

COUNTING ELLIPTIC PLANE CURVES WITH FIXED j -INVARIANT

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ABSTRACT. The number of degree d elliptic plane curves with fixed j -invariant passing through $3d - 1$ general points in \mathbf{P}^2 is computed.

0. SUMMARY

Let N_d be the number of irreducible, reduced, nodal, degree d *rational plane curves* passing through $3d - 1$ general points in the complex projective plane \mathbf{P}^2 . The numbers N_d satisfy a beautiful recursion relation ([K-M], [R-T]):

$$N_1 = 1,$$

$$\forall d > 1, \quad N_d = \sum_{i+j=d, i,j>0} N_i N_j \left(i^2 j^2 \binom{3d-4}{3i-2} - i^3 j \binom{3d-4}{3i-1} \right).$$

Let $E_{d,j}$ be the number of irreducible, reduced, nodal, degree d *elliptic plane curves with fixed j -invariant j* passing through $3d - 1$ general points \mathbf{P}^2 . $E_{d,j}$ is defined for $d \geq 3$ and $\infty \neq j \in \overline{M}_{1,1}$. In this note, the following relations are established:

$$\forall j \neq 0, 1728, \infty, \quad E_{d,j} = \binom{d-1}{2} N_d,$$

$$j = 0, \quad E_{d,0} = \frac{1}{3} \binom{d-1}{2} N_d,$$

$$j = 1728, \quad E_{d,1728} = \frac{1}{2} \binom{d-1}{2} N_d.$$

If $d \equiv 0 \pmod{3}$, then $3 \nmid \binom{d-1}{2}$. Since $E_{3l,0}$ is an integer, $N_{3l} \equiv 0 \pmod{3}$ for $l \geq 1$. In fact, a check of values in [DF-I] shows $N_d \equiv 0 \pmod{3}$ if and only if $d \equiv 0 \pmod{3}$ for $3 \leq d \leq 12$. P. Aluffi has calculated $E_{3,j}$ for $j < \infty$ in [Al]. Aluffi's results agree with the above formulas.

Thanks are due to Y. Ruan for discussions about Gromov-Witten invariants and quantum cohomology. The question of determining the numbers $E_{d,j}$ was first considered by the author in a conversation with him. In [K-Q-R], the approach of this paper is studied in the genus 2 case where a degeneration to a reducible union

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of rational curves replaces the nodal degeneration considered here. An enumerative formula for genus 2 plane curves with fixed complex structure is derived. A symplectic determination of $E_{d,j}$ was found independently in [I].

1. KONTSEVICH'S SPACE OF STABLE MAPS

1.1. **The quasi-projective subvarieties** $U_C(\Gamma, c, \bar{w})$, $U_{j=\infty}(\Lambda, \bar{w})$. Fix $d \geq 3$ for the entire paper. Let C be a nonsingular elliptic curve or an irreducible, nodal rational curve of arithmetic genus 1. Consider the coarse moduli space of $3d - 1$ -pointed stable maps from C to \mathbf{P}^2 of degree $d \geq 3$, $\overline{M}_{C,3d-1}(\mathbf{P}^2, d)$. The points of $\overline{M}_{C,3d-1}(\mathbf{P}^2, d)$ correspond to stable maps of degree d from nodal, $3d - 1$ -pointed degenerations of C to \mathbf{P}^2 . Two such pointed maps are equivalent if they differ by an isomorphism of the pointed domain curves. For convenience, the notation $\overline{M}_C(d) = \overline{M}_{C,3d-1}(\mathbf{P}^2, d)$ will be used. Let $S_d = \{1, 2, \dots, 3d - 1\}$ be the marking set. Constructions of $\overline{M}_C(d)$ can be found in [A], [B-M], [K], and [F-P]. Let Γ be a tree consisting of a distinguished vertex c , $k \geq 0$ other vertices v_1, \dots, v_k , and $3d - 1$ marked legs. Let $0 \leq e \leq d$. Weight the vertex c by e . Let w_1, \dots, w_k be non-negative integral weights of the vertices v_1, \dots, v_k satisfying:

$$e + w_1 + \dots + w_k = d.$$

Denote the weighting by $\bar{w} = (e, w_1, \dots, w_k)$. The marked, weighted tree with distinguished vertex (Γ, c, \bar{w}) is stable if the following implication holds for all $1 \leq i \leq k$:

$$w_i = 0 \Rightarrow \text{valence}(v_i) \geq 3.$$

Two marked, weighted trees with distinguished vertex (Γ, c, \bar{w}) and (Γ', c', \bar{w}') are isomorphic if there is an isomorphism of marked trees $\Gamma \rightarrow \Gamma'$ sending c to c' and respecting the weights.

A quasi-projective subvariety $U_C(\Gamma, c, \bar{w})$ of $\overline{M}_C(d)$ is associated to each isomorphism class of stable, marked, weighted tree with distinguished vertex (Γ, c, \bar{w}) . The subvariety $U_C(\Gamma, c, \bar{w})$ consists of stable maps $\mu : (D, p_1, \dots, p_{3d-1}) \rightarrow \mathbf{P}^2$ satisfying the following conditions. The domain D is equal to a union:

$$D = C \cup \mathbf{P}_1^1 \cup \dots \cup \mathbf{P}_k^1.$$

The marked, weighted dual graph with distinguished vertex of the map μ is isomorphic to (Γ, c, \bar{w}) . The distinguished vertex of the dual graph of μ corresponds to the (unique) component of D isomorphic to C . Weights of the dual graph of μ are obtained by the degree of μ on the components. Note $U_C(\Gamma, c, \bar{w}) = \emptyset$ if and only if $e = 1$.

Let (Γ, c, \bar{w}) be a stable, marked, weighted tree with distinguished vertex. Assume $e \neq 1$. The dimension of $U_C(\Gamma, c, \bar{w})$ is determined as follows. If $e \geq 2$, then

$$\dim U_C(\Gamma, c, \bar{w}) = 6d - 2 - k.$$

If $e = 0$, then

$$\dim U_C(\Gamma, c, \bar{w}) = 6d - k$$

(where k is the number of non-distinguished vertices of Γ). These calculations are straightforward.

Let C be a nonsingular elliptic curve. Every stable map in $\overline{M}_C(d)$ has domain obtained by attaching a finite number of marked trees to C . By the definition of tree and map stability:

$$\bigcup_{(\Gamma, c, \overline{w})} U_C(\Gamma, c, \overline{w}) = \overline{M}_C(d).$$

Let C be an irreducible, 1-nodal rational curve. The quasi-projective varieties $U_C(\Gamma, c, \overline{w})$ do not cover $\overline{M}_C(d)$. The curve C can degenerate into a simple circuit of \mathbf{P}^1 's. Let Λ be a graph with 1 circuit (1st Betti number equal to 1, no self edges), $k \geq 1$ vertices v_0, \dots, v_k , and $3d - 1$ marked legs. Note the different vertex numbering convention. At least 2 vertices are required to make a circuit, so $k \geq 1$. Let w_0, w_1, \dots, w_k be non-negative, integral weights summing to d . The marked, weighted graph with 1 circuit (Λ, \overline{w}) is stable if each zero weighted vertex has valence at least 3. A quasi-projective subvariety $U_C(\Lambda, \overline{w})$ of $\overline{M}_C(d)$ is associated to each isomorphism class of stable, marked, weighted graph with 1 circuit (Λ, \overline{w}) . $U_C(\Lambda, \overline{w})$ consists of stable maps with marked, weighted dual graphs isomorphic to (Λ, \overline{w}) . The union

$$\bigcup_{(\Gamma, c, \overline{w})} U_C(\Gamma, c, \overline{w}) \cup \bigcup_{(\Lambda, \overline{w})} U_C(\Lambda, \overline{w}) = \overline{M}_C(d)$$

holds by the definition of stability.

Finally, the dimensions of the loci $U_C(\Lambda, \overline{w})$ will be required. Let (Λ, \overline{w}) be a stable, marked, weighted graph with 1 circuit. Let c_1, \dots, c_l be the unique circuit of vertices of Λ . Let e be the sum of the weights of the circuit vertices. $U_C(\Lambda, \overline{w}) = \emptyset$ if and only if $e = 1$. If $e \geq 2$, then

$$\dim U_C(\Lambda, \overline{w}) = 6d - 2 - k.$$

If $e = 0$, then

$$\dim U_C(\Lambda, \overline{w}) = 6d - k$$

(where $k + 1$ is the total number of vertices of Λ). Again, these results are straightforward.

1.2. The component $\overline{W}_1(d)$. Let $\overline{M}_1(d) = \overline{M}_{1,3d-1}(\mathbf{P}^2, d)$ be Kontsevich's space of $3d - 1$ -pointed stable maps from genus 1 curves to \mathbf{P}^2 . There is canonical morphism

$$\pi : \overline{M}_1(d) \rightarrow \overline{M}_{1,1}$$

obtained by forgetting the map and all the markings except $1 \in S_d$ (the $3d - 1$ possible choices of marking in S_d all yield the same morphism π). Let $j \in \overline{M}_{1,1}$. By the definitions of the moduli spaces, there is a canonical bijection of points:

$$\overline{M}_{C_j}(d) \rightarrow \pi^{-1}(j),$$

where C_j is the elliptic curve (possibly nodal rational) with j -invariant j . Define an open locus $W_1(d) \subset \overline{M}_1(d)$ by

$$[\mu : (D, p_1, \dots, p_{3d-1}) \rightarrow \mathbf{P}^2] \in W_1(d)$$

if and only if D is irreducible. Similarly, an open locus $M_{1,3d-1}^{irr} \subset \overline{M}_{1,3d-1}$ of irreducible pointed curves is defined. Since $d \geq 3$, $M_{1,3d-1}^{irr}$ is a nonsingular fine moduli space with a universal curve. There is a canonical forgetful map $\rho : W_1(d) \rightarrow M_{1,3d-1}^{irr}$. It is easily seen that $W_1(d)$ is an open set of a tautological \mathbf{P}^{3d-1} -bundle

over the universal Picard variety of degree d line bundles over $M_{1,3d-1}^{irr}$. Therefore, $W_1(d)$ is a nonsingular irreducible variety of dimension $6d - 1$ and the morphism ρ is smooth. Let $\overline{W}_1(d)$ be the closure of $W_1(d)$ in $\overline{M}_1(d)$.

Let $\pi_W : W_1(d) \rightarrow \overline{M}_{1,1}$ be the restriction of π . Let $\pi_{M^{irr}} : M_{1,3d-1}^{irr} \rightarrow \overline{M}_{1,1}$ be the canonical forgetful map. Then, $\pi_W = \pi_{M^{irr}} \circ \rho$. It is well known that $\pi_{M^{irr}}^{-1}(j)$ is a nonsingular irreducible divisor in $M_{1,3d-1}^{irr}$ for $0, 1728 \neq j \in \overline{M}_{1,1}$. The scheme theoretic inverse images $\pi_{M^{irr}}^{-1}(0)$ and $\pi_{M^{irr}}^{-1}(1728)$ are easily seen to be irreducible divisors of multiplicities 3 and 2 respectively. Since ρ is smooth with irreducible fibers, $\pi_W^{-1}(j)$ is a nonsingular irreducible divisor in $W_1(d)$ for $j \neq 0, 1728$ and a divisor of multiplicity 3, 2 for $j = 0, 1728$ respectively.

2. A DEFORMATION RESULT

Let Φ be a stable, marked, weighted tree with distinguished vertex c determined by the data: $k = 1, (e, w_1) = (0, d)$. There are 2^{3d-1} isomorphism classes of such Φ determined by the marking distribution. Let $j \in \overline{M}_{1,1}$. The dimension of $U_j(\Phi)$ is $6d - 1$. A point $[\mu] \in U_j(\Phi)$ has domain $C_j \cup \mathbf{P}^1$. There are $3d - 1$ dimensions of the map $\mu|_{\mathbf{P}^1} : \mathbf{P}^1 \rightarrow \mathbf{P}^2$. The incidence point $p = C_j \cap \mathbf{P}^1$ moves in a 1-dimensional family on \mathbf{P}^1 . The remaining $3d - 1$ markings move in $3d - 1$ dimensions on C_j and \mathbf{P}^1 (specified by the marking distribution). $6d - 1 = 3d - 1 + 1 + 3d - 1$. A technical result is needed in the computation of the numbers $E_{d,j}$.

Lemma 1. *Let $I(\Phi, j) = \overline{W}_1(d) \cap U_j(\Phi) \subset \overline{M}_1(d)$. The dimension of $I(\Phi, j)$ is bounded by $\dim I(\Phi, j) \leq 6d - 3$.*

Proof. Let $[\mu] \in I(\Phi, j)$ be a point. Let $D = C_j \cup \mathbf{P}^1$ be the domain of μ as above. The following condition will be shown to hold: the linear series on \mathbf{P}^1 determined by $\mu|_{\mathbf{P}^1}$ has vanishing sequence $\{0, \geq 2, *\}$ at the incidence point $p = C_j \cap \mathbf{P}^1$. The existence of a point with vanishing sequence $\{0, \geq 2, *\}$ is a 1-dimensional condition on the linear series. The condition that the incidence point p has this vanishing sequence is an additional 1-dimensional constraint on p . Therefore, the dimension of $I(\Phi, j)$ is at most $6d - 1 - 1 - 1 = 6d - 3$. The vanishing sequence $\{0, \geq 2, *\}$ is equivalent to $d(\mu|_{\mathbf{P}^1}) = 0$ at p .

It remains to establish the vanishing sequence $\{0, \geq 2, *\}$ at p . This result is easily seen in explicit holomorphic coordinates. Let Δ_t be a disk at the origin in \mathbb{C} with coordinate t . Let $\eta : \mathcal{E} \rightarrow \Delta_t$ be a flat family of curves of arithmetic genus 1 satisfying:

- (i) $\eta^{-1}(0) \cong C_j$.
- (ii) $\eta^{-1}(t \neq 0)$ is irreducible, reduced, and (at worst) nodal.

For each $1 \leq i \leq d$, let $\mathcal{G}_i = \mathcal{H}_i \subset \mathcal{E}$ be the open subset of \mathcal{E} on which the morphism η is smooth. Consider the fiber product:

$$X = \mathcal{G}_1 \times_{\Delta_t} \cdots \times_{\Delta_t} \mathcal{G}_d \times_{\Delta_t} \mathcal{H}_1 \times_{\Delta_t} \cdots \times_{\Delta_t} \mathcal{H}_d.$$

X is a nonsingular open set of the $2d$ -fold fiber product of \mathcal{E} over Δ_t . Let $Y \subset X$ be the subset of points $y = (g_1, \dots, g_d, h_1, \dots, h_d)$ where the two divisors $\sum g_i$ and $\sum h_i$ are linearly equivalent on the curve $\mathcal{E}_{\eta(y)}$. Y is a nonsingular divisor in X .

Let $p \in C_j = \eta^{-1}(0)$ be a nonsingular point of C_j . Certainly $p \in \mathcal{G}_i, \mathcal{H}_i$ for all i . Let $\gamma : \Delta_t \rightarrow \mathcal{E}$ be any local holomorphic section of η such that $\gamma(0) = p$. Let V be a local holomorphic field of vertical tangent vectors to \mathcal{E} on an open set containing p . The section γ and the vertical vector field V together determine local

holomorphic coordinates (t, v) on \mathcal{E} at p . Let $\phi_V : \mathcal{E} \times \mathbb{C} \rightarrow \mathcal{E}$ be the holomorphic flow of V defined locally near $(p, 0) \in \mathcal{E} \times \mathbb{C}$. The coordinate map

$$\psi : (t, v) \rightarrow \mathcal{E}$$

is determined by $\psi(t, v) = \phi_V(\gamma(t), v)$.

Local coordinates on X near the point $x_p = (p, \dots, p, p, \dots, p) \in X$ are given by

$$(t, v_1, \dots, v_d, w_1, \dots, w_d).$$

The coordinate map is determined by:

$$\psi_X(t, v_1, \dots, v_d, w_1, \dots, w_d) =$$

$$(\psi(t, v_1), \dots, \psi(t, v_d), \psi(t, w_1), \dots, \psi(t, w_d)) \in X.$$

Note $x_p \in Y$. Let $f(t, v_1, \dots, v_d, w_1, \dots, w_d)$ be a local equation of Y at x_p . Since f is identically 0 on the line $(t, 0, \dots, 0, 0, \dots, 0)$,

$$(1) \quad \forall k \geq 0, \quad \frac{\partial^k f}{\partial t^k} \Big|_{x_p} = 0.$$

The tangent directions in the plane $t = 0$ correspond to divisors on the fixed curve C_j . Here, it is well known (up to a \mathbb{C}^* -factor)

$$(2) \quad \frac{\partial f}{\partial v_i} \Big|_{x_p} = +1, \quad \frac{\partial f}{\partial w_i} \Big|_{x_p} = -1.$$

Equations (1) and (2) are the only properties of f that will be used.

Let $\hat{\eta} : \hat{\mathcal{E}} \rightarrow \Delta_t$ be the family obtained by blowing-up \mathcal{E} at p and adding $3d - 1$ marking. Let $\mu : \hat{\mathcal{E}} \rightarrow \mathbf{P}^2$ be a morphism. Let $\hat{\eta}^{-1}(0) = D = C_j \cup \mathbf{P}^1$. Assume the following conditions are satisfied:

- (i) $\mu, \hat{\eta}$, and the $3d - 1$ markings determine a family of Kontsevich stable pointed maps to \mathbf{P}^2 .
- (ii) The markings of D are distributed according to Φ .
- (iii) $\text{deg}(\mu|_{C_j}) = 0, \text{deg}(\mu|_{\mathbf{P}^1}) = d$.

Let L_1, L_2 be general divisors of $\mu^*(\mathcal{O}_{\mathbf{P}^2}(1))$ that each intersect \mathbf{P}^1 transversely at d distinct points. For $1 \leq \alpha \leq 2, L_\alpha$ breaks into holomorphic sections $s_{\alpha,1} + \dots + s_{\alpha,d}$ of $\hat{\eta}$ over a holomorphic disk at $0 \in \Delta_t$. These sections $s_{\alpha,i}$ ($1 \leq \alpha \leq 2, 1 \leq i \leq d$) determine a map $\lambda : \Delta_t \rightarrow Y$ locally at $0 \in \Delta_t$. Let an affine coordinate on \mathbf{P}^1 be given by ξ corresponding to the normal direction

$$(3) \quad \frac{d\gamma}{dt} \Big|_{t=0} + \xi \cdot V(p).$$

Let $s_{1,i}(0) = \nu_i \in \mathbb{C} \subset \mathbf{P}^1, s_{2,i}(0) = \omega_i \in \mathbb{C} \subset \mathbf{P}^1$ be given in terms of the affine coordinate ξ . The map λ has the form

$$\lambda(t) = (t, \nu_1 t, \dots, \nu_d t, \omega_1 t, \dots, \omega_d t)$$

to first order in t (written in the coordinates determined by ψ_X). Equations (1), (2), and the condition $f(\lambda(t)) = 0$ imply

$$(4) \quad \sum_{i=1}^d \nu_i = \sum_{i=1}^d \omega_i.$$

$L_1 \cap \mathbf{P}^1$ is a degree d polynomial with roots at ν_i . Condition (4) implies that the sums of the roots (in the coordinates (3)) of general elements of the linear series $\mu|_{\mathbf{P}^1}$ are the same. Therefore, a constant K exists with the following property. If

$$\beta_0 + \beta_1\xi + \dots + \beta_{d-1}\xi^{d-1} + \beta_d\xi^d$$

is an element of the linear series $\mu|_{\mathbf{P}^1}$, then $\beta_{d-1} + K \cdot \beta_d = 0$. The vanishing sequence at $\xi = \infty$ is therefore $\{0, \geq 2, *\}$. The point $\xi = \infty$ is the intersection $C_j \cap \mathbf{P}^1$.

Suppose $\tilde{\eta} : \tilde{\mathcal{E}} \rightarrow \Delta_t$ is obtained from \mathcal{E} by a sequence of n blow-ups over p . The fiber $\tilde{\eta}^{-1}(0)$ is assumed to be C_j union a chain of \mathbf{P}^1 's of length n . Each blow-up occurs in the exceptional divisor of the previous blow-up. Let \mathbf{P} denote the extreme exceptional divisor. Let $\mu : \tilde{\mathcal{E}} \rightarrow \mathbf{P}^2$ be of degree d on \mathbf{P} and degree 0 on the other components of the special fiber $\tilde{\eta}^{-1}(0)$. Let there be $3d - 1$ markings as before. It must be again concluded that the linear series on \mathbf{P} has vanishing sequence $\{0, \geq 2, *\}$ at the node.

Let γ be section of η such that the lift of γ to $\tilde{\eta}$ meets P . Let the coordinates (t, v) on \mathcal{E} be determined by this γ (and any V). An affine coordinate ξ is obtained on \mathbf{P} in the following manner. Let γ_ξ be the section of η determined in (t, v) coordinates by

$$\gamma_\xi(t) = (t, \xi t^n).$$

Let $\tilde{\gamma}_\xi$ be the lift of γ_ξ to a section of $\tilde{\eta}$. The association

$$\mathbb{C} \ni \xi \mapsto \tilde{\eta}(0) \in \mathbf{P}$$

is an affine coordinate on \mathbf{P} . Let L_1, L_2 be divisors in the linear series μ intersecting \mathbf{P} transversely. As before, L_α breaks into holomorphic sections $s_{\alpha,1}$. Let $s_{1,i} = \nu_i \in \mathbb{C} \subset \mathbf{P}$, $s_{2,i} = \omega_i \in \mathbb{C} \subset \mathbf{P}$. As before, a map $\lambda : \Delta_t \rightarrow Y$ is obtained from the sections $s_{\alpha,i}$. In the coordinates determined by ψ_X ,

$$\lambda(t) = (t, \nu_1 t^n + O(t^{n+1}), \dots, \nu_d t^n + O(t^{n+1}), \omega_1 t^n + O(t^{n+1}), \dots, \omega_d t^n + O(t^{n+1})).$$

As before $f(\lambda(t)) = 0$. The term of leading order in t of $f(\lambda(t))$ is

$$\left(\sum_{i=1}^d \nu_i - \sum_{i=1}^d \omega_i \right) \cdot t^n.$$

This follows from equations (1) and (2). The vanishing sequence $\{0, \geq 2, *\}$ is obtained as before.

By definition, an element $[\mu] \in I(\Phi, j)$ can be obtained as the special fiber of family of Kontsevich stable maps where the domain is a smoothing of the node p . After resolving the singularity in the total space at the node p by blowing-up, a family $\tilde{\mathcal{E}}$ is obtained. The above results show the linear series on \mathbf{P}^1 has vanishing sequence $\{0, \geq 2, *\}$ at p . □

The markings play no role in the preceding proof. An identical argument establishes the following:

Lemma 2. *Let Φ be a stable, marked, weighted tree with distinguished vertex satisfying $e = 0$ and $w_i = d$ for some i . Let k be the number of non-distinguished vertices of Φ . Let $j \in \overline{M}_{1,1}$. Let $I(\Phi, j) = \overline{W}_1(d) \cap U_j(\Phi)$. The dimension of $I(\Phi, j)$ is bounded by $\dim I(\Phi, j) \leq 6d - k - 2$.*

Lemma 3. *Let Ω be a stable, marked, weighted graph with 1 circuit. Let v_i be a non-circuit vertex with weight $w_i = d$ (this implies $e = 0$). Let $k + 1$ be the total number of vertices of Ω . Let $I(\Omega, \infty) = \overline{W}_1(d) \cap U_\infty(\Omega)$. The dimension of $I(\Omega, \infty)$ is bounded by $\dim I(\Omega, \infty) \leq 6d - k - 2$.*

The vanishing sequence $\{0, \geq 2, *\}$ condition reduces the dimensions of $U_C(\Phi)$, $U_\infty(\Omega)$ by 2.

3. THE NUMBERS $E_{d,j}$

The space of maps $\overline{M}_1(d)$ is equipped with $3d - 1$ evaluation maps corresponding to the marked points. For $i \in S_d$, let $e_i : \overline{W}_1(d) \rightarrow \mathbf{P}^2$ be the restriction of the i^{th} evaluation map to $\overline{W}_1(d)$. let $\mathcal{L}_i = e_i^*(\mathcal{O}_{\mathbf{P}^2})$ Let

$$Z = c_1(\mathcal{L}_1)^2 \cap \dots \cap c_1(\mathcal{L}_{3d-1})^2$$

Let $\pi_{\overline{W}} : \overline{W}_1(d) \rightarrow \overline{M}_{1,1} \cong \mathbf{P}^1$ be the restriction of π to $\overline{W}_1(d)$. Let

$$T = c_1(\pi_{\overline{W}}^*(\mathcal{O}_{\mathbf{P}^1}(1))).$$

Note $\overline{W}_1(d)$ is an irreducible, projective variety of dimension $6d - 1$. The top intersection of line bundles on $\overline{W}_1(d)$, $Z \cap T$, is an integer.

Lemma 4.

$$\begin{aligned} \forall j \neq 0, 1728, \infty, & \quad Z \cap T = E_{d,j} , \\ j = 0, & \quad Z \cap T = 3 \cdot E_{d,0} , \\ j = 1728, & \quad Z \cap T = 2 \cdot E_{d,1728} . \end{aligned}$$

Proof. Via pull-back, lines in \mathbf{P}^2 yield representative classes of $c_1(\mathcal{L}_i)$. Therefore $3d - 1$ general points in \mathbf{P}^2 , $\overline{x} = (x_1, \dots, x_{3d-1})$, determine a representative cycle $Z_{\overline{x}}$ of the the class Z . Let $\infty > j \in \overline{M}_{1,1}$. It is first established for a general representative $Z_{\overline{x}}$,

$$(5) \quad Z_{\overline{x}} \cap \pi_{\overline{W}}^{-1}(j) \subset \pi_{\overline{W}}^{-1}(j).$$

Statement (5) is proven by considering the quasi-projective strata of $\overline{M}_{C_j}(d)$.

Note $\pi_{\overline{W}}^{-1}(j)$ is the strata $U_{C_j}(\Gamma, c, \overline{w})$ where $(\Gamma, c, \overline{w})$ is the trivial, stable, marked, weighted tree with distinguished vertex. Assume now $(\Gamma, c, \overline{w})$ is not the trivial tree. By the equations for the dimension of $(\Gamma, c, \overline{w})$,

$$\dim U_{C_j}(\Gamma, c, \overline{w}) \leq 6d - 3$$

unless $e = 0$ and $k = 1, 2$. Since the linear series determined by the evaluation maps are base point free, the general intersection (5) will miss all loci of dimension less than $6d - 2$.

It remains to consider the trees $(\Gamma, c, \overline{w})$ where $e = 0$ and $k = 1, 2$. If $k = 1$, $(\Gamma, c, \overline{w}) = \Phi$ satisfies the conditions of Lemma (1). By Lemma (1),

$$\dim I(\Phi, j) \leq 6d - 3.$$

Hence, the general intersection (5) will miss all the loci $U_C(\Phi, c, (0, d))$.

If $k = 2$, there are two cases to consider. If there exists a vertex of weight d , then $(\Gamma, c, \overline{w}) = \Phi$ satisfies the conditions of Lemma (2). By Lemma (2),

$$\dim I(\Phi, j) \leq 6d - 4.$$

If $w_1 + w_2 = d$ is a positive partition, then the image of every map $[\mu] \in U_C(\Gamma, c, \bar{w})$ is the union of two rational curves of degrees w_1 and w_2 . No such unions pass through $3d - 1$ general points. The proof of claim (5) is complete.

For $\infty > j \neq 0, 1728$, $\pi_W^{-1}(j)$ is a nonsingular irreducible divisor of $W_1(d)$. Since the linear series determined by the evaluation maps are base point free, the general intersection cycle

$$(6) \quad Z_{\bar{x}} \cap \pi_W^{-1}(j)$$

is a reduced collection of $Z \cap T$ points (Bertini’s theorem). The general intersection cycle (6) consists exactly of the reduced, nodal, degree d elliptic plane curves with j -invariant j passing through the points \bar{x} .

The argument for $j = 0, 1728$ is identical except that $\pi_W^{-1}(0)$, and $\pi_W^{-1}(1728)$ are divisors in $W_1(d)$ with multiplicity 3, 2 respectively. These multiplicities arise from the extra automorphisms for $j = 0, 1728$. Therefore the cycle (6) is a collection of

$$\frac{1}{3} \cdot Z \cap T, \quad \frac{1}{2} \cdot Z \cap T$$

triple and double points respectively. □

It remains to evaluate $Z \cap T$.

Lemma 5. $Z \cap T = \binom{d-1}{2} N_d$.

Proof. It is first established for a general representative $Z_{\bar{x}}$,

$$(7) \quad Z_{\bar{x}} \cap \pi_W^{-1}(\infty) \subset \pi_W^{-1}(\infty).$$

The statement (7) is proven by considering the quasi-projective strata of $\overline{M}_\infty(d)$.

By arguments of Lemma (4), all the loci $U_\infty(\Gamma, c, \bar{w})$ where (Γ, c, \bar{w}) is not the trivial tree are avoided in the general intersection (7). Only the strata $U_\infty(\Lambda, \bar{w})$ remain to be considered. Let $k + 1 \geq 2$ be the total number of vertices of Λ . By the equations for the dimensions of $U_\infty(\Lambda, \bar{w})$,

$$\dim U_\infty(\Lambda, \bar{w}) \leq 6d - 2 - k \leq 6d - 3$$

unless all the circuit vertices have weight zero. If all circuit vertices have weight zero, $k + 1 \geq 3$. Now

$$\dim U_\infty(\Lambda, \bar{w}) \leq 6d - k \leq 6d - 3$$

unless $k = 2$.

Only one stable, marked, weighted, graph with 1-circuit (Λ, \bar{w}) need be considered. Vertices c_1, c_2 form a weightless circuit. Vertex v_3 is connected to c_2 and $w_3 = d$. $(\Lambda, \bar{w}) = \Omega$ satisfies the conditions of Lemma (3). Therefore,

$$\dim I(\Omega, \infty) \leq 6d - 4.$$

Claim (7) is now proven.

The divisor $\pi_W^{-1}(\infty)$ is nonsingular and irreducible in $W_1(d)$. As above,

$$(8) \quad Z_{\bar{x}} \cap \pi_W^{-1}(\infty)$$

is a reduced collection of $Z \cap T$ points. The general intersection cycle (8) also consists exactly of degree d maps of the 1-nodal rational curve C_∞ passing through \bar{x} . The image of such a map must be one of the N_d degree d , nodal, rational

plane curves passing through \bar{x} . The number of distinct birational maps (up to isomorphism) from C_∞ to a $\binom{d-1}{2}$ -nodal plane curve is exactly $\binom{d-1}{2}$. Therefore, $Z \cap T = \binom{d-1}{2} N_d$. \square

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