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STRONG CONVERGENCE OF APPROXIMATED SEQUENCES FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study the convergence of the sequence defined by

$$x_0 \in C, \;\; x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n, \;\; n = 0, 1, 2, \dots ,$$

where $0 \le \alpha_n \le 1$ and T is a nonexpansive mapping from a closed convex subset of a Banach space into itself.

1. Introduction

Let C be a closed, convex subset of a Banach space E and let T be a nonexpansive mapping from C into C, i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We deal with the iterative process

$$x_0 \in C, \;\; x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n, \;\; n = 0, 1, 2, \dots , \tag{1.1}$$

where $0 \le \alpha_n \le 1$ and $\alpha_n \to 0$. Concerning this process, Reich [5] posed the following problem:

Problem. Let E be a Banach space. Is there a sequence $\{\alpha_n\}$ such that whenever a weakly compact, convex subset C of E possesses the fixed point property for nonexpansive mappings, then the sequence $\{x_n\}$ defined by (1.1) converges to a fixed point of T for all x in C and all nonexpansive $T : C \to C$?

Though Reich [4, 5] showed an answer in the case when E is uniformly smooth and $\alpha_n = n^{-a}$ with $0 < a < 1$, the problem has been generally open. Recently, Wittmann [7] solved the problem in the case when E is a Hilbert space and $\{\alpha_n\}$ satisfies

$$0 \le \alpha_n \le 1, \; \lim_{n\to\infty} \alpha_n = 0, \; \sum_{n=0}^{\infty} \alpha_n = \infty \;\text{ and }\; \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{1.2}$$

In this paper, we extend Wittmann's result to Banach spaces. Our result is the following:

Theorem. *Let E be a Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed, convex subset of E. Let T be a nonexpansive mapping from C into C such that the set $F(T)$ of fixed points of T is nonempty. Let $\{\alpha_n\}$*

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be a sequence which satisfies (1.2). *Let $x \in C$ and let $\{x_n\}$ be the sequence defined by* (1.1). *Assume that $\{z_t\}$ converges strongly to $z \in F(T)$ as $t \downarrow 0$, where for $0 < t < 1$, z_t is a unique element of C which satisfies $z_t = tx + (1-t)Tz_t$. Then $\{x_n\}$ converges strongly to z.*

If C satisfies additional assumptions then $\{z_t\}$ defined above converges strongly to a fixed point of T. We know the following [4, 6]:

Let E be a Banach space whose norm is uniformly Gâteaux differentiable, let C be a weakly compact, convex subset of E and let T be a nonexpansive mapping from C into C. Let $x \in C$ and let z_t be a unique element of C which satisfies $z_t = tx+(1-t)Tz_t$ for $0 < t < 1$. Assume that each nonempty, T-invariant, closed, convex subset of C contains a fixed point of T. Then $\{z_t\}$ converges strongly to a fixed point of T.

So our theorem gives an answer to Reich's problem in the case when the norm of E is uniformly Gâteaux differentiable and each nonempty, closed, convex subset of C possesses the fixed point property for nonexpansive mappings.

2. Preliminaries and notations

Throughout this paper, all vector spaces are real and we denote by $\mathbb{N}$ and $\mathbb{N}_+$, the set of all nonnegative integers and the set of all positive integers, respectively. Let E be a Banach space and let E' be its dual. The value of $y \in E'$ at $x \in E$ will be denoted by $\langle x, y\rangle$. We also denote by J the duality mapping from E into $2^{E'}$, i.e.,

$$Jx = \{y \in E' : \langle x, y\rangle = \|x\|^2 = \|y\|^2\}, \quad x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if, for each $y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists uniformly for $x \in U$. E is said to be uniformly smooth if the limit (2.1) exists uniformly for $x, y \in U$. It is well known that if the norm of E is uniformly Gâteaux differentiable then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of E.

Let μ be a continuous, linear functional on l^∞ and let $(a_0, a_1, \cdots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \cdots))$. We call μ a Banach limit [1] when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \cdots) \in l^\infty$.

To prove our result, we need the following propositions, which can be deduced by the same lines as those in [3]. For the sake of completeness, we give the proofs in our appendix.

Proposition 1. *Let a be a real number and let $(a_0, a_1, \cdots) \in l^\infty$. Then $\mu_n(a_n) \le a$ for all Banach limits μ if and only if for each $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}_+$ such that*

$$\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \varepsilon \quad \textit{for all } p \ge p_0 \textit{ and } n \in \mathbb{N}. \tag{2.2}$$

Proposition 2. *Let a be a real number and let $(a_0, a_1, \cdots) \in l^\infty$ such that $\mu_n(a_n) \le a$ for all Banach limits μ and $\overline{\lim}_{n\to\infty} (a_{n+1} - a_n) \le 0$. Then $\overline{\lim}_{n\to\infty} a_n \le a$.*

3. Proof of Theorem

The following is obtained in [7]. For the sake of completeness, we give the proof.

Lemma 1. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$

Proof. We remark that $\{x_n\}$ and $\{Tx_n\}$ are bounded by $F(T) \neq \emptyset$. Set $M = \sup\{\|Tx_n\| : n \in \mathbb{N}\}$. Then since $\|x_{n+1} - x_n\| \leq |\alpha_n - \alpha_{n-1}|(\|x\| + M) + (1-\alpha_n)\|x_n - x_{n-1}\|$ for each $n \in \mathbb{N}_+$, we have

$$\begin{aligned}
&\|x_{n+m+1} - x_{n+m}\| \\
&\leq \left(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\right)(\|x\| + M) + \left(\prod_{k=m}^{n+m-1} (1 - \alpha_{k+1})\right)\|x_{m+1} - x_m\| \\
&\leq \left(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\right)(\|x\| + M) + \exp\left(-\sum_{k=m}^{n+m-1} \alpha_{k+1}\right)\|x_{m+1} - x_m\|
\end{aligned}$$

for all $m, n \in \mathbb{N}$. So the boundedness of $\{x_n\}$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$ yield

$$\varlimsup_{n\to\infty} \|x_{n+1} - x_n\| = \varlimsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| \leq \left(\sum_{k=m}^{\infty} |\alpha_{k+1} - \alpha_k|\right)(\|x\| + M)$$

for all $m \in \mathbb{N}$. Hence by $\sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$, we get the conclusion. □

Using Proposition 2, we obtain the following.

Lemma 2. $\varlimsup_{n\to\infty} \langle x - z, J(x_n - z)\rangle \leq 0.$

Proof. Let μ be a Banach limit and let $0 < t < 1$. Since $\{\alpha_n\}$ converges to 0, T is nonexpansive and μ is a Banach limit, we get

$$\mu_n \|x_n - Tz_t\|^2 \leq \mu_n \|x_n - z_t\|^2.$$

From $(1-t)(x_n - Tz_t) = (x_n - z_t) - t(x_n - x)$, we have

$$\begin{aligned}
(1-t)^2 \|x_n - Tz_t\|^2 &\geq \|x_n - z_t\|^2 - 2t\langle x_n - x, J(x_n - z_t)\rangle \\
&= (1-2t)\|x_n - z_t\|^2 + 2t\langle x - z_t, J(x_n - z_t)\rangle
\end{aligned}$$

for each $n \in \mathbb{N}$. These inequalities yield

$$\frac{t}{2}\, \mu_n \|x_n - z_t\|^2 \geq \mu_n \langle x - z_t, J(x_n - z_t)\rangle.$$

Tending t to 0, we get

$$0 \geq \mu_n \langle x - z, J(x_n - z)\rangle,$$

because E has a uniformly Gâteaux differentiable norm. On the other hand, we have

$$\lim_{n\to\infty} \left|\langle x - z, J(x_{n+1} - z)\rangle - \langle x - z, J(x_n - z)\rangle\right| = 0$$

by Lemma 1. Hence by Proposition 2, we obtain

$$\varlimsup_{n\to\infty} \langle x - z, J(x_n - z)\rangle \leq 0. \qquad \square$$

Now we can prove our theorem.

Proof of Theorem. Since $(1-\alpha_n)(Tx_n - z) = (x_{n+1} - z) - \alpha_n(x - z)$, we have

$$\|(1-\alpha_n)(Tx_n - z)\|^2 \geq \|x_{n+1} - z\|^2 - 2\alpha_n\langle x - z, J(x_{n+1} - z)\rangle,$$

which yields

$$\|x_{n+1} - z\|^2 \leq (1-\alpha_n)\|x_n - z\|^2 + 2(1-(1-\alpha_n))\langle x - z, J(x_{n+1} - z)\rangle$$

for each $n \in \mathbb{N}$. Let $\varepsilon > 0$. By Lemma 2, there exists $m \in \mathbb{N}$ such that

$$\langle x - z, J(x_n - z)\rangle \leq \frac{\varepsilon}{2}$$

for all $n \geq m$. Then we have

$$\|x_{n+m} - z\|^2 \leq \left(\prod_{k=m}^{n+m-1}(1-\alpha_k)\right)\|x_m - z\|^2 + \left(1 - \prod_{k=m}^{n+m-1}(1-\alpha_k)\right)\varepsilon$$

for all $n \in \mathbb{N}$. Hence by $\sum_{k=0}^{\infty}\alpha_k = \infty$, we get

$$\overline{\lim_{n\to\infty}}\|x_n - z\|^2 = \overline{\lim_{n\to\infty}}\|x_{n+m} - z\|^2 \leq \varepsilon.$$

Since ε is an arbitrary positive real number, $\{x_n\}$ converges strongly to z. □

Remark. Halpern [2] showed that $\alpha_n \to 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$ are necessary conditions for the convergence of the sequence $\{x_n\}$ defined by (1.1). The condition $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ is used only to show $x_{n+1} - x_n \to 0$. For other conditions which ensure $x_{n+1} - x_n \to 0$, see [7].

Appendix

In this appendix, we prove Proposition 1 and Proposition 2.

Proof of Proposition 1. First we shall prove the only if part. Assume that $\mu_n(a_n) \leq a$ for all Banach limits μ. Define a sublinear functional q from l^∞ into the set of real numbers by

$$q((b_0, b_1, \cdots)) = \overline{\lim_{p\to\infty}}\sup_{n\in\mathbb{N}}\frac{1}{p}\sum_{i=n}^{n+p-1} b_i, \qquad (b_0, b_1, \cdots) \in l^\infty.$$

We write $q_n(b_n)$ instead of $q((b_0, b_1, \cdots))$ for $(b_0, b_1, \cdots) \in l^\infty$. By the Hahn-Banach theorem, there exists a linear functional μ from l^∞ into the set of real numbers such that $\mu \leq q$ and $\mu_n(a_n) = q_n(a_n)$. It is easy to see that μ is a Banach limit. From the assumption, we have $q_n(a_n) \leq a$. So for each $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}_+$ which satisfies (2.2).

Next we shall prove the if part. Assume that for each $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}_+$ which satisfies (2.2). Let μ be a Banach limit and let $\varepsilon > 0$. By the hypothesis, there exists $p_0 \in \mathbb{N}_+$ which satisfies (2.2). So we have

$$\mu_n(a_n) = \mu_n\left(\frac{a_n + a_{n+1} + \cdots + a_{n+p_0-1}}{p_0}\right) \leq a + \varepsilon.$$

Since ε is an arbitrary positive real number, we get $\mu_n(a_n) \leq a$. □

Proof of Proposition 2. Let $\varepsilon > 0$. By Proposition 1, there exists $p \geq 2$ such that

$$\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ such that $a_{n+1} - a_n < \varepsilon/(p-1)$ for all $n \geq n_0$. Let $n \geq n_0 + p$. Then we have

$$\begin{aligned} a_n =& a_{n-i} + (a_{n-i+1} - a_{n-i}) + (a_{n-i+2} - a_{n-i+1}) + \cdots + (a_n - a_{n-1}) \\ \leq& a_{n-i} + \frac{i\varepsilon}{p-1} \end{aligned}$$

for each $i = 0, 1, \cdots, p-1$. So we get

$$a_n \leq \frac{a_n + a_{n-1} + \cdots + a_{n-p+1}}{p} + \frac{1}{p} \cdot \frac{p(p-1)}{2} \cdot \frac{\varepsilon}{p-1} \leq a + \varepsilon.$$

Hence we have

$$\overline{\lim_{n\to\infty}}\, a_n \leq a + \varepsilon.$$

Since ε is an arbitrary positive real number, we get the conclusion. □

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References

[1] S. Banach, *Théorie des opérations linéaires,* Monografie Mat., PWN, Warszawa, 1932.
[2] B. Halpern, *Fixed points of nonexpanding maps,* Bull. Amer. Math. Soc. **73** (1967), 957-961. MR **36:**2022
[3] G. G. Lorentz, *A contribution to the theory of divergent series,* Acta Math. **80** (1948), 167-190. MR **10:**367e
[4] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces,* J. Math. Anal. Appl. **75** (1980), 287-292. MR **82a:**47050
[5] S. Reich, *Some problems and results in fixed point theory,* Contemp. Math. **21** (1983), 179-187. MR **85e:**47082
[6] W. Takahashi and Y. Ueda, *On Reich's strong convergence theorems for resolvents of accretive operators,* J. Math. Anal. Appl. **104** (1984), 546-553. MR **86c:**47070
[7] R. Wittmann, *Approximation of fixed points of nonexpansive mappings,* Arch. Math. **58** (1992), 486-491. MR **93c:**47069

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