CESÀRO TRANSFORMS OF FOURIER COEFFICIENTS
OF $L^\infty$-FUNCTIONS

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(Communicated by J. Marshall Ash)

Abstract. In this note, we show that Cesàro transforms of Fourier cosine or sine coefficients of any $L^\infty(0, \pi)$-function are Fourier cosine or sine coefficients of some $BMO(0, \pi)$-function.

Let $p \in [1, \infty)$ and $L^p(0, \pi)$ denote the space of Lebesgue measurable functions $f : (0, \pi) \to (-\infty, \infty)$ with the usual norm $\|f\|_p < \infty$. As is well known, $L^\infty(0, \pi)$, the space of essentially bounded functions $f : (0, \pi) \to (-\infty, \infty)$ with the usual norm $\|f\|_\infty < \infty$, is viewed as a limit space $L^p(0, \pi)$ as $p \to \infty$ in sense of duality. However, in the situation of Hardy space, $L^\infty(0, \pi)$ is substituted by $BMO(0, \pi)$—the space of functions $f \in L^1(0, \pi)$ with bounded mean oscillation:

$$\|f\|_* = \sup_{I} \frac{1}{|I|} \int_I |f(x) - f_I| dx < \infty,$$

where the supremum is taken over all subintervals $I$ of $(0, \pi)$, $f_I$ stands for the mean value of $f$ on $I$: $1/|I| \int_I f(x) dx$ and $|I|$ denotes the length of $I$: $|I| = \int_I dx$.

The following inclusion chain is helpful for us to understand the relation between those spaces mentioned above:

$$L^\infty(0, \pi) \subsetneq BMO(0, \pi) \subsetneq \bigcap_{1 \leq p < \infty} L^p(0, \pi).$$

Now, suppose that $f \in L^1(0, \pi)$ and $a = \{a_n\}$ or $b = \{b_n\}$ is the sequence of Fourier cosine or sine coefficients of $f$ extended to $(-\pi, \pi)$ as an even or odd function, namely,

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx, n = 0, 1, 2, \ldots,$$

or

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx, n = 1, 2, \ldots.$$
In other words, the even or odd extension of \( f \in L^1(0, \pi) \) has a Fourier series below:

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx
\]

or

\[
f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx
\]

The Cesàro transform of \( a = \{a_n\} \) or \( b = \{b_n\} \) is defined by \( Ca = \{A_n\} \) or \( Cb = \{B_n\} \), where

\[
A_0 = a_0, A_n = \frac{\sum_{k=1}^{n} a_k}{n}, n = 1, 2, ...
\]

or

\[
B_n = \frac{\sum_{k=1}^{n} b_k}{n}, n = 1, 2, ...
\]

A very natural question is raised here: If \( f \in L^p(0, \pi) \) with Fourier cosine or sine coefficients \( a = \{a_n\} \) or \( b = \{b_n\} \) then must \( Ca = \{A_n\} \) or \( Cb = \{B_n\} \) be Fourier cosine of sine coefficients of a function also in \( L^p(0, \pi) \)?

In 1928, Hardy gave a positive answer for the question in the case: \( p \in [1, \infty) \). Since then, there have been some further generalizations, \([1, \ 2]\). But there has been no satisfactory result about the case: \( p = \infty \), just like the case \( p \in [1, \infty) \), \([4]\). For instance, if taking a bounded function \( f(x) = \cos x \) with Fourier cosine coefficients \( a = \{0, 1, 0, 0, \ldots\} \), then we immediately find that \( Ca = \{0, 1, 1/2, 1/3, \ldots\} \) is the sequence of Fourier cosine coefficients of function \( F(x) = \log 1/(2\sin(x/2)) \). However, this \( F \) is unbounded, i.e., \( F \notin L^\infty(0, \pi) \). Through a careful observation, we, on the other hand, discover that the function \( F \) is of BMO property, that is to say, \( F \in BMO(0, \pi) \). More importantly, we are motivated by the above argument to enable us to answer the question in the case of \( p = \infty \).

**Theorem.** Let \( f \in L^\infty(0, \pi) \) with Fourier cosine or sine coefficients \( a = \{a_n\} \) or \( b = \{b_n\} \). Then \( Ca = \{A_n\} \) or \( Cb = \{B_n\} \) are Fourier cosine or sine coefficients of some function \( F \in BMO(0, \pi) \).

**Proof.** It is sufficient to verify this fact for Fourier cosine coefficients.

First of all, we define a linear operator \( \sigma \) on \( L^\infty(0, \pi) \), which may be called the Cesàro operator on \( L^\infty(0, \pi) \), and is exactly given by

\[
(\sigma f)(x) = \int_{x}^{\pi} \frac{f(t)}{\tan \frac{t}{2}} dt, f \in L^\infty(0, \pi).
\]

Also let

\[
(\lambda f)(x) = 2 \int_{x}^{\pi} \frac{f(t)}{t} dt, f \in L^\infty(0, \pi).
\]

Then \( \lambda f - \sigma f \) is a bounded function, i.e., \( \lambda f - \sigma f \in L^\infty(0, \pi) \). In fact, for \( f \in L^\infty(0, \pi) \), it is easy to get that \( |(\lambda f)(x) - (\sigma f)(x)| \leq C_1 ||f||_{\infty} \), where \( C_1 = 2 \int_{0}^{\pi} \frac{1}{2 \tan \frac{t}{2}} - \frac{1}{t} dt < \infty \).

Next, assuming that \( K(t) = -\log|2\sin(t/2)| \) for \( t \in (-\pi, \pi) \), we get that \( c = \{c_n\} \), where \( c_0 = 0, c_n = a_n/n, n = 1, 2, \ldots \), is the sequence of Fourier cosine
coefficients of function \( g(x) = \frac{1}{2} \int_{-\pi}^{\pi} f(x + t)K(t)dt \) [7, p.180]. As Hardy showed in [3], \( \sigma a \) is the sequence of Fourier cosine coefficients of function \( F(x) = ((\sigma f)(x) + g(x))/2 \).

Finally, we prove that \( F \in BMO(0, \pi) \). For this end, it suffices to check that \( \lambda f \) is in \( BMO(0, \pi) \) since \( \lambda f - \sigma f \in L^\infty(0, \pi) \) and \( \|g\|_\infty \leq C_2\|f\|_\infty \), where \( C_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |\log|2\sin \frac{t}{2}||dt < \infty \). At this time, taking any interval \( I = (\alpha, \beta) \subset (0, \pi) \) and \( C_I = (\lambda f)(\beta) \) for \( f \in L^\infty(0, \pi) \), we obtain that \( |I| = \beta - \alpha \) and

\[
\int_I |(\lambda f)(x) - C_I| dx = \int_\alpha^\beta | \int_x^\beta \frac{f(t)}{t} dt | dx \\
\leq \|f\|_\infty \int_\alpha^\beta \log \frac{\beta}{x} dx \\
\leq |I|\|f\|_\infty.
\]

Furthermore,

\[
\frac{1}{|I|} \int_I |(\lambda f)(x) - (\lambda f)_I| dx \leq \frac{2}{|I|} \int_I |(\lambda f)(x) - C_I| dx \\
\leq 2\|f\|_\infty.
\]

That is to say, \( \lambda f \in BMO(0, \pi) \). Hence the proof is completed.

**Remarks.** 1. \( L^\infty(0, \pi) \) in Theorem cannot be replaced by \( BMO(0, \pi) \). Otherwise, it will follow that \( \log^2 |x| \) is a function in \( BMO(0, \pi) \), which results in a contradiction. Indeed, if \( f \in BMO(0, \pi) \) then the statement that \( Ca \) or \( Cb \) is a sequence of Fourier cosine or sine coefficients of some function in \( BMO(0, \pi) \) holds if and only if the operator \( \lambda \) is bounded from \( BMO(0, \pi) \) to \( BMO(0, \pi) \). Yet, if picking \( f(x) = \log(\pi/x) \) then we see that \( (\lambda f)(x) = \log^2(\pi/x) \) is outside \( BMO(0, \pi) \) due to the unboundedness of \( \log(\pi/x) \) on \((0, \pi) \) and Stegenga’s multiplier theorem applied to this \( (\lambda f) \), [5]. Of course, we have here used a fact that \( F \in BMO(0, \pi) \) once \( f \in BMO(0, \pi) \), which is easily derived. Actually, if we write \( H^1_R(\pi, \pi) \) and \( BMO(\pi, \pi) \) as the real Hardy space and \( BMO \) (bounded mean oscillation) space on \((\pi, \pi) \) respectively then Fefferman’s duality theorem tells us that \( [H^1_R(\pi, \pi)]^* = BMO(\pi, \pi) \) under the inner pair: \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx \), [6]. To show that \( F \in BMO(0, \pi) \), we only need to prove that the following function

\[
F_*(x) = \begin{cases} 
F(x), & x \in (0, \pi), \\
0, & x \in (-\pi, 0),
\end{cases}
\]

is in \( BMO(-\pi, \pi) \). For this, by Fubini’s theorem we find a constant \( C_3 \) such that for any \( G \in H^1_R(\pi, \pi) \),

\[
|\int_{-\pi}^{\pi} F_*(x)G(x)dx| = \frac{1}{\pi} |\int_0^{\pi} \int_{-\pi}^{\pi} f(x + t)G(x)dx]K(t)dt| \\
\leq C_3\|f\|_1\|G\|_1 \int_0^{\pi} |K(t)|dt.
\]

Equivalently, \( F_* \in BMO(-\pi, \pi) \) and hence \( F \in BMO(0, \pi) \).
2. From the above proof it is turns out that \( \sigma \) is bounded linear operator from 
\( L^\infty(0, \pi) \) (not from \( BMO(0, \pi) \)) to \( BMO(0, \pi) \). This operator looks very much like
the conjugate operator below:

\[
\tilde{f}(x) = \frac{1}{\pi} \int_0^\pi \frac{f(x - t) - f(x + t)}{2 \tan \frac{t}{2}} dt.
\]

Nevertheless, we should note that \( \tilde{f} \in BMO(0, \pi) \) if \( f \in BMO(0, \pi) \).

References

5. D.A. Stegenga, *Bounded Toeplitz operators on \( H^1 \) and applications on the duality between \( H^1 \) and the functions of bounded mean oscillation*, Amer. J. Math. 98 (1976), 573-589. MR 54:8340

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