STRONG CONVERGENCE OF APPROXIMATED SEQUENCES FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study the convergence of the sequence defined by

\[ x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2, \ldots, \]

where \( 0 \leq \alpha_n \leq 1 \) and \( T \) is a nonexpansive mapping from a closed convex subset of a Banach space into itself.

1. Introduction

Let \( C \) be a closed, convex subset of a Banach space \( E \) and let \( T \) be a nonexpansive mapping from \( C \) into \( C \), i.e., \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x, y \in C \). We deal with the iterative process

\[ x_0 \in C, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2, \ldots, \]

where \( 0 \leq \alpha_n \leq 1 \) and \( \alpha_n \to 0 \). Concerning this process, Reich [5] posed the following problem:

Problem. Let \( E \) be a Banach space. Is there a sequence \( \{\alpha_n\} \) such that whenever a weakly compact, convex subset \( C \) of \( E \) possesses the fixed point property for nonexpansive mappings, then the sequence \( \{x_n\} \) defined by (1.1) converges to a fixed point of \( T \) for all \( x \) in \( C \) and all nonexpansive \( T : C \to C \)?

Though Reich [4, 5] showed an answer in the case when \( E \) is uniformly smooth and \( \alpha_n = n^{-a} \) with \( 0 < a < 1 \), the problem has been generally open. Recently, Wittmann [7] solved the problem in the case when \( E \) is a Hilbert space and \( \{\alpha_n\} \) satisfies

\[ 0 \leq \alpha_n \leq 1, \quad \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \]

In this paper, we extend Wittmann’s result to Banach spaces. Our result is the following:

Theorem. Let \( E \) be a Banach space whose norm is uniformly Gâteaux differentiable and let \( C \) be a closed, convex subset of \( E \). Let \( T \) be a nonexpansive mapping from \( C \) into \( C \) such that the set \( F(T) \) of fixed points of \( T \) is nonempty. Let \( \{\alpha_n\} \)

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be a sequence which satisfies (1.2). Let \( x \in C \) and let \( \{x_n\} \) be the sequence defined by (1.1). Assume that \( \{z_i\} \) converges strongly to \( z \in F(T) \) as \( t \downarrow 0 \), where for \( 0 < t < 1 \), \( z_1 \) is a unique element of \( C \) which satisfies \( z_1 = tx + (1-t)Tz_1 \). Then \( \{x_n\} \) converges strongly to \( z \).

If \( C \) satisfies additional assumptions then \( \{z_i\} \) defined above converges strongly to a fixed point of \( T \). We know the following [4, 6]:

Let \( E \) be a Banach space whose norm is uniformly Gâteaux differentiable, let \( C \) be a weakly compact, convex subset of \( E \) and let \( T \) be a nonexpansive mapping from \( C \) into \( C \). Let \( x \in C \) and let \( z_i \) be a unique element of \( C \) which satisfies \( z_i = tx + (1-t)Tz_i \) for \( 0 < t < 1 \). Assume that each nonempty, \( T \)-invariant, closed, convex subset of \( C \) contains a fixed point of \( T \). Then \( \{z_i\} \) converges strongly to a fixed point of \( T \).

So our theorem gives an answer to Reich’s problem in the case when the norm of \( E \) is uniformly Gâteaux differentiable and each nonempty, closed, convex subset of \( C \) possesses the fixed point property for nonexpansive mappings.

2. Preliminaries and notations

Throughout this paper, all vector spaces are real and we denote by \( \mathbb{N} \) and \( \mathbb{N}_+ \), the set of all nonnegative integers and the set of all positive integers, respectively. Let \( E \) be a Banach space and let \( E' \) be its dual. The value of \( y \in E' \) at \( x \in E \) will be denoted by \( \langle x, y \rangle \). We also denote by \( J \) the duality mapping from \( E \) into \( 2^{E'} \), i.e.,

\[
Jx = \{ y \in E' : \langle x, y \rangle = \|x\|^2 = \|y\|^2 \}, \quad x \in E.
\]

Let \( U = \{ x \in E : \|x\| = 1 \} \). The norm of \( E \) is said to be uniformly Gâteaux differentiable if, for each \( y \in U \), the limit

\[
(2.1) \quad \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists uniformly for \( x \in U \). \( E \) is said to be uniformly smooth if the limit (2.1) exists uniformly for \( x, y \in U \). It is well known that if the norm of \( E \) is uniformly Gâteaux differentiable then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of \( E \).

Let \( \mu \) be a continuous, linear functional on \( l^\infty \) and let \( (a_0, a_1, \cdots) \in l^\infty \). We write \( \mu_n(a_n) \) instead of \( \mu((a_0, a_1, \cdots)) \). We call \( \mu \) a Banach limit [1] when \( \mu \) satisfies \( \|\mu\| = \mu_n(1) = 1 \) and \( \mu_n(a_{n+1}) = \mu_n(a_n) \) for all \( (a_0, a_1, \cdots) \in l^\infty \).

To prove our result, we need the following propositions, which can be deduced by the same lines as those in [3]. For the sake of completeness, we give the proofs in our appendix.

**Proposition 1.** Let \( a \) be a real number and let \( (a_0, a_1, \cdots) \in l^\infty \). Then \( \mu_n(a_n) \leq a \) for all Banach limits \( \mu \) if and only if for each \( \varepsilon > 0 \), there exists \( p_0 \in \mathbb{N}_+ \) such that

\[
(2.2) \quad \frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \varepsilon \quad \text{for all} \quad p \geq p_0 \quad \text{and} \quad n \in \mathbb{N}.
\]

**Proposition 2.** Let \( a \) be a real number and let \( (a_0, a_1, \cdots) \in l^\infty \) such that \( \mu_n(a_n) \leq a \) for all Banach limits \( \mu \) and \( \lim_{n \to \infty} (a_{n+1} - a_n) \leq 0 \). Then \( \lim_{n \to \infty} a_n \leq a \).
3. Proof of Theorem

The following is obtained in [7]. For the sake of completeness, we give the proof.

**Lemma 1.** \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0. \)

**Proof.** We remark that \( \{x_n\} \) and \( \{Tx_n\} \) are bounded by \( F(T) \neq \emptyset \). Set \( M = \sup\{\|Tx_n\| : n \in \mathbb{N}\} \). Then since \( \|x_{n+1} - x_n\| \leq |\alpha_n - \alpha_n-1| (\|x\| + M) + (1-\alpha_n)\|x_n - x_{n-1}\| \) for each \( n \in \mathbb{N}_+ \), we have

\[
\|x_{n+m+1} - x_{n+m}\| \\
\leq \left( \sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k| \right) (\|x\| + M) + \left( \prod_{k=m}^{n+m-1} (1 - \alpha_{k+1}) \right) \|x_{m+1} - x_m\| \\
\leq \left( \sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k| \right) (\|x\| + M) + \exp\left( - \sum_{k=m}^{n+m-1} \alpha_{k+1} \right) \|x_{m+1} - x_m\|
\]

for all \( m, n \in \mathbb{N} \). So the boundedness of \( \{x_n\} \) and \( \sum_{k=0}^{\infty} \alpha_k = \infty \) yield

\[
\lim_{n \to \infty} \| x_{n+1} - x_n \| = \lim_{n \to \infty} \| x_{n+m+1} - x_{n+m} \| \leq \left( \sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k| \right) (\|x\| + M)
\]

for all \( m \in \mathbb{N} \). Hence by \( \sum_{k=0}^{\infty} \alpha_{k+1} - \alpha_k < \infty \), we get the conclusion. \( \square \)

Using Proposition 2, we obtain the following.

**Lemma 2.** \( \lim_{n \to \infty} \langle x - z, J(x_n - z) \rangle \leq 0. \)

**Proof.** Let \( \mu \) be a Banach limit and let \( 0 < t < 1 \). Since \( \{\alpha_n\} \) converges to 0, \( T \) is nonexpansive and \( \mu \) is a Banach limit, we get

\[
\mu_n \|x_n - Tz_t\|^2 \leq \mu_n \|x_n - z_t\|^2.
\]

From \( (1-t)(x_n - Tz_t) = (x_n - z_t) - t(x_n - x) \), we have

\[
(1-t)^2 \|x_n - Tz_t\|^2 \geq \|x_n - z_t\|^2 - 2t(x_n - x, J(x_n - z_t)) \\
= (1-2t)\|x_n - z_t\|^2 + 2t(x - z_t, J(x_n - z_t))
\]

for each \( n \in \mathbb{N} \). These inequalities yield

\[
\frac{t}{2} \mu_n \|x_n - z_t\|^2 \geq \mu_n \langle x - z_t, J(x_n - z_t) \rangle.
\]

Tending \( t \) to 0, we get

\[
0 \geq \mu_n \langle x - z, J(x_n - z) \rangle,
\]

because \( E \) has a uniformly Gâteaux differentiable norm. On the other hand, we have

\[
\lim_{n \to \infty} \| (x - z, J(x_{n+1} - z)) - (x - z, J(x_n - z)) \| = 0
\]

by Lemma 1. Hence by Proposition 2, we obtain

\[
\lim_{n \to \infty} \langle x - z, J(x_n - z) \rangle \leq 0. \quad \square
\]
Now we can prove our theorem.

**Proof of Theorem.** Since \((1 - \alpha_n)(Tx_n - z) = (x_{n+1} - z) - \alpha_n(x - z)\), we have
\[
\|(1 - \alpha_n)(Tx_n - z)\|^2 \geq \|x_{n+1} - z\|^2 - 2\alpha_n\langle x - z, J(x_{n+1} - z)\rangle,
\]
which yields
\[
\|x_{n+1} - z\|^2 \leq (1 - \alpha_n)\|x_n - z\|^2 + 2(1 - \alpha_n)\langle x - z, J(x_{n+1} - z)\rangle
\]
for each \(n \in \mathbb{N}\). Let \(\varepsilon > 0\). By Lemma 2, there exists \(m \in \mathbb{N}\) such that
\[
\langle x - z, J(x_n - z)\rangle \leq \frac{\varepsilon}{2}
\]
for all \(n \geq m\). Then we have
\[
\|x_{n+m} - z\|^2 \leq \left(\prod_{k=m}^{n+m-1} (1 - \alpha_k)\right)\|x_m - z\|^2 + \left(1 - \prod_{k=m}^{n+m-1} (1 - \alpha_k)\right)\varepsilon
\]
for all \(n \in \mathbb{N}\). Hence by \(\sum_{k=0}^{\infty} \alpha_k = \infty\), we get
\[
\lim_{n \to \infty} \|x_n - z\|^2 = \lim_{n \to \infty} \|x_{n+m} - z\|^2 \leq \varepsilon.
\]
Since \(\varepsilon\) is an arbitrary positive real number, \(\{x_n\}\) converges strongly to \(z\). \(\square\)

**Remark.** Halpern [2] showed that \(\alpha_n \to 0\) and \(\sum_{n=0}^{\infty} \alpha_n = \infty\) are necessary conditions for the convergence of the sequence \(\{x_n\}\) defined by (1.1). The condition \(\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty\) is used only to show \(x_{n+1} - x_n \to 0\). For other conditions which ensure \(x_{n+1} - x_n \to 0\), see [7].

**APPENDIX**

In this appendix, we prove Proposition 1 and Proposition 2.

**Proof of Proposition 1.** First we shall prove the only if part. Assume that \(\mu_n(a_n) \leq a\) for all Banach limits \(\mu\). Define a sublinear functional \(q\) from \(l^\infty\) into the set of real numbers by
\[
q((b_0, b_1, \cdots)) = \lim_{p \to \infty} \sup_{n \in \mathbb{N}} \frac{1}{p} \sum_{i=n}^{n+p-1} b_i, \quad (b_0, b_1, \cdots) \in l^\infty.
\]
We write \(q_n(b_n)\) instead of \(q((b_0, b_1, \cdots))\) for \((b_0, b_1, \cdots) \in l^\infty\). By the Hahn-Banach theorem, there exists a linear functional \(\mu\) from \(l^\infty\) into the set of real numbers such that \(\mu \leq q\) and \(\mu_n(a_n) = q_n(a_n)\). It is easy to see that \(\mu\) is a Banach limit. From the assumption, we have \(q_n(a_n) \leq a\). So for each \(\varepsilon > 0\), there exists \(p_0 \in \mathbb{N}_+\) which satisfies (2.2).

Next we shall prove the if part. Assume that for each \(\varepsilon > 0\), there exists \(p_0 \in \mathbb{N}_+\) which satisfies (2.2). Let \(\mu\) be a Banach limit and let \(\varepsilon > 0\). By the hypothesis, there exists \(p_0 \in \mathbb{N}_+\) which satisfies (2.2). So we have
\[
\mu_n(a_n) = \mu_n\left(\frac{a_n + a_{n+1} + \cdots + a_{n+p_0-1}}{p_0}\right) \leq a + \varepsilon.
\]
Since \(\varepsilon\) is an arbitrary positive real number, we get \(\mu_n(a_n) \leq a\). \(\square\)
Proof of Proposition 2. Let $\varepsilon > 0$. By Proposition 1, there exists $p \geq 2$ such that
\[
\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \frac{\varepsilon}{2}
\]
for all $n \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ such that $a_{n+1} - a_n < \varepsilon/(p-1)$ for all $n \geq n_0$. Let $n \geq n_0 + p$. Then we have
\[
a_n = a_{n-i} + (a_{n-i+1} - a_{n-i}) + (a_{n-i+2} - a_{n-i+1}) + \cdots + (a_n - a_{n-1})
\leq a_{n-i} + \frac{i\varepsilon}{p-1}
\]
for each $i = 0, 1, \cdots, p-1$. So we get
\[
a_n \leq \frac{a_n + a_{n-1} + \cdots + a_{n-p+1}}{p} + \frac{1}{p} \cdot \frac{p(p-1)}{2} \cdot \frac{\varepsilon}{p-1} \leq a + \varepsilon.
\]
Hence we have
\[
\lim_{n \to \infty} a_n \leq a + \varepsilon.
\]
Since $\varepsilon$ is an arbitrary positive real number, we get the conclusion. \qed

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