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EQUIVARIANT ACYCLIC MAPS

AMIYA MUKHERJEE AND ANIRUDDHA C. NAOLEKAR

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Abstract. In this paper we apply a recently developed new version of the Bredon-Illman cohomology theory to obtain an equivariant analogue of a result of Kan and Thurston, which implies that a connected CW-complex has the homotopy type of a space obtained by applying the plus construction of Quillen to certain Eilenberg-Mac Lane spaces.

1. Statement of results

A space X is acyclic if its reduced integral homology $\widetilde{H}_*(X) = 0$. The universal coefficient theorem then implies that X is acyclic if and only if the reduced cohomology $\widetilde{H}^*(X; G) = 0$ for every coefficient group G. Also a map $f : X \longrightarrow Y$ is acyclic if its homotopy fibre is acyclic. We say that a G-space X is G-acyclic if its reduced Bredon-Illman cohomology $\widetilde{H}^*_G(X; \lambda) = 0$ for every abelian O_G-group λ, and a G-map $f : X \longrightarrow Y$ is G-acyclic if its G-homotopy fibre is G-acyclic.

Here O_G denotes the category of orbit spaces G/H and G-maps, and an O_G-group is a contravariant functor $O_G \longrightarrow \mathbf{Grp}$. Other notions like O_G-space, O_G-fibration, etc. have similar meaning (terminology depending on the nature of codomain of the functors). The homotopy O_G-group $\underline{\pi}_n(X)$ of a G-space X with a stationary point $x^0 \in X^G$ as base point is defined by $\underline{\pi}_n(X)(G/H) = \pi_n(X^H, x^0)$ and $\underline{\pi}_n(X)(\widehat{g}) = \pi_n(g)$, where $\widehat{g} : G/H \longrightarrow G/K$ is a morphism in O_G arising from a subconjugacy relation $g^{-1}Hg \subseteq K$, and $g : X^K \longrightarrow X^H$ is the left translation by g. A G-map $f : X \longrightarrow Y$ induces a morphism of O_G-groups $\underline{\pi}_n(f) : \underline{\pi}_n(X) \longrightarrow \underline{\pi}_n(Y)$ defined by $\underline{\pi}_n(f)(G/H) = \pi_n(f^H)$, where $f^H = f|X^H$.

Given an O_G-group λ (where G is a compact Lie group) and an integer $n \geq 1$, there is a G-space X such that $\underline{\pi}_n(X) = \lambda$ and $\underline{\pi}_j(X) = \underline{0}$ if $j \neq n$. This G-space is the classifying space for the Bredon-Illman cohomology, and is called an equivariant Eilenberg-Mac Lane space $K(\lambda, n)$ of type (λ, n) (see [5]).

For a G-space X, there is a concept of an equivariant local coefficients system M on X, and also of an equivariant cohomology $H^*_G(X; M)$ (see [8]). This cohomology reduces to the equivariant singular cohomology of Bredon and Illman [2], [7] when M is simple in a certain sense, and to the Steenrod cohomology with the classical local coefficients system when G is trivial. In Section 2 we present an alternative description of $H^*_G(X; M)$ in a way which is best suited in the context of G-acyclicity.

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Now suppose that G is finite, and consider G-spaces X which are compactly generated weakly Hausdorff with base point $x^0 \in X^G$ such that X has the G-homotopy type of a G-connected G-CW-complex. Then, in line of Kan and Thurston [6], our first main theorem is

Theorem 1.1. *For a G-space X, there exist an O_G-group λ with a perfect normal O_G-subgroup η and a G-acyclic map*

$$f : K(\lambda, 1) \longrightarrow X,$$

which is natural with respect to X, such that $Ker\, \underline{\pi}_1(f) = \eta$, and

$$f^* : H^*_G(X; M) \longrightarrow H^*_G(K(\lambda, 1); f^*M)$$

is an isomorphism for every equivariant local coefficients system M on X.

Given a G-space X and a perfect normal O_G-subgroup η of $\underline{\pi}_1(X)$, it is possible to construct a G-space X^+_η by applying the plus construction of Quillen [9] to each X^H with respect to the group $\eta(G/H)$, and then combining the resulting spaces together by means of a functorial bar construction. It turns out that the G-space X^+_η is completely determined by the pair $(\underline{\pi}_1(X), \eta)$ up to G-homotopy equivalence. More specifically, we have the following two theorems which provide a classification of G-acyclic maps from a given G-space.

Theorem 1.2. *If X is a G-space and η a perfect normal O_G-subgroup of $\underline{\pi}_1(X)$, then there exist a G-space X^+_η and a G-acyclic map $f : X \longrightarrow X^+_\eta$ such that $Ker\, \underline{\pi}_1(f) = \eta$.*

Theorem 1.3. *If $f : X \longrightarrow Y$ and $f' : X \longrightarrow Y'$ are G-maps, where f is G-acyclic, then there is a G-map $h : Y \longrightarrow Y'$ with $hf \simeq_G f'$ if and only if $Ker\, \underline{\pi}_1(f) \subseteq Ker\, \underline{\pi}_1(f')$; moreover, any two such h are G-homotopic. In addition, if f' is G-acyclic, then h is also G-acyclic, and h is a G-homotopy equivalence if and only if $Ker\, \underline{\pi}_1(f) = Ker\, \underline{\pi}_1(f')$.*

Finally, we obtain as an application our second main theorem which is

Theorem 1.4. *Given a G-space X, there exists an O_G-group λ with a perfect normal O_G-subgroup η such that X has the G-homotopy type of $K(\lambda, 1)^+_\eta$.*

We note that the condition of G-connectivity of X is a necessary condition for each of the main theorems to be true, and therefore cannot be avoided.

The proofs of the theorems appear in Section 3.

2. Criteria for G-acyclicity

The proofs of our theorems are based on the following two propositions. The first implies that a G-map $f : X \longrightarrow Y$ is G-acyclic if and only if each $f^H : X^H \longrightarrow Y^H$ is acyclic, and then the second gives the cohomological assertion of Theorem 1.1.

Proposition 2.1. *A G-space X is G-acyclic if and only if each X^H is acyclic.*

Proposition 2.2. *If a G-map $f : X \longrightarrow Y$ is G-acyclic , then f induces an isomorphism*

$$f^* : H^*_G(Y; M) \longrightarrow H^*_G(X; f^*M)$$

for every equivariant local coefficients system M on Y.

Proof of Proposition 2.1. There is a spectral sequence

$$E_2^{p,q} = Ext^p(\underline{\widetilde{H}}_q(X), \lambda) \Longrightarrow \widetilde{H}_G^{p+q}(X; \lambda),$$

obtained by means of an injective resolution of the O_G-group λ, where $\underline{\widetilde{H}}_q(X)$ is the O_G-group whose value at G/H is the reduced integral homology $\widetilde{H}_q(X^H)$ (cf. [2, I, §10]). Since the category of abelian O_G-groups has sufficiently many injectives, we can embed the O_G-group $\underline{\widetilde{H}}_q(X)$ in an injective O_G-group λ_q. Then, we have in the corresponding spectral sequence $E_2^{p,q} = 0$ for $p > 0$. Therefore, if X is G-acyclic, then

$$0 = \widetilde{H}_G^q(X, \lambda_q) \cong Ext^0(\underline{\widetilde{H}}_q(X), \lambda_q) = Hom(\underline{\widetilde{H}}_q(X), \lambda_q).$$

This implies that $\underline{\widetilde{H}}_q(X) = 0$ as we have already a monomorphism $\underline{\widetilde{H}}_q(X) \longrightarrow \lambda_q$. Since this happens for every q, each X^H is acyclic.

The converse follows easily again from the same spectral sequence. This completes the proof.

Turning now to Proposition 2.2, let us recall briefly from [8, §8] an alternative description of the equivariant cohomology $H_G^*(X; M)$.

First note that an equivariant local coefficients system M on X is a contravariant functor $M : \Pi X \longrightarrow \mathbf{Ab}$, where ΠX is the following category. An object of ΠX is a G-map $x_H : G/H \longrightarrow X$, and a morphism $[\widehat{g}, \phi] : x_H \longrightarrow y_K$ is a certain equivalence class of pairs $(\widehat{g}, \phi)$, where $\widehat{g} : G/H \longrightarrow G/K$, $g^{-1}Hg \subseteq K$, is a G-map, and $\phi : G/H \times I \longrightarrow X$ is a G-homotopy from x_H to $y_K \circ \widehat{g}$.

Given M, we define an O_G-group $M_0 : O_G \longrightarrow \mathbf{Ab}$ by sending G/H to $M(x_H^0)$, and sending a G-map $\widehat{g} : G/H \longrightarrow G/K$ to $M([\widehat{g}, k])$, where x_H^0 is an object in ΠX given by the constant G-map $G/H \longrightarrow x^0 \in X$, and $[\widehat{g}, k] : x_H^0 \longrightarrow x_K^0$ is a morphism in ΠX given by the constant homotopy k on x^0. Note that the bijection $b : X^H \longrightarrow Map_G(G/H, X)$, $b(x)(gH) = gx$, is implicit in the definition. In fact, this makes M_0 a $\underline{\pi}_1(X)$-module with action $\rho : \underline{\pi}_1(X) \times M_0 \longrightarrow M_0$ given by $\rho(G/H)(\alpha, m) = M(b(\alpha))(m)$, where $\alpha \in \pi_1(X^H, x^0)$ and $b(\alpha) : x_H^0 \longrightarrow x_H^0$ is an equivalence in ΠX.

Next, consider the family of universal covering spaces $p_H : \widetilde{X}^H \longrightarrow X^H$, $H \subseteq G$. Then, for a G-map $\widehat{g} : G/H \longrightarrow G/K$, the left translation $g : X^K \longrightarrow X^H$ lifts to a map $\widetilde{g} : \widetilde{X}^K, \longrightarrow \widetilde{X}^H$ which is unique up to the choice of base points over x^0 in $\widetilde{X}^K$ and $\widetilde{X}^H$.

Finally, let $\mathbb{Z}\underline{\pi}_1(X)$ denote the O_G-group, where $\mathbb{Z}\underline{\pi}_1(X)(G/H)$ is the integral group ring $\mathbb{Z}\pi_1(X^H, x^0)$.

Then the cohomology $H_G^*(X; M)$ for a finite group G may be obtained by means of a cochain complex $S_{\pi,G}^*(\mathcal{U}; M_0)$, where $\mathcal{U}$ is what we call the universal O_G-covering space of X. The nth group $S_{\pi,G}^n(\mathcal{U}; M_0)$ of this cochain complex is a subgroup of

$$\bigoplus_{H \subseteq G} Hom_{\mathbb{Z}\underline{\pi}_1(X)(G/H)}(C_n(\widetilde{X}^H), M_0(G/H))$$

consisting of elements $c = \{c_H\}_{H \subseteq G}$ which satisfy the condition : if two equivariant singular n-simplexes $\sigma : \triangle_n \longrightarrow \widetilde{X}^H$ and $\tau : \triangle_n \longrightarrow \widetilde{X}^K$ are connected by a G-map $\widehat{g} : G/H \longrightarrow G/K$ such that $\sigma = \widetilde{g} \circ \tau$, then $M_0(\widehat{g})(c_K(\tau)) = c_H(\sigma)$. Note that the condition is a simplified version of a general case where G is a compact Lie group (see [8, (8.3)]).

The following definitions and notations are preparatory to our next lemma which provides yet another description of $H^*_G(X;M)$.

Let L be a right $\underline{\pi}_1(X)$-module which acts on M_0 with actions $\theta : L \times \underline{\pi}_1(X) \longrightarrow L$ and $\omega : L \times M_0 \longrightarrow M_0$ such that $\omega \circ (\theta \times id) = \omega \circ (id \times \rho)$.

Here are two examples of L which will be important in the proof of Proposition 2.2.

Example 2.3. Take $L = \underline{\pi}_1(X)$, $\omega = \rho$ as defined above, and $\theta =$ multiplication.

Example 2.4. Let $f : X \longrightarrow Y$ be a G-map and M an equivariant local coefficients system on Y. Then $(f^*M)_0 = M_0$. Take $L = \underline{\pi}_1(Y)$, and $\theta : \underline{\pi}_1(Y) \times \underline{\pi}_1(X) \longrightarrow \underline{\pi}_1(Y)$ as $\theta(G/H)(\beta, \alpha) = \beta \cdot f^H_*(\alpha)$. Let $\omega : \underline{\pi}_1(Y) \times M_0 \longrightarrow M_0$ be as in Example 2.3, and $\rho : \underline{\pi}_1(X) \times M_0 \longrightarrow M_0$ be given by $\rho(G/H)(\alpha, m) = \omega(G/H)(f^H_*(\alpha), m)$.

We shall denote the L of this example by $f^*\underline{\pi}_1(Y)$.

Consider the O_G-group $\underline{C}_n(X;L) : O_G \longrightarrow \mathbf{Ab}$, where

$$\underline{C}_n(X;L)(G/H) = L(G/H) \otimes_{\mathbb{Z}\underline{\pi}_1(X)(G/H)} C_n(\widetilde{X}^H),$$

and, for a G-map $\widehat{g} : G/H \longrightarrow G/K$, $\underline{C}_n(X;L)(\widehat{g}) = L(\widehat{g}) \otimes C_n(\widetilde{g})$. Clearly, these give rise to a chain complex $\underline{C}_*(X;L)$ in the abelian category of abelian O_G-groups. Then, $Hom_L(\underline{C}_*(X;L), M_0)$ becomes a cochain complex of groups whose nth group consists of L-invariant natural transformations $\underline{C}_n(X;L) \longrightarrow M_0$.

Lemma 2.5. *There is an isomorphism*

$$\Psi : S^*_{\pi,G}(\mathcal{U}; M_0) \longrightarrow Hom_L(\underline{C}_*(X;L), M_0)$$

of cochain complexes.

Proof. Define Ψ and its inverse Ψ' in the following way. Let $c = \{c_H\}_{H \subseteq G} \in S^n_{\pi,G}(\mathcal{U}; M_0)$, $T \in Hom_L(\underline{C}_n(X;L), M_0)$, $l \in L(G/H)$, and $\sigma : \triangle_n \longrightarrow \widetilde{X}^H$ be a singular n-simplex. Then, set

$$\Psi(c)(G/H)(l \otimes \sigma) = \omega(G/H)(l, c_H(\sigma)), \text{ and } (\Psi'(T))_H(\sigma) = T(G/H)(1 \otimes \sigma).$$

It does not pose any difficulty to verify that Ψ and Ψ' are cochain maps inverse to one other (cf. [8, §9]). □

The point to note here is that G has to be finite for Ψ' to be well defined.

Proof of Proposition 2.2. The category of abelian L-invariant O_G-groups possesses sufficiently many injectives. Let M^*_0 be an injective resolution of M_0 in this category. Then, in view of Lemma 2.5, the bicomplex $Hom_L(\underline{C}_*(X;L), M^*_0)$ provides a spectral sequence $E(X, L, M)$ in which

$$E^{p,q}_2 = Ext^p(\underline{H}_q(X, L), M_0) \Longrightarrow H^{p+q}_G(X;M),$$

where $\underline{H}_q(X;L) : O_G \longrightarrow \mathbf{Ab}$ is given by $\underline{H}_q(X;L)(G/H) = H_q(X^H; L(G/H))$ which is the ordinary cohomology of X^H with local coefficients $L(G/H)$.

Now if $f : X \longrightarrow Y$ is a G-map and M is an equivariant local coefficients system on Y, then f induces a map of the spectral sequences $f^* : E(Y, \underline{\pi}_1(Y), M) \longrightarrow E(X, f^*\underline{\pi}_1(Y), f^*M)$, where $\underline{\pi}_1(Y)$ is as in Example 2.3, and $f^*\underline{\pi}_1(Y)$ is as in Example 2.4. If f is G-acyclic, then f^* is an isomorphism at the E_2-level, by Proposition 2.1 and Proposition (4.3) of [1]. Consequently, $f^* : H^*_G(Y;M) \longrightarrow H^*_G(X; f^*M)$ is an isomorphism. This completes the proof.

3. Proof of the theorems

Proof of Theorem 1.1. It is possible to convert a G-space X into an O_G-space by means of a functor $\mathcal{R}$ defined by $\mathcal{R}(X)(G/H) = X^H$, $\mathcal{R}(X)(\widehat{g}) = g$ (left translation). Conversely, Elmendorf [5] defined a functor $\mathcal{S} : O_G$-spaces $\longrightarrow G$-spaces, and a natural transformation $N : \mathcal{RS} \longrightarrow id$ such that, for each O_G-space T and each $H \subseteq G$, $N(T)(G/H) : (\mathcal{S}T)^H \longrightarrow T(G/H)$ is a homotopy equivalence. In particular, $N(\mathcal{R}(X))(G/\{e\}) : \mathcal{SR}X \longrightarrow X$ is a natural G-homotopy equivalence.

Now, if X is a G-space, then using the Kan-Thurston theorem [6] for each X^H, we get a group $\lambda(G/H)$ with a perfect normal subgroup $\eta(G/H)$, and a fibration $p(G/H) : K(\lambda(G/H), 1) \longrightarrow X^H$ satisfying the conditions that $p(G/H)$ is acyclic, and $Ker\,\pi_1(p(G/H)) = \eta(G/H)$ (note that here we are using O_G as an indexing set). By naturality, these fibrations produce an O_G-fibration $p : E \longrightarrow B$, where $E = \mathcal{R}K(\lambda, 1)$ and $B = \mathcal{R}X$. Applying the Elmendorf's functor $\mathcal{S}$ to it, we get a G-map $\mathcal{S}p : \mathcal{S}E \longrightarrow \mathcal{S}B$ so that $(\mathcal{S}E)^H$ and $(\mathcal{S}B)^H$ have the homotopy types of $K(\lambda(G/H), 1)$ and X^H respectively. This gives Theorem 1.1 immediately. □

Proof of Theorem 1.2. First note that the plus construction $W \longrightarrow W_P^+$, where W is a CW-space and P is a perfect normal subgroup of $\pi_1(W)$, is not functorial, but functorial up to homotopy. However, it is possible to choose W_P^+ from its homotopy type so that $W \longrightarrow W_P^+$ becomes functorial. This may be done in the following way. Let $\alpha : \widetilde{W}_P \longrightarrow W$ be the covering space of W corresponding to the subgroup P so that $Im\,\pi_1(\alpha) = P$, and let $\beta : A(\widetilde{W}_P) \longrightarrow \widetilde{W}_P$ be the natural fibration obtained by applying the acyclic functor A of Dror [4]. Then the cofibre $i : W \longrightarrow C_\alpha$ of $\alpha \circ \beta : A(\widetilde{W}_P) \longrightarrow W$, where C_a is the mapping cone of α, is homotopically equivalent to $W \longrightarrow W_P^+$ (over W). These cofibres provide a functor which may be called the functorial plus construction.

Now if X is a G-space and η is a perfect normal O_G-subgroup of $\underline{\pi}_1(X)$, then applying the functorial plus construction to each X^H we get an acyclic map $f(G/H) : X^H \longrightarrow (X^H)^+_{\eta(G/H)}$ such that $Ker\,\pi_1(f(G/H)) = \eta(G/H)$. These maps give a morphism of O_G-spaces which turns into a G-map $f' : \mathcal{SR}X \longrightarrow X^+_\eta$ by means of the Elmendorf's functor $\mathcal{S}$. Then a composition of a G-homotopy equivalence $X \longrightarrow \mathcal{SR}X$ with f' gives the required G-acyclic map $f : X \longrightarrow X^+_\eta$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. If h exists, then $\underline{\pi}_1(f') = \underline{\pi}_1(h) \circ \underline{\pi}_1(f)$, and therefore $Ker\,\underline{\pi}_1(f) \subseteq Ker\,\underline{\pi}_1(f')$. Conversely, consider the G-push out diagram, and its restriction to each H-fixed point set

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"] & Y \arrow[d, "g'"] \\ Y' \arrow[r, "g"] & Y \cup_X Y' \end{tikzcd} \qquad \begin{tikzcd} X^H \arrow[r, "f^H"] \arrow[d, "f'^H"] & Y^H \arrow[d, "g'^H"] \\ Y'^H \arrow[r, "g^H"] & Y^H \cup_{X^H} Y'^H \end{tikzcd}$$

The second diagram implies that g^H is acyclic, since f^H is so, and, by the van Kampen theorem, $\pi_1(g^H)$ is an isomorphism, since $Ker\,\pi_1(f^H) \subseteq Ker\,\pi_1(f'^H)$. Therefore g^H is a homotopy equivalence, and hence g is a G-homotopy equivalence, by the equivariant Whitehead theorem [3, p. 107]. Then, if g_1 is a G-homotopy inverse of g, $h = g_1 \circ g' : Y \longrightarrow Y'$ is the required G-map with $h \circ f \simeq_G f'$.

Clearly h is G-acyclic if f' is so, and, since $\underline{\pi}_1(h)$ is an isomorphism if and only if $Ker\,\underline{\pi}_1(f) = Ker\,\underline{\pi}_1(f')$, the last assertion follows.

To see that h is unique up to G-homotopy equivalence, suppose that $j : F \longrightarrow X$ is the G-homotopy fibre of $f : X \longrightarrow Y$. Then, since $f \circ j \simeq_G y^0$, f extends to a G-map $k : X \cup_j CF \longrightarrow Y$ over the equivariant mapping cone of j. The G-map k is actually a G-homotopy equivalence, because its restriction to each H-fixed point set $k^H : X^H \cup CF^H \longrightarrow Y^H$ is acyclic and $\pi_1(k^H)$ is an isomorphism. Thus we have an equivariant coexact sequence

$$F \longrightarrow X \longrightarrow Y \longrightarrow \Sigma F,$$

where ΣF is the equivariant suspension of F. Since ΣF^H is simply connected and $\widetilde{H}_*(\Sigma F^H;\mathbb{Z}) = 0$, ΣF^H is contractible. This implies that ΣF is G-contractible by the equivariant Whitehead theorem. Thus the map $f^* : [Y, Y']^0_G \longrightarrow [X, Y']^0_G$ in the equivariant Barratt-Puppe sequence [3, p. 142] is injective, where $[Y, Y']^0_G$ denotes the set of base point preserving G-homotopy classes of G-maps $Y \longrightarrow Y'$. This ensures the uniqueness of h, and the proof of Theorem 1.3 is complete.

The assertion of Theorem 1.4 is now straightforward.

In conclusion, we remark that the proofs appearing in this section remain valid if G is a compact Lie group and X is a G-CW-space with each X^H a connected CW-space.

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Stat-Math Division, Indian Statistical Institute, 203 B. T. Road, Calcutta 700 035, India

E-mail address: amiya@isical.ernet.in

School of Mathematics, SPIC Science Foundation, 92, G. N. Chetty Road, Madras 600 017, India

E-mail address: anirudha@ssf.ernet.in