ON A THEOREM OF Ossa

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(Communicated by Thomas Goodwillie)

Abstract. If $V$ is an elementary abelian 2-group, Ossa proved that the connective $K$-theory of $BV$ splits into copies of $\mathbb{Z}/2$ and of the connective $K$-theory of the infinite real projective space. We give a brief proof of Ossa’s theorem.

Introduction

We have been asked whether our work, [1] and [2] on the Brown-Peterson homology of $BV$, $V$ an elementary $p$-group, gives a nice structure of the connective $K$-theory of $BV$. The answer is that the approach of [1] leads to the elegant structure theorem of Ossa [4]. Although the approach is motivated by our [1] and [2], the proof is independent of that work. In this reproof of an established theorem we shall limit our exposition to the $p = 2$ case. For us, the notation makes this the easiest case, but for Ossa, it was the more difficult one. With obvious modifications, the odd-primary version of our argument follows the same outline. We thank Don Davis for the Liulevicius reference.

Notation. Let $bu$ be the connective $K$-theory spectrum and let $P$ denote $B\mathbb{Z}/2$ (also known as infinite real projective space). Let $H\mathbb{Z}/2$ be the $\mathbb{Z}/2$ Eilenberg-MacLane spectrum.

Theorem 1 (Ossa). With the above notation, there is a homotopy equivalence of spectra

$$bu \wedge P \wedge P \simeq \bigvee_{0 < i, j} \Sigma^{2i+2j-2} H\mathbb{Z}/2 \lor [\Sigma^2 bu \wedge P].$$

Eric Ossa has kindly pointed out that our proof gives this as a homotopy equivalence of $BP$-module spectra.

Note that $H\mathbb{Z}/2 \wedge P \simeq \bigvee_{0 < i} \Sigma^i H\mathbb{Z}/2$. (The proof of this is like that of Lemma 3.) Thus the theorem can be used inductively to split $bu \wedge P \cdot \cdot \cdot \wedge P$ into suspended copies of $H\mathbb{Z}/2$ and one suspended copy of $bu \wedge P$. Since $bu \wedge BV = bu \wedge (P \times \cdot \cdot \cdot \times P)$ is a wedge sum of $bu \wedge P \wedge \cdot \cdot \cdot \wedge P$’s, we get the following corollary.

Corollary 2. Let $V$ be an elementary abelian $p$-group. Then $bu_*(BV)$ is isomorphic to a sum of suspended copies of $\mathbb{Z}/2$ and of $bu_*(P)$. \qed

Received by the editors January 11, 1996 and, in revised form, July 19, 1996.

1991 Mathematics Subject Classification. Primary 55P10, 55N20; Secondary 55N15, 55S10.

Key words and phrases. $K$-theory, real projective space, elementary abelian group.

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Lemma 3. There is a homotopy equivalence $bu \wedge CP^\infty \simeq \bigvee_{0 < n} bu \wedge S^{2n}$. In particular, there is a projection $\rho : bu \wedge CP^\infty \to bu \wedge S^2$.

Proof. Choose $f_n : S^{2n} \to bu \wedge CP^\infty$ representing the $bu_*$ generators of $bu_*(CP^\infty)$. Define $f : \bigvee_{0 < n} S^{2n} \to bu \wedge CP^\infty$ by $f|S^{2n} = f_n$. We have the composition

$$F : \bigvee_{0 < n} bu \wedge S^{2n} \xrightarrow{bu \wedge f} bu \wedge bu \wedge CP^\infty \xrightarrow{\mu \wedge CP^\infty} bu \wedge CP^\infty$$

where $\mu$ is the pairing of the $bu$ spectrum. $F$ induces an isomorphism in homotopy and thus is an equivalence.

The Proof of Theorem 1

Let $\pi : P \to CP^\infty$ represent the nonzero second dimensional integral homology class of $P$. Define $g_1$ to be the composition

$$g_1 : bu \wedge P \wedge P \xrightarrow{bu \wedge \pi \wedge P} bu \wedge CP^\infty \wedge P \xrightarrow{\rho \wedge \rho} bu \wedge S^2 \wedge P.$$

Let $H^*(P \wedge P; \mathbb{Z}/2) \cong \mathbb{Z}/2[s,t](st)$ be the mod 2 cohomology of $P \wedge P$. For $b = s^{2i-1} \wedge t^{2j-1} \in H^{2i+2j-2}(P \wedge P; \mathbb{Z}/2)$, let $g_b : P \wedge P \to \Sigma^{dim(b)} H\mathbb{Z}/2$ represent $b$. Now construct the map $g_0$ by the following composition:

$$g_0 : bu \wedge P \wedge P \xrightarrow{bu \wedge \pi \wedge \pi} bu \wedge \bigvee_{0 < i,j} [\Sigma^{2i+2j-2} H\mathbb{Z}/2] \xrightarrow{\vee \cdot t^i} \bigvee_{0 < i,j} [\Sigma^{2i+2j-2} H\mathbb{Z}/2]$$

where $\nu : bu \wedge H\mathbb{Z}/2 \to H\mathbb{Z}/2$ is the standard pairing making mod 2 homology a module theory over connective $K$-theory. The map

$$g = g_0 \vee g_1 : bu \wedge P \wedge P \to \bigvee_{0 < i,j} [\Sigma^{2i+2j-2} H\mathbb{Z}/2] \vee [\Sigma^2 bu \wedge P]$$

is our candidate for the equivalence.

Let $A$ be the mod 2 Steenrod algebra and $E = E[Q_0, Q_1]$ ($Q_0 = S^1$ and $Q_1 = S^1 \wedge S^1 \wedge S^1$). Then $H^*(bu; \mathbb{Z}/2) \cong A/A(Q_0, Q_1) \cong A \otimes_{E} \mathbb{Z}/2$. In $H^*(P \wedge P; \mathbb{Z}/2)$, the classes $\{s^i \wedge t^j : i > 0\}$ give a basis for an $E$-module $D^*$ isomorphic to $H^*(S^2 \wedge P; \mathbb{Z}/2)$. Let $M \cong H^*(P \wedge P; \mathbb{Z}/2)/D^*$. It is isomorphic to a free $E$-module with basis $\{s^{2i-1} \wedge t^{2j-1} : i, j > 0\}$. Clearly in dimension 2,

$$(bu \wedge \pi)^* \circ \rho^* : H^2(bu \wedge S^2; \mathbb{Z}/2) \to H^2(bu \wedge P; \mathbb{Z}/2)$$

is an isomorphism. Thus $g_1^*$ takes $H^*(bu \wedge S^2 \wedge P; \mathbb{Z}/2)$ isomorphically onto $A/A(Q_0, Q_1) \otimes D^*$. By the construction of the composition $g_0$, we see that $g_0^*$ takes $H^*(\bigvee_{0 < i,j} \Sigma^{2i+2j-2} H\mathbb{Z}/2, \mathbb{Z}/2)$ onto the $A$-module generated by $\{1 \wedge s^{2i-1} \wedge t^{2j-1} : i, j > 0\}$.

The composition of the projection of

$$H^*(bu \wedge P \wedge P) \to H^*(bu \wedge P \wedge P)/(A/A(Q_0, Q_1) \otimes D^*) \cong \left(A \otimes_{E} \mathbb{Z}/2\right) \otimes M$$

with $g_0^*$ gives an isomorphism. Although this is obvious it does require a proof. A generalization from the literature is Proposition 1.7 of Arunas Liulevicius [3]. Let his $N$ be $\mathbb{Z}/2$, his $A$ our $A$, his $B$ our $E$ and his $M$ our $M$. He shows:

$$M \otimes \left(A \otimes_{E} \mathbb{Z}/2\right) \cong A \otimes_{E} M.$$
REFERENCES


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