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HYPERBOLIC SURFACES IN $\mathbb{P}^3(\mathbb{C})$

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Abstract. We show a class of perturbations X of the Fermat hypersurface such that any holomorphic curve from $\mathbb{C}$ into X is degenerate. Applying this result, we give explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ of arbitrary degree $d \geq 22$, and of curves of arbitrary degree $d \geq 19$ in $\mathbb{P}^2(\mathbb{C})$ with hyperbolic complements.

1. Introduction

A holomorphic curve in a projective variety X is said to be degenerate if it is contained in a proper algebraic subset of X. In 1979 ([GG]) M. Green and Ph. Griffiths conjectured that every holomorphic curve in a complex projective variety of general type is degenerate. Up to now this conjecture still seems far from completely proved, but there has been some progress. M. Green ([G]) proved the degeneracy of holomorphic curves, in the Fermat variety of large degree. In [N] A. M. Nadel gives a class of projective hypersurfaces for which the conjecture is valid. Using the results on degeneracy of holomorphic curves, Nadel constructed some explicit examples of hyperbolic hypersurfaces in $\mathbb{P}^3$.

In this note, we first consider perturbations X of the Fermat hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$ such that for some fixed $k \geq 0$ each monomial in the defining polynomial of X contains every homogeneous coordinate of power either 0, or at least $d-k$. We show that if d is large enough with respect to n and to the number of non-zero monomials in the defining polynomial, then for such a hypersurface, any holomorphic map $\mathbb{C} \to X$ is degenerate.

Secondly, we apply the above result to give explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ of arbitrary degree $d \geq 22$, and curves in $\mathbb{P}^2(\mathbb{C})$ with hyperbolic complements of arbitrary degree $d \geq 19$. Notice that up to now all known explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ are of degree d divided by some integer > 1 (2 in Brody-Green's example, 3 in Nadel's example, 3 or 4 in Masuda-Noguchi's examples). Indeed, in [MN] an algorithm is given to construct hyperbolic surfaces of degree $d > 54$.

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2. Degeneracy of holomorphic curves

Let

$$M_j = z_1^{a_{j,1}} \cdots z_{n+1}^{a_{j,n+1}}, \qquad 1 \le j \le s,$$

be distinct monomials of degree d with non-negative exponents. Let X be a hypersurface of degree d of $\mathbb{P}^n(\mathbb{C})$ defined by

$$X\colon c_1 M_1 + \cdots + c_s M_s = 0$$

where $c_j \in \mathbb{C}^*$ are non-zero constants. We call X a *perturbation of the Fermat hypersurface of degree d* if $s \ge n+1$ and

$$M_j = z_j^d, \qquad j = 1, \dots, n+1.$$

Theorem 2.1. *Suppose that there is an integer $k \ge 0$ such that X satisfies the following conditions*:

(i) *For $j \ge n+2$, $m = 1, \dots, n+1$, the exponent $\alpha_{j,m}$ is either 0, or $\alpha_{j,m} \ge d-k$.*

(ii) $d > k + s(s-2)$.

Then every holomorphic curve in X is degenerate.

To prove Theorem 2.1 let us recall Cartan's defect relation for holomorphic curves ([C], see also [MN]).

Let f be a holomorphic curve and H be a hyperplane of $\mathbb{P}^n(\mathbb{C})$ which does not contain the image of f. We denote by $\deg_z f^*H$ the degree of the pull-backed divisor f^*H at $z \in \mathbb{C}$. We say that f ramifies at least d (> 0) over H if $\deg_z f^*H \ge d$ for all $z \in f^{-1}H$. In case $f^{-1}H = 0$, we set $d = \infty$.

Lemma 2.2 (H. Cartan [C]). *Assume that f is linearly non-degenerate and ramifies at least d over H_j, $1 \le j \le q$, where the hyperplanes H_j are in general position. Then*

$$\sum_{j=1}^{q} \left(1 - \frac{n}{d_j}\right) \le n+1.$$

Now let X be a hypersurface satisfying the hypothesis of Theorem 2.1, and let $f = (f_1, \dots, f_{n+1}) : \mathbb{C} \to X$ be a holomorphic curve. We are going to show that $\{f_1^d, \dots, f_{n+1}^d, M_{n+2} \circ f, \dots, M_s \circ f\}$ are linearly dependent. Suppose that it is not the case. Consider a holomorphic curve g in $\mathbb{P}^{s-2}(\mathbb{C})$ defined by

$$g\colon z \in \mathbb{C} \mapsto (f_1^d(z), \dots, f_{n+1}^d(z), M_{n+2} \circ f(z), \dots, M_{s-1} \circ f(z)) \in \mathbb{P}^{s-2}(\mathbb{C}).$$

Take the following hyperplanes in general position:

$$H_1 = \{z_1 = 0\}, \dots, H_{s-1} = \{z_{s-1} = 0\},\ H_s = \{c_1 z_1 + \cdots + c_{s-1} z_{s-1} = 0\}.$$

By the hypothesis of Theorem 2.1 we see that g ramifies at least $d-k$ over H_j for all $1 \le j \le s$. It follows from Lemma 2.2 that

$$\sum_{j=1}^{s} \left(1 - \frac{s-2}{d-k}\right) \le s-1. \tag{1}$$

Hence $d \le k + s(s-2)$, a contradiction. Then the image of f is contained in the proper algebraic subset of X defined by the equation

$$a_1 z_1^d + \cdots + a_{n+1} z_{n+1}^d + a_{n+2} M_{n+2} + \cdots + a_{s-1} M_{s-1} = 0,$$

where not all a_j are zeros. Theorem 2.1 is proved. □

Corollary 2.3 (M. Green [G]). *Let X be the Fermat hypersurface*

$$X\colon z_1^d + \cdots + z_{n+1}^d = 0,$$

and let $f = (f_1, \ldots, f_{n+1})$ be a holomorphic curve in X. If $d > n^2 - 1$, there is a decomposition of indices $\{1, \ldots, n+1\} = \bigcup I_\xi$ such that:

(i) *If $i, j \in I_\xi$, $f_i/f_j = \text{const.}$*

(ii) *$\sum_{i\in I_\xi} f_i^d = 0$ for any ξ.*

Proof. It suffices to take $k = 0$, $s = n+1$ in Theorem 2.1, and apply Theorem 2.1 repeatedly. The corollary is then proved by induction. Notice that the hypothesis of Theorem 2.1 is fulfilled after every step of induction. □

The following more precise form of Theorem 2.1 is very useful in applications to surfaces in $\mathbb{P}^3(\mathbb{C})$.

Theorem 2.4. *Let X be a hypersurface satisfying the hypothesis of Theorem 2.1, where the inequality* (ii) *is replaced by a weaker one*:

$$\frac{(n+1)(s-2)}{d} + \frac{(s-2)(s-n-1)}{d-k} < 1.$$

Then any holomorphic curve in X is degenerate.

Proof. We can repeat the proof of Theorem 2.1, but instead of (1) we use the inequality

$$\sum_{j=1}^{n+1}\left(1 - \frac{s-2}{d}\right) + \sum_{j=n+2}^{s}\left(1 - \frac{s-2}{d-k}\right) \le s-1. \tag{2}$$

□

3. Hyperbolic surfaces

In this section we use Theorems 2.1 and 2.4 to give explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ and of curves in $\mathbb{P}^2(\mathbb{C})$ with hyperbolic complements.

Theorem 3.1. *Let X be a surface in $\mathbb{P}^3(\mathbb{C})$ of degree d defined by the equation*

$$X\colon z_1^d + z_2^d + z_3^d + z_4^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0, \tag{3}$$

where $c \neq 0$, $\sum_{i=1}^3 \alpha_i = d$, $\alpha_i \ge 7$. Then X is hyperbolic if $d \ge 22$.

Proof. Take $k = d-7$. Then X satisfies the hypothesis of Theorem 2.4, and every holomorphic curve in X is degenerate.

Now let $f = (f_1, f_2, f_3, f_4)\colon \mathbb{C} \to X$ be a holomorphic curve in X. Consider the following possible cases:

1) For some $i = 1, 2, 3$, $f_i \equiv 0$. Then f is a constant map by Corollary 2.3.

2) $f_4 \equiv 0$. Then the image of (f_1, f_2, f_3) is contained in the curve defined by the equation

$$Y\colon z_1^d + z_2^d + z_3^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0.$$

From the proof of Theorem 2.1 it follows that $\{f_1^d, f_2^d, f_3^d\}$ are linearly dependent. Then at least two of $\{f_1, f_2, f_3\}$, say, f_1 and f_2, have a constant ratio. Substitute this relation into (3) and show that f is a constant map (note that $\alpha_i \neq 0$ for all $i = 1, 2, 3$).

3) Assume that any $f_i \not\equiv 0$. From the proof of Theorem 2.1 it follows that $\{f_1^d, \dots, f_4^d\}$ are linearly dependent:

$$a_1 f_1^d + \cdots + a_4 f_4^d \equiv 0,$$

where not all a_i are zeros. Consider the following possible cases:

(i) $a_i \neq 0$, $i = 1, \dots, 4$. By Corollary 2.3, f is a constant map, or we can assume that $f_1 = c_1 f_2, f_3 = c_2 f_4$. Then we can substitute this relation into (3) and show that f is a constant map.

(ii) Only one of $a_i = 0$, say $a_4 = 0$. Then (f_1, f_2, f_3) is a constant map by Corollary 2.3, and it is easy to show that f is a constant map.

(iii) $a_4 \neq 0$, and two coefficients, say, $a_1 = a_2 = 0$. Then we have $f_3 = c_3 f_4$. Substitute this relation into (3) and we obtain

$$f_1^d + f_2^d + \varepsilon_1 f_3^d + \varepsilon_2 f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} \equiv 0, \tag{4}$$

where $\varepsilon_2 \neq 0$. We return to case 2).

(iv) $a_4 = 0$, and one of a_1, a_2, a_3, say, $a_1 = 0$. Then f_2/f_3 is a constant, and we obtain:

$$f_1^d + A f_3^d + f_4^d + B f_1^{\alpha_1} f_3^{\alpha_2+\alpha_3} = 0,$$

where $B \neq 0$. If $A \neq 0$, then $\{f_1^d, f_3^d, f_4^d\}$ are linearly dependent, again by the proof of Theorem 2.1 and we return to the case similar to 2).

Now suppose that $A = 0$. Then the image of the map (f_1, f_3, f_4) is contained in the following curve in $\mathbb{P}^2(\mathbb{C})$ (with homogeneous coordinates (z_1, z_3, z_4)):

$$Y\colon z_1^d + z_4^d + B z_1^{\alpha_1} z_3^{\alpha_2+\alpha_3} = 0.$$

We are going to show that under the hypothesis of Theorem 3.1, the genus of Y is at least 2.

The genus of Y is equal to the number of integer points in the triangle with the vertices $(d,0), (0,d)$ and $(\alpha_1, 0)$ (see, for example, [Ho]). It is easy to see that this triangle contains at least two integer points, if $\alpha_1 < d-2$. Here, by the hypothesis we have $\alpha_1 = d - (\alpha_2 + \alpha_3) \leq d - 14$. The proof is completed. □

Remark 1. In [MN] K. Masuda and J. Noguchi proved that for every n there is a number $d(n)$ such that for every $d \geq d(n)$ there are hyperbolic hypersurfaces of degree d in $\mathbb{P}^n(\mathbb{C})$. They pointed out that $d(3) \leq 54$. In [N] M. Nadel gives explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ of degree $d = 3e$, $e \geq 7$. From Theorem 3.1 it follows that $d(3) \leq 22$. Combining Theorem 3.1 with Nadel's results ([N]) we have $d(3) \leq 21$.

Remark 2. It is clear that we can take $c z_i^{\alpha_i} z_j^{\alpha_j} z_l^{\alpha_l}$ in the equation (3) instead of $c z_1^{\alpha_1} z_2^{\alpha_2}, z_3^{\alpha_3}$, for any triple (z_i, z_j, z_l) from (z_1, z_2, z_3, z_4).

Remark 3. From the proof of Theorem 3.1 it follows that the following surfaces are hyperbolic:

$$X\colon z_1^d + z_2^d + z_3^d + z_4^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} = 0, \tag{5}$$

where $c \neq 0$, $\sum_{i=1}^4 \alpha_i = d$, $\alpha_i \geq 6$, $d > 24$. In fact, it suffices to take $k = d - 6$ and repeat the proof of Theorem 3.1.

Theorem 3.2. *Let X be a curve in $\mathbb{P}^2(\mathbb{C})$ defined by the equation*

$$X\colon z_1^d + z_2^d + z_3^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0,$$

where $c \neq 0$, $\alpha_i \geq 6$. If $d \geq 19$, then $\mathbb{P}^2(\mathbb{C})\backslash X$ is completely hyperbolic and hyperbolically imbedded into $\mathbb{P}^2(\mathbb{C})$.

Proof. Due to R. Brody and M. Green ([BG]), X is hyperbolic and the complement $\mathbb{P}^2(\mathbb{C})\backslash X$ is completely hyperbolic and hyperbolically imbedded into $\mathbb{P}^2(\mathbb{C})$ if and only if neither X nor $\mathbb{P}^2(\mathbb{C})$ admits a non-constant holomorphic curve from $\mathbb{C}$. By the proof of Theorem 3.1 it suffices to prove that any holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^2\backslash X$ is constant.

Let $f = (f_1, f_2, f_3)$ be such a curve. Consider the surface Y defined by the following equation:

$$Y\colon z_1^d + z_2^d + z_3^d + z_4^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0.$$

Let

$$\varphi\colon Y\backslash\{z_4 = 0\} \to \mathbb{P}^2(\mathbb{C})\backslash X$$

be the projection of the first three homogeneous coordinates. Then φ is an unramified covering, and f may be lifted to $\tilde{f} = (f_1, f_2, f_3, f_4)\colon \mathbb{C} \to Y\backslash\{z_4 = 0\}$.

Now we will show that under the hypothesis of Theorem 3.2, $\tilde{f}$ is degenerate in Y. In fact, if it is not the case, then we take $k = d - 6$ and repeat the proof of Theorem 2.4. Note that $f_4 \neq 0$, and making use of Lemma 2.2, we take $d_4 = \infty$. Therefore, instead of the inequality (2) we obtain

$$3\left(1 - \frac{5-2}{d}\right) + 1 + \left(1 - \frac{5-2}{6}\right) \leq 5 - 1.$$

It is impossible when $d \geq 19$.

Hence, by the proof of Theorem 3.1, if Y is hyperbolic, then $\tilde{f}$ is constant, and so is f. The theorem is proved. □

Remark 4. M. G. Zaidenberg ([Z1]) proved that for $d \geq 5$ there are hyperbolic curves of degree d such that their complements are completely hyperbolic and hyperbolically imbedded into $\mathbb{P}^2(\mathbb{C})$. In [MN] K. Masuda and J. Noguchi give the construction of such curves with $d \geq 48$. Here we have examples with $d \geq 19$.

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