}\alpha_{k,p}C_{p,p}. \end{aligned} \tag{22}$$

But we have

$$\frac{p-k}{k}\alpha_{k,p} = \frac{(-1)^{p-k}2(2p-1)!}{(p-k-1)!(p+k)!} = -\frac{p+k+1}{k+1}\alpha_{k+1,p}.$$

So from (21) and (22)

$$\begin{aligned} -\frac{1}{k}S'_k(0) &= \lambda_k - 2C_{k+1,k+1} - \sum_{p=k+2}^{n} \frac{p+k+1}{k+1}\alpha_{k+1,p}C_{p,p} \\ &= \lambda_k - 2\left(C_{k+1,k+1} + \sum_{p=k+2}^{n}\alpha_{k+1,p}C_{p,p}\right) - \sum_{p=k+2}^{n} \frac{p-k-1}{k+1}\alpha_{k+1,p}C_{p,p} \\ &= \lambda_k - 2\lambda_{k+1} - 2\sum_{p=k+2}^{n}(-1)^{p-k-1}\lambda_p \\ &= \lambda_k + 2\sum_{p=k+1}^{n}(-1)^{p-k}\lambda_p. \end{aligned}$$

Thus (19) is valid. □

Proof of the Theorem. We replace de Branges special system of functions by the special system of functions $\{S_k(t)\}_{k=1}^{n}$ and use his method to complete the proof. □

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