CHARACTERIZATION OF CHAOTIC ORDER
AND ITS APPLICATION TO FURUTA INEQUALITY

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Abstract. In this note, we give a simple characterization of the chaotic order
\( \log A \geq \log B \) among positive invertible operators \( A, B \) on a Hilbert space. As an application, we discuss Furuta’s type operator inequality.

1. Introduction

A (bounded linear) operator \( A \) on a Hilbert space \( H \) is positive, in symbols \( A \geq 0 \), if \( (Ax, x) \geq 0 \) for all \( x \in H \). And \( A > 0 \) means that \( A \) is positive invertible. It is well-known that \( A \geq B \geq 0 \) does not assure \( A^2 \geq B^2 \) in general, but the Löwner-Heinz inequality says that the function \( t \mapsto t^\alpha \) on \([0, \infty)\) is operator monotone for \( 0 \leq \alpha \leq 1 \), i.e.,

\[
A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha,
\]

(1)

cf. [8]. Furuta [5] gave it an ingenious extension which is called the Furuta inequality (cf. [2], [7] and [6] for an elementary and one-page proof):

If \( A \geq B \geq 0 \), then

\[
(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}
\]

and

\[
(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}
\]

holds for \( r \geq 0, \ p \geq 0 \) and \( q \geq 1 \) with \( (1 + 2r)q \geq p + 2r \) (see Figure 1).

Since \( \log t \) is operator monotone, i.e., \( \log A \geq \log B \) for \( A \geq B > 0 \), it induces a weaker order \( \gg \) among positive invertible operators than the usual one \( \geq \), which is called the chaotic order, cf. [3]. Now Ando’s theorem [1] is rephrased as a characterization of the chaotic order via a form of (2): For \( A, B > 0 \), \( A \gg B \) if and only if

\[
(A^r B^p A^r)^{1/2} \leq A^p
\]

holds for all \( p \geq 0 \).

Afterwards, it is extended to the following result [3], cf. [4].
Theorem A. For $A, B > 0$, $A \gg B$ if and only if
\[(A^r B^p A^r)^{\frac{2}{r+p}} \leq A^{2r}\]
holds for all $p, r \geq 0$.

In this note, we give a simple characterization of the chaotic order. Precisely, $\log A > \log B$ if and only if there is an $\alpha > 0$ such that
\[A^\alpha > B^\alpha.\]
As an application, we can obtain Furuta’s type operator inequality implying Theorem A.

2. Characterization of chaotic order

We begin by stating a simple lemma which is the heart of this note:

Lemma 1. If $A$ and $B$ are selfadjoint and $A > B$, then there exists an $\alpha \in (0, 1]$ such that
\[e^{\alpha A} > e^{\alpha B}.\]

Proof. The assumption $A > B$ means that $A - B \geq \varepsilon > 0$ for some $\varepsilon$. We here take $0 < \alpha < \varepsilon/(\|A\| + \varepsilon\|B\|)$ and $\alpha \leq 1$. Then we have
\[e^{\alpha A} - e^{\alpha B} = \alpha(A - B) + \sum_{n=2}^{\infty} \frac{\alpha^n}{n!} (A^n - B^n)\]
\[\geq \alpha \varepsilon + \alpha^2 \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!} (A^n - B^n)\]
\[\geq \alpha \varepsilon - \alpha^2 \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!} (A^n - B^n)\|
\[\geq \alpha \varepsilon - \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!} (\|A\|^n + \|B\|^n)\]
\[\geq \alpha(\varepsilon - \alpha (e^{\|A\|} + e^{\|B\|})) > 0.\]
Lemma 1 implies the following basic inequality:

**Corollary 2.** If \( A, B > 0 \), then \( \log A > \log B \) if and only if there exists an \( \alpha \in (0, 1) \) such that \( A^{\alpha} > B^{\alpha} \).

**Proof.** If \( \log A > \log B \), then \( \log A^{\alpha} > \log B^{\alpha} \) for some \( \alpha \in (0, 1) \) by Lemma 1. Conversely, if \( A^{\alpha} > B^{\alpha} \) for some \( \alpha \in (0, 1) \), then \( A^{\alpha} \geq B^{\alpha} + \delta \) for some \( \delta > 0 \) and

\[
\alpha \log A = \log A^{\alpha} \geq \log(B^{\alpha} + \delta) > \log B^{\alpha} = \alpha \log B.
\]

By the above discussion, we have the following simple characterization of the chaotic order:

**Theorem 3.** For \( A, B > 0 \), \( A \gg B \), i.e., \( \log A \geq \log B \), if and only if for any \( \delta \in (0, 1] \) there exists an \( \alpha = \alpha_\delta > 0 \) such that

\[
(8) \quad (e^\delta A)^{\alpha} > B^{\alpha}.
\]

**Proof.** Since \( A \gg B \) is equivalent to \( \log e^\delta A = \log A + \delta > \log B \) for any \( \delta > 0 \), Corollary 2 implies that \( A \gg B \) is equivalent to saying that for any \( \delta > 0 \) there exists an \( \alpha = \alpha_\delta \in (0, 1] \) such that \( (e^\delta A)^{\alpha} > B^{\alpha} \).

We comment that some inequalities related to the chaotic order can be obtained from our result. Among others, we here discuss the Furuta inequality under the chaotic order. Combining Theorem 3 and the Furuta inequality, we have the following lemma:

**Lemma 4.** If \( A, B > 0 \) and \( A \gg B \), then for any \( \delta > 0 \) there exists an \( \alpha = \alpha_\delta \in (0, 1] \) such that

\[
(9) \quad (A^p B^q A^r)^{\frac{\delta}{q}} \leq e^{\frac{\delta r}{q}} A^{\frac{p+2r}{q}}
\]

holds for \( p \geq 0 \), \( r \geq 0 \) and \( q \geq 1 \) with \( (\alpha + 2r)q \geq p + 2r \).

Thus we have the following result equivalent to Theorem A:

**Theorem 5.** If \( A, B > 0 \) and \( A \gg B \), then

\[
(10) \quad (A^p B^q A^r)^{\frac{\delta}{q}} \leq A^{\frac{p+2r}{q}}
\]

holds for \( p \geq 0 \), \( r \geq 0 \) and \( q \geq 1 \) with \( 2rq \geq p + 2r \).

The point of the proof is that if \( p, q, r \) satisfy the above condition, then \( (\alpha + 2r)q \geq p + 2r \) for all \( \alpha > 0 \).

**References**

5. T. Furuta, *A \geq B \geq 0 assures \((B^p A^p B^p)^{1/q} \geq B^{(p+2r)/q}\) for \( r \geq 0, p \geq 0, q \geq 1 \) with \((1 + 2r)q \geq p + 2r\)*, Proc. Amer. Math. Soc., 101 (1987), 85-88. MR 89b:47028

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