THE RANK STABLE TOPOLOGY OF INSTANTONS ON $\mathbb{CP}^2$

JIM BRYAN AND MARC SANDERS

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Abstract. Let $\mathcal{M}_k^n$ be the moduli space of based (anti-self-dual) instantons on $\mathbb{CP}^2$ of charge $k$ and rank $n$. There is a natural inclusion $\mathcal{M}_k^n \hookrightarrow \mathcal{M}_k^{n+1}$. We show that the direct limit space $\mathcal{M}_k^\infty$ is homotopy equivalent to $BU(k) \times BU(k)$. Let $\ell_\infty$ be a line in the complex projective plane and let $\mathbb{CP}^2$ be the blow-up at a point away from $\ell_\infty$. $\mathcal{M}_k^n$ can be alternatively described as the moduli space of rank $n$ holomorphic bundles on $\mathbb{CP}^2$ with $c_1 = 0$ and $c_2 = k$ and with a fixed holomorphic trivialization on $\ell_\infty$.

1. Introduction

In his 1989 paper [Ta], Taubes studied the stable topology of the based instanton moduli spaces. He showed that if $\mathcal{M}_k^n(X)$ denotes the moduli space of based $SU(n)$-instantons of charge $k$ on $X$, then there is a map $\mathcal{M}_k^n(X) \to \mathcal{M}_k^{n+1}(X)$ and, in the direct limit topology, $\mathcal{M}_k^\infty(X)$ has the homotopy type of $\text{Map}_0(X, BSU(n))$.

There is also a map $\mathcal{M}_k^n(X) \hookrightarrow \mathcal{M}_k^{n+1}(X)$ given by the direct sum of a connection with the trivial connection on a trivial line bundle and one can consider the direct limit $\mathcal{M}_k^\infty(X)$. For the case of $X = S^4$ with the round metric, it was shown by Kirwan and also by Sanders ([Kir], [Sa]) that the direct limit has the homotopy type of $BU(k)$.

In this note we consider the case of $X = \mathbb{CP}^2$ where $\mathbb{CP}^2$ denotes the complex projective plane with the Fubini-Study metric and the opposite orientation of the one induced by the complex structure. Our result is:

Theorem 1.1. $\mathcal{M}_k^\infty(\mathbb{CP}^2)$ has the homotopy type of $BU(k) \times BU(k)$.

The main tool in the proof of the theorem is a construction of the moduli spaces $\mathcal{M}_k^\infty(\mathbb{CP}^2)$ due to King [Ki]. In general, Buchdahl [Bu] has shown that, for appropriate metrics on the $N$-fold connected sum $\#_N \mathbb{CP}^2$, the moduli spaces $\mathcal{M}_k^\infty(\#_N \mathbb{CP}^2)$ are diffeomorphic to certain spaces of equivalence classes of holomorphic bundles on $\mathbb{CP}^2$ blown-up at $N$ points. The universal $U(k) \times U(k)$ bundle that appears giving the homotopy equivalence of Theorem 1.1 can be constructed as higher direct image bundles (see section 3).

Remark 1.1. The cofibration $S^2 \to \mathbb{CP}^2 \to S^4$ gives rise to the fibration of mapping spaces $\Omega^4 BSU(n) \to \text{Map}_\ast(\mathbb{CP}^2, BSU(n)) \to \Omega^2 BSU(n)$ which for K-theoretic

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reasons is a trivial fibration in the limit over \( n \). The total space of this fibration is homotopy equivalent to the space of based gauge equivalence classes of all connections on \( \mathbb{CP}^2 \). Thus, from Taubes’ result, \( \mathcal{M}_k^\infty \) must have the property that taking the limit over \( k \) gives \( BU \times BU \). For \( S^4 \), similar remarks imply that \( \lim_{k \to \infty} \mathcal{M}_k^\infty(S^4) \simeq BU \) and the inclusion of \( \mathcal{M}_k^\infty(S^4) \) into this limit has been shown to be (up to homotopy) the natural inclusion \( BU(k) \hookrightarrow BU \) ([Sa]). Theorem 1.1 and these results for \( S^4 \) suggest a general conjecture which is supported by the fact that the higher direct image bundle giving our homotopy equivalence generalizes in an appropriate way.

**Conjecture 1.1.** For appropriate metrics on \( \#_N \mathbb{CP}^2 \), \( \mathcal{M}_k^\infty(\#_N \mathbb{CP}^2) \) has the homotopy type of a product \( BU(k) \times \cdots \times BU(k) \) with \( N + 1 \) factors.

**Remark 1.2.** Combining Theorem 1.1 with Taubes’ stabilization result leads to an alternate proof of Bott periodicity for the unitary group. There is a natural map \( \Phi : M \to \Omega \), and their relationships to those on \( S \). As an amusing corollary, we will be able to rederive many of the Bott periodicity relationships among \( Sp, U, SO \), and their homogeneous spaces.

2. **The Construction of \( \mathcal{M}_k^n(\mathbb{CP}^2) \)**

Let \( x_0 \in \mathbb{CP}^2 \) be the base point. Since \( \mathbb{CP}^2 \setminus \{x_0\} \) is conformally equivalent to \( \hat{C}^2 \), the complex plane blown-up at the origin, \( \mathcal{M}_k^n(\mathbb{CP}^2) \), can be regarded as instantons on \( \hat{C}^2 \) based “at infinity”. Buchdahl [Bu] proved an analogue in this non-compact setting of Donaldson’s theorem relating instantons to holomorphic bundles: Let \( \hat{C}^2_N \) be the complex plane blown-up at \( N \) points with a Kähler metric. Then \( \hat{C}^2_N \) has a “conformal compactification” to \( \#_N \hat{\mathbb{CP}^2} \) and a “complex compactification” to \( \hat{\mathbb{CP}^2}_N \) (the projective plane blown-up at \( N \) points). We have added a point \( x_0 \) in the former case and a complex projective line \( \ell_\infty \) in the latter.

Define \( \mathcal{M}_{\text{alg},k}^n(\mathbb{CP}^2_N) \) to be the moduli space consisting of pairs \( (\mathcal{E}, \tau) \) where \( \mathcal{E} \) is a rank \( n \) holomorphic bundle on \( \mathbb{CP}^2_N \) with \( c_1(\mathcal{E}) = 0, c_2(\mathcal{E}) = k \), and where \( \tau : \mathcal{E}|_{\ell_\infty} \to \mathbb{C}^n \otimes \mathcal{O}_{\ell_\infty} \) is a holomorphic trivialization of \( \mathcal{E} \) on \( \ell_\infty \).

There is a natural map \( \Phi : \mathcal{M}_k^n(\#_N \mathbb{CP}^2) \to \mathcal{M}_{\text{alg},k}^n(\mathbb{CP}^2_N) \) defined as follows. Let \( p : \mathbb{CP}^2_N \to \#_N \mathbb{CP}^2 \) be the map that collapses \( \ell_\infty \mapsto x_0 \). If \( [A] \in \mathcal{M}_k^n \).
then the $\bar{\partial}$ operator that defines the holomorphic bundle $\mathcal{V} = \Phi(A)$ is taken to be $(\hat{d}_{\nu}(A))^{(0,1)}$, the anti-holomorphic part of the covariant derivative defined by the pullback of the connection. The anti-self-duality of $A$ implies that the curvature of $p^*(A)$ is a $(1,1)$-form and so $\bar{\partial}^2 = 0$.

Buchdahl’s theorem is then

**Theorem 2.1.** The map $\Phi : \mathcal{M}^n_{\mathbb{C}}(\#_N \mathbb{CP}^2) \rightarrow \mathcal{M}^n_{\text{alg}, k}(\mathbb{CP}^2)$ is a diffeomorphism.

The case $N = 1$ was first proved by King [Ki]. We now restrict ourselves to that case and simply write $\mathcal{M}^n_k$ for $\mathcal{M}^n(\mathbb{CP}^2)$ and $\mathcal{M}^n_{\text{alg}, k}(\mathbb{CP}^2)$.

King constructed $\mathcal{M}^n_k$ explicitly in terms of linear algebra data. We recall his construction. Consider configurations of linear maps:

$$
\begin{array}{c}
W_0 \\
\downarrow \quad a_1, a_2 \\
\downarrow x \\
W_1 \\
\downarrow c \\
V_\infty
\end{array}
$$

where $W_0$, $W_1$ and $V_\infty$ are complex vector spaces of dimensions $k$, $k$, and $n$ respectively.

A configuration $(a_1, a_2, b, c, x)$ is called **integrable** if it satisfies the equation

$$a_1 xa_2 - a_2 xa_1 + bc = 0.$$

A configuration $(a_1, a_2, b, c, x)$ is **non-degenerate** if it satisfies the following conditions:

$$\forall (\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathbb{C}^2 \text{ such that } \lambda_1 \mu_1 + \lambda_2 \mu_2 = 0 \text{ and } (\mu_1, \mu_2) \neq (0,0),$$

$$\forall v \in W_1 \text{ such that } \begin{cases} xa_1 v = \lambda_1 v, & (\mu_1 a_1 + \mu_2 a_2) v = 0, \\ xa_2 v = \lambda_2 v, & cv = 0, \end{cases}$$

and $\forall w \in W_0^*$ such that

$$\begin{cases} x^* a_1^* w = \lambda_1 w, & (\mu_1 a_1^* + \mu_2 a_2^*) w = 0, \\ x^* a_2^* w = \lambda_2 w, & b^* w = 0. \end{cases}$$

Let $A^n_k$ be the space of all integrable non-degenerate configurations. $G = \text{Gl}(W_0) \times \text{Gl}(W_1)$ acts canonically on $A^n_k$. The action is explicitly given by

$$(g_0, g_1) \cdot (a_1, a_2, b, c, x) = (g_0 a_1 g_1^{-1}, g_0 a_2 g_1^{-1}, g_0 b, c g_1^{-1}, g_1 x g_0^{-1}).$$

**Theorem 2.2.** The moduli space $\mathcal{M}^n_k$ is isomorphic to $A^n_k / G$.

**Proof.** King uses such configurations to determine monads that in turn determine holomorphic bundles. Configurations in the same $G$ orbit determine the same bundle. For the sake of brevity we refer the reader to [Ki] or [Br] for details. The construction identifies the vector spaces $W_0$ and $W_1$ canonically as $H^1(\mathcal{E}(-\ell_{\infty}))$ and $H^1(\mathcal{E}(-\ell_{\infty} + E))$ respectively, where $E \subset \mathbb{CP}^2$ is the exceptional divisor. The vector space $V_\infty$ is identified with the fiber over $\ell_{\infty}$. 
3. Proof of Theorem 1.1

We prove the theorem in two steps: We first show that the space of monad data $A_k^n$ forms a principal $G = Gl(k) \times Gl(k)$ bundle over $M_k^n$. We then show that the induced $G$-equivariant inclusion $A_k^n \hookrightarrow A_k^{n+2}$ is null-homotopic so that we can conclude that $A_k^\infty$ is contractible.

Lemma 3.1. $G$ acts freely on the space of monad data $A_k^n$.

Proof. This is essentially proved in [Ki] where it is implicitly shown that the non-degeneracy conditions are precisely the conditions that guarantee freeness. We point out that this also follows more conceptually from the existence of a universal family $E \to M_k^n \times \mathbb{C}P^2$ and the cohomological interpretation of $W_0$ and $W_1$:

First, the existence of a universal family can be shown via the gauge theoretic construction: Let $V$ be a smooth hermitian vector bundle on $\mathbb{C}P^2$ with $c_1(V) = 0$ and $c_2(V) = k$. Let $A_{0,1}^{1,1}$ denote unitary connections on $V$ with curvature of pure type $(1, 1)$ and that restrict to the trivial connection on $\ell_\infty$ and let $\mathcal{G}_0^C$ denote the complex gauge transformations of $V$ that are the identity restricted to $\ell_\infty$. Then $M_k^n = A_{0,1}^{1,1}/\mathcal{G}_0^C$. The quotient

$$(A_{0,1}^{1,1} \times V)/\mathcal{G}_0^C \to M_k^n \times \mathbb{C}P^2$$

will form a universal bundle if the moduli space is smooth and no $E \in M_k^n$ has non-trivial automorphisms (cf. [Fr-Mo] Chapt. IV):

Lemma 3.2. $M_k^n$ is smooth and any $E \in M_k^n$ has no non-trivial automorphisms preserving $\tau : E|_{\ell_\infty} \to C^n \otimes O_{\ell_\infty}$.

By Serre duality $H^2(E \otimes E^*) = H^0(E \otimes E^* \otimes K)^*$. Since $E \otimes E^*$ is trivial on $\ell_\infty$, it is also trivial on nearby lines. Any section of $E \otimes E^* \otimes K$ restricts to a section of $C^n \otimes O_{\ell_\infty}(-3)$ and so must vanish on $\ell_\infty$. Likewise, it must vanish on nearby lines and so it is 0 on an open set and must be identically 0. Thus $H^2(E \otimes E^*) = 0$ and smoothness follows once we show there are no automorphisms.

Suppose that there exists an automorphism $\phi \in H^0(E \otimes E^*)$ such that $\phi \neq 1$ and $\phi$ preserves $\tau$ so that $\phi|_{\ell_\infty} = 1|_{\ell_\infty}$. Then $\phi - 1$ is a non-zero section of $E \otimes E^*$ vanishing on $\ell_\infty$. We then get an injection $0 \to O(\ell_\infty) \to E \otimes E^*$. Restricting this sequence to $\ell_\infty$ we get an injection $0 \to O_{\ell_\infty}(1) \to O_{\ell_\infty} \otimes C^n^2$ which is a contradiction.

Let $\pi : M_k^n \times \mathbb{C}P^2 \to M_k^n$. The higher direct image sheaves $R^1\pi_*(E(-\ell_\infty))$ and $R^1\pi_*(E(-\ell_\infty + E))$ are locally free and rank $k$. This follows from the index theorem and the vanishing of the $H^0$ and $H^2$ cohomology of $E(-\ell_\infty)$ and $E(-\ell_\infty + E)$. The $H^0$ vanishing follows by again considering the restriction of a section of the bundles to lines nearby to $\ell_\infty$. Using Serre duality and the same argument, one gets the vanishing for $H^2$.

Consequently the vector spaces $W_0$ and $W_1$ are the fibers of the vector bundles $R^1\pi_*(E(-\ell_\infty))$ and $R^1\pi_*(E(-\ell_\infty + E))$. The $G$-orbit of a configuration giving a bundle $\mathcal{E}$ can be identified with the group of isomorphisms $g_0 : H^1(\mathcal{E}(\ell_\infty)) \to C^k$ and $g_1 : H^1(\mathcal{E}(\ell_\infty + E)) \to C^k$. Thus $A_k^n$ is realized precisely as the total space of
the principal $\text{Gl}(k) \times \text{Gl}(k)$ bundle associated to

$$R^1\pi_*(\mathcal{E}(-\ell_\infty)) \oplus R^1\pi_*(\mathcal{E}(-\ell_\infty + E)).$$

Recall that the map $\mathcal{M}_k^n \hookrightarrow \mathcal{M}_k^{n+1}$ is defined by the direct sum with the trivial connection: $[A] \mapsto [A \oplus \theta]$. In terms of holomorphic bundles this is $\mathcal{E} \hookrightarrow \mathcal{E} \oplus \mathcal{O}$. Tracing through the monod construction, it is easy to see that the inclusion induces the $G$-equivariant map $A_k^n \hookrightarrow A_k^{n+1}$ given by $(a_1, a_2, x, b, c) \mapsto (a_1, a_2, x, b', c')$ where $b'$ is $b$ with an extra first column of zeroes and $c'$ is $c$ with an extra first row of zeroes. Define $A_k^\infty$ to be the direct limit $\lim_{n \to \infty} A_k^n$ so that there is a homeomorphism between $\mathcal{M}_k^n$ and $A_k^n/G$.

**Lemma 3.3.** $A_k^\infty$ is a contractible space.

**Proof.** Since the $A_k^n$'s are algebraic varieties and the maps $A_k^n \to A_k^{n+1}$ are algebraic, they admit triangulations compatible with the maps. Thus $A_k^\infty$ inherits the structure of a CW-complex and so it is sufficient to show that all of its homotopy groups are zero. To this end we prove that for any $k$ and $l$ there is an $r > l$ such that the natural inclusion from $A_k^n \hookrightarrow A_k^r$ is homotopically trivial.

Consider the homotopy $H_t : A_k^n \to A_k^{2k+n}$ defined as follows:

$$H_t((a_1, a_2, x, b, c)) = ((1-t)a_1, (1-t)a_2, (1-t)x, b_t, c_t)$$

where

$$c_t = \begin{pmatrix} tI_k \\ 0_{k,k} \\ (1-t)c \end{pmatrix}, \quad b_t = (0_{k,k}, tI_k, (1-t)^2b),$$

$I_k$ is the $k \times k$ identity matrix and $0_{k,k}$ is the $k \times k$ zero matrix. To see that $H_t(v) \in A_k^{2k+n}$ for any $v \in A_k^n$, we check that the integrability and non-degeneracy conditions are satisfied for all $0 \leq t \leq 1$. Integrability holds because $b_t c_t = (1-t)^3bc$. Non-degeneracy is satisfied for all $t \neq 0$ because there is a full rank $k \times k$ block, $tI_k$, in both $c_t$ and $b_t$. Furthermore, $H_0$ is just the inclusion $A_k^n \hookrightarrow A_k^{2k}$, so non-degeneracy also holds when $t = 0$. Finally, note that $H_1$ is a constant map.

These lemmas show that $A_k^\infty$ is a contractible space acted on freely by $G = \text{Gl}(k) \times \text{Gl}(k)$ and $A_k^\infty/G = \mathcal{M}_k^\infty$. Thus $\mathcal{M}_k^\infty$ is homotopic to $BG$ which in turn has the homotopy type of $BU(k) \times BU(k)$. We end by remarking that the proof shows that the universal $U(k) \times U(k)$ bundle is the bundle that restricts to any of the finite $\mathcal{M}_k^n$'s as $R^1\pi_*(\mathcal{E}(-\ell_\infty)) \oplus R^1\pi_*(\mathcal{E}(-\ell_\infty + E))$.

**References**


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Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, California 94720-5070

E-mail address: jbryan@msri.org

Department of Mathematics and Computer Science, Dickinson College, Carlisle, Pennsylvania 17013

E-mail address: sandersm@dickinson.edu