PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 12, December 1997, Pages 3655–3658
S 0002-9939(97)04057-4

CHARACTERIZATION OF CHAOTIC ORDER AND ITS APPLICATION TO FURUTA INEQUALITY

MASATOSHI FUJII, JIAN FEI JIANG, AND EIZABURO KAMEI

(Communicated by Palle E. T. Jorgensen)

Dedicated to Professor P. R. Halmos on his 80th Birthday

Abstract. In this note, we give a simple characterization of the chaotic order $\log A \geq \log B$ among positive invertible operators A, B on a Hilbert space. As an application, we discuss Furuta's type operator inequality.

1. Introduction

A (bounded linear) operator A on a Hilbert space H is positive, in symbols $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. And $A > 0$ means that A is positive invertible. It is well-known that $A \geq B \geq 0$ does not assure $A^2 \geq B^2$ in general, but the Löwner-Heinz inequality says that the function $t \to t^\alpha$ on $[0, \infty)$ is operator monotone for $0 \leq \alpha \leq 1$, i.e.,

$$A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha, \tag{1}$$

cf. [8]. Furuta [5] gave it an ingenious extension which is called the Furuta inequality (cf. [2], [7] and [6] for an elementary and one-page proof):

If $A \geq B \geq 0$, then

$$(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q} \tag{2}$$

and

$$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q} \tag{3}$$

holds for $r \geq 0,\ p \geq 0$ and $q \geq 1$ with $(1 + 2r)q \geq p + 2r$ (see Figure 1).

Since $\log\ t$ is operator monotone, i.e., $\log\ A \geq \log\ B$ for $A \geq B > 0$, it induces a weaker order $\gg$ among positive invertible operators than the usual one $\geq$, which is called the chaotic order, cf. [3]. Now Ando's theorem [1] is rephrased as a characterization of the chaotic order via a form of (2): For $A, B > 0$, $A \gg B$ if and only if

$$(A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} \leq A^p \tag{4}$$

holds for all $p \geq 0$.

Afterwards, it is extended to the following result [3], cf. [4].

Received by the editors July 16, 1996.
1991 *Mathematics Subject Classification.* Primary 47A63, 47B15.
Key words and phrases. Positive operators, chaotic order, Löwner-Heinz inequality, Furuta inequality.

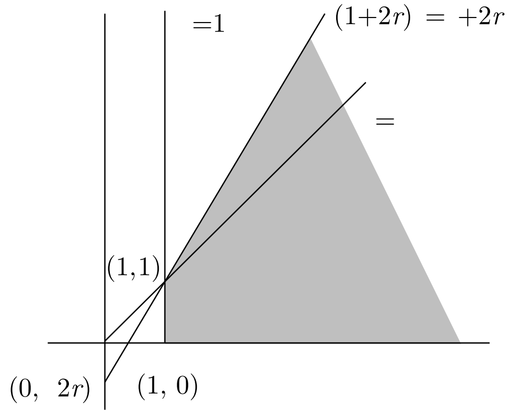

Figure 1

Theorem A. *For $A, B > 0$, $A \gg B$ if and only if*

$$(A^r B^p A^r)^{\frac{2r}{p+2r}} \leq A^{2r} \tag{5}$$

holds for all $p, r \geq 0$.

In this note, we give a simple characterization of the chaotic order. Precisely, $\log\ A > \log\ B$ if and only if there is an $\alpha > 0$ such that

$$A^\alpha > B^\alpha. \tag{6}$$

As an application, we can obtain Furuta's type operator inequality implying Theorem A.

2. Characterization of chaotic order

We begin by stating a simple lemma which is the heart of this note:

Lemma 1. *If A and B are selfadjoint and $A > B$, then there exists an $\alpha \in (0, 1]$ such that*

$$e^{\alpha A} > e^{\alpha B}. \tag{7}$$

Proof. The assumption $A > B$ means that $A - B \geq \varepsilon > 0$ for some ε. We here take $0 < \alpha < \varepsilon/(e^{\|A\|} + e^{\|B\|})$ and $\alpha \leq 1$. Then we have

$$\begin{aligned} e^{\alpha A} - e^{\alpha B} &= \alpha(A - B) + \sum_{n=2}^{\infty} \frac{\alpha^n}{n!}(A^n - B^n) \\ &\geq \alpha\varepsilon + \alpha^2 \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!}(A^n - B^n) \\ &\geq \alpha\varepsilon - \alpha^2 \Big\| \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!}(A^n - B^n) \Big\| \\ &\geq \alpha\varepsilon - \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!}(\|A\|^n + \|B\|^n) \\ &\geq \alpha(\varepsilon - \alpha(e^{\|A\|} + e^{\|B\|})) > 0. \end{aligned}$$

□

Lemma 1 implies the following basic inequality:

Corollary 2. *If* $A, B > 0$, *then* $\log\ A > \log\ B$ *if and only if there exists an* $\alpha \in (0,1]$ *such that* $A^\alpha > B^\alpha$.

Proof. If $\log\ A > \log\ B$, then $A^\alpha > B^\alpha$ for some $\alpha \in (0,1]$ by Lemma 1. Conversely, if $A^\alpha > B^\alpha$ for some $\alpha \in (0,1]$, then $A^\alpha \geq B^\alpha + \delta$ for some $\delta > 0$ and

$$\alpha \log A = \log A^\alpha \geq \log(B^\alpha + \delta) > \log B^\alpha = \alpha \log B.$$

□

By the above discussion, we have the following simple characterization of the chaotic order:

Theorem 3. *For* $A, B > 0$, $A \gg B$, *i.e.,* $\log\ A \geq \log\ B$, *if and only if for any* $\delta \in (0,1]$ *there exists an* $\alpha = \alpha_\delta > 0$ *such that*

$$(e^\delta A)^\alpha > B^\alpha. \tag{8}$$

Proof. Since $A \gg B$ is equivalent to $\log e^\delta A = \log A + \delta > \log B$ for any $\delta > 0$, Corollary 2 implies that $A \gg B$ is equivalent to saying that for any $\delta > 0$ there exists an $\alpha = \alpha_\delta \in (0,1]$ such that $(e^\delta A)^\alpha > B^\alpha$. □

We comment that some inequalities related to the chaotic order can be obtained from our result. Among others, we here discuss the Furuta inequality under the chaotic order. Combining Theorem 3 and the Furuta inequality, we have the following lemma:

Lemma 4. *If* $A, B > 0$ *and* $A \gg B$, *then for any* $\delta > 0$ *there exists an* $\alpha = \alpha_\delta \in (0,1]$ *such that*

$$(A^r B^p A^r)^{\frac{1}{q}} \leq e^{\frac{\delta p}{q}} A^{\frac{p+2r}{q}} \tag{9}$$

holds for $p \geq 0$, $r \geq 0$ *and* $q \geq 1$ *with* $(\alpha + 2r)q \geq p + 2r$.

Thus we have the following result equivalent to Theorem A:

Theorem 5. *If* $A, B > 0$ *and* $A \gg B$, *then*

$$(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}} \tag{10}$$

holds for $p \geq 0$, $r \geq 0$ *and* $q \geq 1$ *with* $2rq \geq p + 2r$.

The point of the proof is that if p, q and r satisfy the above condition, then $(\alpha + 2r)q \geq p + 2r$ for all $\alpha > 0$.

References

1. T.Ando, *On some operator inequalities*, Math. Ann., **279** (1987), 157-159. MR **89c:**47019
2. M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator theory, **23** (1990), 67-72. MR **91g:**47012
3. M.Fujii,T.Furuta and E.Kamei, *Operator functions associated with Furuta's inequality*, Linear Alg. and its Appl., **179** (1993), 161-169. MR **93j:**47026
4. M.Fujii and E.Kamei, *Furuta's inequality and a generalization of Ando's Theorem*, Proc. Amer. Math. Soc., **115** (1992), 409-413. MR **92i:**47018
5. T.Furuta, $A \geq B \geq 0$ *assures* $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ *for* $r \geq 0, p \geq 0, q \geq 1$ *with* $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc., **101** (1987), 85-88. MR **89b:**47028
6. T.Furuta, *Elementary proof of an order preserving inequality*, Proc. Japan Acad., **65** (1989), 126. MR **90g:**47029

7. E.Kamei, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883-886. MR **89m:**47011
8. G.K.Pedersen, *Some operator monotone functions*, Proc. Amer. Math. Soc., **36** (1972), 309–310. MR **46:**6078

Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582, Japan

E-mail address: mfujii@cc.osaka-kyoiku.ac.jp

Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582, Japan

permanent address: Department of Basic Science and Technology, China Textile University, Shanghai, China, Postal code 200051

Momodani Senior High School, Ikuno, Osaka 544, Japan

Current address: Maebashi Institute of Technology, Kamisadori, Maebashi, Gunma 371, Japan

E-mail address: kamei@maebashi-it.ac.jp