

## SUBLATTICES OF THE BANACH ENVELOPE OF WEAK $L^1$

HEINRICH P. LOTZ AND N. T. PECK

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ABSTRACT. We prove that every separable Banach lattice is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .

For a finite nonatomic separable measure space  $(X, \Sigma, \mu)$  we study lattice isometric and order isometric embeddings into the Banach envelope  $\mathbb{W}$  of the Lorentz space  $L(1, \infty)$ . It was shown in [4] that  $L^1$  and  $\ell^\infty$  are lattice isometric to closed sublattices of  $\mathbb{W}$ , and it was shown in [9] that a separable Banach lattice with order continuous norm is lattice isometric to a closed sublattice of  $\mathbb{W}$ . Using a different approach than in [4] and [9], we show in this paper that *every* separable Banach lattice is lattice isometric to a closed sublattice of  $\mathbb{W}$  and that  $L^\infty$  as well as Weak  $L^p$ ,  $1 < p < \infty$ , of a separable measure space is lattice isometric to a closed sublattice of  $\mathbb{W}$ . See also [7], in which the first author has given a characterization of the dual space of  $L(1, \infty)$ .

Let  $(X, \Sigma, \mu)$  be a measure space. The Lorentz space  $L(1, \infty)$  consists of all (equivalence classes of)  $\mu$ -measurable functions  $f$  on  $X$  for which

$$\| \| f \| \| = \sup_{s>0} s\mu\{|f| > s\} < \infty.$$

This space with the topology defined by the quasinorm  $\| \| \cdot \| \|$  is called Weak  $L^1$ . Let  $U$  be the convex hull of  $V = \{f : \| \| f \| \| \leq 1\}$  and let  $\|\cdot\|$  be the Minkowski functional of  $U$ . Since  $V$  is solid, i.e.,  $f \in V, |g| \leq |f|$  implies  $g \in V$ ,  $U$  is solid. Hence  $\|\cdot\|$  is a lattice seminorm on  $L(1, \infty)$  and  $\|\cdot\| \leq \| \| \cdot \| \|$ . Let  $I$  be the ideal  $\{f : \| \| f \| \| = 0\}$ . Then  $L(1, \infty)/I$  with the quotient order and the norm  $\| \| f + I \| \| = \| \| f \| \|$  is a normed vector lattice. Its completion  $\mathbb{W}$  is a Banach lattice, which is called the Banach envelope of Weak  $L^1$ . By identifying  $f$  with  $f + I$ , we can identify Weak  $L^1$  with a dense sublattice of  $\mathbb{W}$ .

It has been shown in [1] and [2] that in the nonatomic case

$$\| \| f \| \| = \limsup_{q/p \rightarrow \infty} \int_{p < |f| \leq q} |f| d\mu / \ln \frac{q}{p}.$$

We will only consider Weak  $L^1$  and its Banach envelope in the case where  $(X, \Sigma, \mu)$  is a finite nonatomic separable measure space. Thus, we can restrict ourselves to the case  $X = (0, 1]$  and  $\mu = \lambda$ , where  $\lambda$  denotes Lebesgue measure on

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$(0, 1]$ . Throughout this paper,  $\lambda$  denotes Lebesgue measure on  $(0, 1]$  and  $L^1$  denotes the space  $L^1((0, 1], \lambda)$ .

Let  $E$  and  $F$  be Banach lattices. A linear operator  $T$  from  $E$  into  $F$  is called a *lattice homomorphism* if  $|Tx| = |x|$  for every  $x \in E$ . If, in addition,  $\|Tx\| = \|x\|$  for all  $x \in E$ , then  $T$  is called a *lattice isometry*.

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We start with embeddings of  $L^1$  and  $L^\infty(0, \infty)$  into Weak  $L^1$  and its envelope  $\mathbb{W}$ . We use the unit square  $X = (0, 1] \times (0, 1]$  with Lebesgue measure  $\mu$  rather than the unit interval (cf. g) of the proof of Theorem 1 below).

**Proposition 1.** For  $f \in L^1$  define  $Tf$  by

$$(Tf)(s, t) = f(t)/s, \quad (s, t) \in X.$$

Then  $T$  is a lattice isometry from  $L^1$  into the Banach envelope of Weak  $L^1$  on  $X$  and a lattice “isometry” into the quasinormed space Weak  $L^1$ .

*Proof.* An easy calculation shows that

$$\|f\| = \sup_{p>0} p \int_0^1 \int_0^{\frac{|f(t)|}{p} \wedge 1} ds dt = \|f\|$$

and that

$$\|Tf\| = \limsup_{\substack{q/p \rightarrow \infty \\ 0 < p < q}} \int_0^1 \int_{\frac{|f(t)|}{q} \wedge 1}^{\frac{|f(t)|}{p} \wedge 1} \frac{|f(t)|}{s} ds dt \Big/ \ln \frac{q}{p} = \|f\|.$$

□

**Proposition 2.** There exists a lattice homomorphism  $T$  from the Banach lattice  $L^\infty(0, \infty)$  into Weak  $L^1$  with

$$\|Tf\| = \|Tf\| = \|f\|$$

for all  $f \in L^\infty$ .

*Proof.* Let  $\{(u_n, v_n) : u_n < v_n\}$  be an enumeration of the subintervals of  $(0, \infty)$  with rational endpoints. Define  $\psi_n : (0, 1] \rightarrow (u_n, v_n]$  by  $\psi_n(t) = u_n + (v_n - u_n)t$ . For  $f \in L^\infty$  let  $f_n = f \circ \psi_n$ . Then

$$\|f\| = \limsup \int |f \circ \psi_n| d\lambda.$$

Choose  $0 < a_n < b_n \leq 1$  with  $b_{n+1} < a_n$  and  $\lim \frac{b_n}{a_n} = \infty$ . For  $f \in L^\infty$  and  $0 < s, t \leq 1$  define

$$(Tf)(s, t) = \begin{cases} \frac{f_n(t)}{s} & \text{for } a_n \leq s \leq b_n, \\ 0 & \text{for } s \notin \cup [a_n, b_n]. \end{cases}$$

Suppose that  $f \in L^\infty$  and that  $\|f\| = 1$ . Then  $|(Tf)(s, t)| \leq 1/s$  and hence  $\|Tf\| \leq 1$ . Fix  $0 < \delta < 1$  and let  $p_n = \delta/b_n$  and  $q_n = 1/a_n$ . If  $a_n \leq s \leq b_n$  and

$\delta < |f(t)|$ , then  $p_n < |(Tf)(s, t)| \leq q_n$ . Hence

$$\begin{aligned} \int_{p_n < |Tf| \leq q_n} |Tf| d\mu &\geq \int_{|f_n| > \delta} \int_{a_n}^{b_n} \frac{|f_n(t)|}{s} ds d\lambda(t) \\ &= \int_{|f_n| > \delta} |f_n| d\lambda \left( \ln \frac{q_n}{p_n} + \ln \delta \right). \end{aligned}$$

Therefore,  $\|Tf\| \geq \limsup \int_{|f_n| > \delta} |f_n| d\lambda$ . Since  $\limsup \int |f_n| d\lambda = 1$  and  $|f_n| \leq 1$ , there exists a subsequence  $(f_{n_i})$  which converges  $\lambda$ -uniformly to the function 1. Hence  $\limsup \int_{|f_n| > \delta} |f_n| d\lambda = 1$ . Consequently,  $1 \leq \|Tf\|$ , and thus  $\|Tf\| = \|Tf\| = 1$ .  $\square$

In order to embed other Banach lattices into  $\mathbb{W}$  a somewhat more elaborate construction is needed. We will construct lattice homomorphisms from nonclosed sublattices of  $(\sum \oplus L^1)_{\ell^\infty}$  into Weak  $L^1$  which are isometries for  $\|\cdot\|$  but which are not continuous for  $\|\cdot\|$  (see Remark 1 below), in contrast to the situation of Propositions 1 and 2.

At first we need a lemma which actually characterizes Weak  $L^1$  functions on any measure space.

**Lemma 1.** *Let  $(X, \Sigma, \mu)$  be a measure space and let  $f$  be measurable,  $f$  finite a.e. If there exist constants  $\gamma$  and  $K, K > 1$ , such that  $\int_{K^i < |f| \leq K^{i+1}} |f| d\mu \leq \gamma$  for all  $i \in \mathbb{Z}$ , then  $f \in L(1, \infty)$ .*

*Proof.* Let  $A_i = \{t : K^i < |f(t)| \leq K^{i+1}\}$ . The condition implies that  $K^i \mu(A_i) \leq \gamma$ . Let  $s > 0$  be given. Then for some  $j \in \mathbb{Z}, K^j \leq s < K^{j+1}$ . Therefore,  $s\mu\{|f| > s\} \leq K^{j+1}\mu\{|f| > K^j\} \leq K^{j+1} \sum_{i=j}^{\infty} \mu(A_i) \leq \gamma K^2 / (K - 1)$ . Hence,  $f \in L(1, \infty)$ .  $\square$

**Theorem 1.** *Let  $E = (\sum \oplus L^1)_{\ell^\infty}$  and let  $M_i \geq 1$  for  $i \in \mathbb{N}$ . Then the closure of the sublattice  $G = \{(h_i) \in E : \sup M_i^{-1} \|h_i\|_\infty < \infty\}$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .*

*Proof.* a) Choose  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  such that for every  $i \in \mathbb{N}$  the set  $\varphi^{-1}(\{i\})$  is infinite. Let  $d_n = \max_{1 \leq j \leq n} M_{\varphi(j)}$ . Then the operator  $S : (h_i) \mapsto (f_n)$ , where  $f_n = h_{\varphi(n)}$ , is a lattice isometry from  $E$  into itself which maps  $G$  into

$$F = \{(f_n) : \sup d_n^{-1} \|f_n\|_\infty < \infty\}.$$

Let  $X = (0, 1] \times (0, 1]$  with Lebesgue measure  $\mu$ . We will construct a lattice homomorphism  $T$  from  $F$  into the Banach envelope of Weak  $L^1$  on  $X$  with  $\|T\| \leq 1$  and  $\|Tf\| \geq \|f\|$  whenever  $f \in SG$ .

b) Choose  $a_n > 0$  and  $b_n > 0$  such that

- i)  $b_1 = 1$ ,
- ii)  $a_{n+1} < b_{n+1} < a_n < b_n$ ,
- iii)  $\lim \frac{b_n}{a_n} = \infty$ ,
- iv)  $\ln \frac{a_l}{b_k d_k} \geq \sum_{i=l}^k \ln \frac{b_i}{a_i}$  for  $l < k$ .

We denote the set  $\{(s, t) : a_n \leq s \leq b_n, 0 < t \leq 1\}$  by  $A_n$ .

c) For  $f = (f_n) \in F$  and  $0 < s, t \leq 1$  define

$$(Tf)(s, t) = \begin{cases} \frac{f_n(t)}{s} & \text{for } (s, t) \in A_n, \\ 0 & \text{for } (s, t) \notin \cup A_n. \end{cases}$$

Obviously,  $T$  defines a lattice homomorphism from  $F$  into the space of measurable functions on the square  $X$ . We note that  $\int_{A_n} |Tf| d\mu = \|f_n\| \ln \frac{b_n}{a_n}$ .

d) Let  $f = (f_n) \in F$  be given and let  $\gamma = \sup d_n^{-1} \|f_n\|_\infty$ . Suppose that there exists  $\delta > 0$  such that  $f_n(t) \geq \delta$  for all  $n$  and all  $t$ . Fix  $p$  and  $q$ ,  $0 < p < q$ , and let  $A = \{(s, t) : p < Tf(s, t) \leq q\}$ . We claim that

$$(\dagger) \quad \int_A Tf d\mu \leq \|f\| \max \left\{ \ln \frac{q}{p}, \ln \frac{q\gamma}{p\delta} \right\}.$$

We may assume that  $A$  is not empty. Let  $B_n = A \cap A_n$ . Then  $A = \cup B_n$ . If  $(s, t) \in B_n$ , then  $\delta/q \leq f_n(t)/q \leq s < f_n(t)/p \leq \gamma d_n/p$ . Let  $l$  and  $k$  be the minimum and maximum respectively of  $\{n : B_n \neq \Phi\}$ . We consider two cases:

1) If  $l = k$ , let  $a(t) = \min\{\max\{a_k, \frac{f_k(t)}{q}\}, b\}$ ,  $b(t) = \max\{\min\{b_k, \frac{f_k(t)}{p}\}, a\}$ ; then

$$\begin{aligned} \int_A Tf d\mu &= \int_0^1 \int_{a(t)}^{b(t)} \frac{f_k(t)}{s} ds dt \\ &\leq \int_0^1 \int_{\frac{f_k(t)}{q}}^{\frac{f_k(t)}{p}} \frac{f_k(t)}{s} ds dt \\ &= \|f_k\| \ln \frac{q}{p}. \end{aligned}$$

2) If  $l < k$ , then  $\delta/q \leq b_k \leq a_l \leq \gamma d_l/p \leq \gamma d_k/p$ . Hence  $a_l/b_k d_k \leq q\gamma/p\delta$ . Therefore,

$$\begin{aligned} \int_A Tf d\mu &\leq \sum_{n=l}^k \int_{B_n} Tf d\mu \leq \sum_{n=l}^k \|f_n\| \ln \frac{b_n}{a_n} \\ &\leq \|f\| \sum_{n=l}^k \ln \frac{b_n}{a_n} \leq \|f\| \ln \frac{a_l}{b_k d_k} \leq \|f\| \ln \frac{q\gamma}{p\delta}. \end{aligned}$$

This proves  $(\dagger)$ . It follows easily from Lemma 1 that  $Tf \in L(1, \infty)$ . Moreover,

$$\begin{aligned} \|Tf\| &= \limsup_{q/p \rightarrow \infty} \int_A |Tf| d\mu / \ln \frac{q}{p} \\ &= \limsup_{q/p \rightarrow \infty} \int_A Tf d\mu / \ln \frac{q}{p} \\ &\leq \|f\|. \end{aligned}$$

e) Let  $g = (g_n) \in F$  be arbitrary. Choose  $\eta > 0$  and let  $f_n = (|g_n| + \eta)$ . Then  $f = (f_n) \in F$  and satisfies the assumptions of d). Hence  $Tf \in L(1, \infty)$  and  $\|Tf\| \leq \|f\| = \|g\| + \eta$ . Since  $|Tg| \leq Tf$ ,  $Tg \in L(1, \infty)$  and  $\|Tg\| \leq \|g\| + \eta$ . Since  $\eta > 0$  is arbitrary,  $\|Tg\| \leq \|g\|$ . This shows that  $T$  maps  $F$  into Weak  $L^1$  and that  $\|Tf\| \leq \|f\|$  for every  $f \in F$ .

f) Let  $f = (f_n) \in F$  be given. Fix  $\alpha$  and  $\beta$ ,  $0 < \alpha < \beta$ , and let  $p_n = \alpha/b_n$  and  $q_n = \beta/a_n$ . Then

$$\begin{aligned} \int_{p_n < |Tf| \leq q_n} |Tf| d\mu &\geq \int_{\alpha < |f_n(t)| \leq \beta} \int_{a_n}^{b_n} \frac{|f_n(t)|}{s} ds dt \\ &= \left( \int_{\alpha < |f_n(t)| \leq \beta} |f_n(t)| dt \right) \ln \frac{b_n}{a_n}. \end{aligned}$$

Since  $q_n/p_n$  tends to  $\infty$ ,

$$\|Tf\| \geq \limsup \int_{\alpha < |f_n(t)| \leq \beta} |f_n(t)| dt .$$

If  $f = Sh$ , where  $h = (h_i) \in G$ , then for every  $i$  there exist infinitely many  $n$ 's with  $f_{\varphi(n)} = h_i$ . Therefore,

$$\|TSh\| \geq \int_{\alpha < |h_i(t)| \leq \beta} |h_i(t)| dt .$$

Consequently,  $\|TSh\| \geq \sup \|h_i\| = \|h\|$ .

g) Finally, let  $\psi : (0, 1] \rightarrow X$  be a measure-preserving transformation. Then  $U : f \mapsto (TSF) \circ \psi$  is a lattice isometry from  $G$  into the Banach envelope of Weak  $L^1$  on  $(0, 1]$  which can be extended to a lattice isometry from the closure of  $G$  into  $\mathbb{W}$ .  $\square$

**Corollary 1.** *The sublattice  $H = \{(f_n) : \{f_n\} \text{ is relatively weakly compact in } L^1\}$  of  $(\sum \oplus L^1)_{\ell^\infty}$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .*

*Proof.* It is easily checked that  $H$  is the closure of the sublattice  $F = \{(f_i) : \sup \|f_i\|_\infty < \infty\}$  of  $(\sum \oplus L^1)_{\ell^\infty}^\infty$ . Now the Corollary follows from Theorem 1 with  $M_i = 1$ .  $\square$

**Corollary 2.** *For every sequence  $(k_i)$  of natural numbers, the Banach lattice  $(\sum \oplus \ell^1(k_i))_{\ell^\infty}$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .*

*Proof.* Clearly, for every  $i$  there exists a lattice isometry  $T_i$  from  $\ell^1(k_i)$  into  $L^1$  with  $\|T_i x\|_\infty \leq k_i \|x\|$  for all  $x \in \ell^1(k_i)$ . Hence,  $T : (x_i) \mapsto (T_i x_i)$  is a lattice isometry from  $(\sum \oplus L^1)_{\ell^\infty}$  into the sublattice  $\{(f_i) : \sup k_i^{-1} \|f_i\|_\infty < \infty\}$  of  $(\sum \oplus L^1)_{\ell^\infty}$ .  $\square$

*Remark 1.* The map  $T$  from  $F$  into Weak  $L^1$  in the proof of Theorem 2 is not  $\|\cdot\|$ -continuous (this follows from Theorem 5 below). However, the restriction to any sublattice  $F_1$  of  $F$  will be continuous if  $F_1$  is closed in  $(\sum \oplus L^1)_{\ell^\infty}$  (indeed, every positive linear map from a Banach lattice into  $(\text{Weak } L^1, \|\cdot\|)$  is continuous). Consequently, if  $G_1$  is a sublattice of the space  $G$  in Theorem 2 and  $G_1$  is closed in  $(\sum \oplus L^1)_{\ell^\infty}$ , then the restriction of  $TS$  is a topological isomorphism from  $G_1$  into  $(\text{Weak } L^1, \|\cdot\|)$ . In particular, the space  $(\sum \oplus \ell^1(k_i))_{\ell^\infty}$  is lattice isomorphic to a closed sublattice of  $(\text{Weak } L^1, \|\cdot\|)$ .

## 2

A positive element  $u$  of a Banach lattice  $E$  is called a *quasi-interior point* if the interval  $[0, u] = \{x : 0 \leq x \leq u\}$  is total in  $E$ . Every separable Banach lattice has a quasi-interior point.

Every Banach lattice is lattice isometric to a closed sublattice of an  $\ell^\infty$ -product of AL-spaces ([5], Lemma 3.4). We will show that if  $E$  is a separable Banach lattice or if  $E$  is the dual of a separable Banach lattice and has a quasi-interior point, then  $E$  is lattice isometric to a closed sublattice of the Banach lattice  $H$  of Corollary 1.

**Lemma 2.** *Let  $E$  be a separable Banach lattice, let  $0 \leq u \in E$  be a quasi-interior point, and let  $0 < x' \in E'$ . Then there exists a lattice homomorphism  $T$  from  $E$  into  $L^1$  such that  $\|Tx\| = \langle |x|, x' \rangle$  for every  $x \in E$  and  $Tu = \langle u, x' \rangle 1$ .*

*Proof.* Let  $I$  be the closed ideal  $\{x : \langle |x|, x' \rangle = 0\}$  and let  $F$  be the completion of  $E/I$  with the norm  $\|x+I\| = \langle |x|, x' \rangle$ . Then  $F$  is an AL-space and the canonical map  $\varphi$  from  $E$  into  $F$  is a lattice homomorphism with  $\|\varphi(x)\| = \langle |x|, x' \rangle$  for every  $x \in E$ . Thus,  $\varphi$  is continuous and therefore  $F$  is separable. Moreover,  $v = \langle u, x' \rangle^{-1} \varphi(u)$  is a weak order unit of  $F$  with  $\|v\| = 1$ . Hence by Kakutani's theorem there is a measure space  $(S, \Sigma, \nu)$ ,  $\nu(s) = 1$ , and a lattice isometry  $T_1$  from  $F$  onto  $G = L^1(S, \Sigma, \nu)$  which maps  $v$  onto the function 1. Since  $G$  is separable, there is a lattice isometry  $T_2$  from  $G$  into  $L^1$  with  $T_2 1 = 1$ . Thus,  $T = T_2 \circ T_1 \circ \varphi$  is the desired lattice homomorphism from  $E$  into  $L^1$ .  $\square$

**Theorem 2.** *Every separable Banach lattice  $E$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .*

*Proof.* We may assume that  $E \neq \{0\}$ . Since  $E$  is separable,  $E$  has a quasi-interior point  $u$ . Also, there is a normalized sequence  $(x'_i) \subset E'$ ,  $x'_i \geq 0$ , such that  $\|x\| = \sup \langle |x|, x'_i \rangle$  for every  $x \in E$ . By Lemma 2, there exist lattice homomorphisms  $T_i : E \rightarrow L^1$  such that  $\|T_i x\| = \langle |x|, x'_i \rangle$  for every  $x \in E$  and  $T_i u = \langle u, x'_i \rangle 1$ . Obviously,  $x \mapsto (T_i x)$  is then a lattice isometry from  $E$  into  $(\sum \oplus L^1)_{\ell^\infty}$ . If  $0 \leq x \leq u$ , then  $0 \leq T_i x \leq \langle u, x'_i \rangle 1 \leq \|u\| 1$ , since  $\sup \|x'_i\| \leq 1$ , and hence  $\{T_i x\}$  is relatively weakly compact. Since the interval  $[0, u]$  is total in  $E$ ,  $T$  maps  $E$  into the sublattice  $H$  of  $(\sum \oplus L^1)_{\ell^\infty}$  of all relatively weakly compact sequences. Now it follows from Corollary 1 that  $E$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .  $\square$

Let  $E$  be a Banach lattice and let  $0 \leq x \in E$ . The linear hull of the order interval  $[0, x]$  is called the *principal ideal generated by  $x$*  and denoted by  $E_x$ .

**Lemma 3.** *Let  $E$  be a separable Banach lattice and let  $0 < x \in E$  and  $0 \leq v' \in E'$ . Then there exists a lattice homomorphism  $T$  from the closure  $G$  of the principal ideal  $E'_x$  into  $L^1$  with  $Tv' = \langle x, v' \rangle 1$  and  $\|Ty'\| = \langle x, |y'| \rangle$  for every  $y' \in G$ .*

*Proof.* By Kakutani's representation theorem for AM-spaces with unit, there exist a compact space  $K$  and a lattice isomorphism  $S$  from  $C(K)$  onto the principal ideal  $E_x$  with  $S1 = x$ . Since  $S$  is a lattice homomorphism and preserves order intervals,  $S'$  is a lattice homomorphism from  $E'$  into the AL-space  $C(K)'$  with  $S'[0, v'] = [0, \nu]$  where  $\nu = S'v'$  ([5], Proposition 1.2). Clearly,  $\|S'x'\| = \langle x, |x'| \rangle$  for every  $x' \in E'$ . Also,  $S'$  maps the ideal  $G$  into the band  $L^1(K, \nu)$ . Since  $E$  is separable,  $[0, v']$  is separable for the weak\* topology. Hence  $[0, \nu]$  is also weak\*-separable. Now,  $[0, \nu]$  is weakly compact, and so weakly separable. Therefore,  $L^1(K, \nu)$  is separable. Thus there exists a lattice isometry  $T_2$  from  $L^1(K, \nu)$  into  $L^1$  with  $T_2 \nu = \langle x, \nu \rangle 1$ . Let  $T_1$  be the restriction of  $S'$  to  $G$ . Then  $T = T_2 \circ T_1$  is the desired lattice homomorphism.  $\square$

**Theorem 3.** *If  $E$  is a separable Banach lattice, then the closure of any principal ideal in  $E'$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .*

*Proof.* Let  $0 \leq v' \in E'$  be given and let  $G$  be the closure of the principal ideal  $E'_v$ . Choose  $0 \leq x_i \in E$  such that  $\|x'\| = \sup \langle x_i, |x'| \rangle$  for every  $x' \in E'$ . By Lemma

3, there exist lattice homomorphisms  $T_i$  from  $G$  into  $L^1$  with  $T_i v' = \langle x_i, v' \rangle 1$  and  $\|T_i x'\| = \langle x_i, |x'| \rangle$  for every  $x' \in G$ . As in the proof of Theorem 2, it follows easily that  $T : x' \mapsto (T_i x')$  is a lattice isometry from  $G$  into the sublattice  $W$  of  $(\sum \oplus L^1)_{\ell^\infty}$  of all relatively weakly compact sequences. Now, Corollary 1 implies that  $G$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .  $\square$

**Corollary 3.** *Let  $E$  be a separable Banach lattice. If the dual  $E'$  has a quasi-interior point, then  $E'$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .*

**Corollary 4.** *Let  $(S, \Sigma, \nu)$  be a separable measure space. Then the Banach lattice  $L^\infty(S, \Sigma, \nu)$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .*

Of course, Corollary 4 also follows from Proposition 2.

Let  $(S, \Sigma, \nu)$  be a measure space. The space Weak  $L^p$  consists of all (equivalence classes of) measurable functions  $f$  on  $S$  for which

$$\sup_{s>0} s(\nu\{|f| > s\})^{1/p} < \infty.$$

If  $1 < p < \infty$ , then Weak  $L^p$  with the norm

$$\|f\| = \sup_{0 < \nu(B) < \infty} \nu(B)^{1/p-1} \int_B |f| d\nu$$

is a dual Banach lattice, which we denote by  $L(p, \infty, \nu)$ . Except in the finite dimensional case, this space does not have a quasi-interior point.

**Theorem 4.** *Let  $(S, \Sigma, \nu)$  be a separable measure space and let  $1 < p < \infty$ . Then Weak  $L^p$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ .*

*Proof.* At first we consider the real interval  $(0, \alpha)$ ,  $0 < \alpha \leq \infty$ , with Lebesgue measure  $\mu$ . Since the predual  $L(p/(p-1), 1)$  is separable, there is a sequence  $(B_n)$  of measurable subsets of  $(0, \alpha)$  such that

$$\|f\| = \sup_n \mu(B_n)^{1/p-1} \int_{B_n} |f| d\mu, \quad f \in L(p, \infty, \mu).$$

For each  $n$  there exists a bijection  $\psi_n$  from  $(0, 1]$  onto  $B_n$  such that  $A \subset (0, 1]$  is measurable if and only if  $\psi_n(A)$  is measurable and such that  $\mu(\psi_n(A)) = \lambda(A)\mu(B_n)$ . An easy calculation shows that for every measurable subset  $A \subset (0, 1]$ ,

$$\begin{aligned} \mu(B_n)^{1/p} \int_A |f \circ \psi_n| d\lambda &= \mu(B_n)^{1/p-1} \int_{\psi_n(A)} |f| d\mu \\ &\leq \mu(B_n)^{1/p-1} \mu(\psi_n(A))^{1-1/p} \|f\| \\ &\leq \lambda(A)^{1-1/p} \|f\|. \end{aligned}$$

This shows that  $\{\mu(B_n)^{1/p} f \circ \psi_n\}$  is relatively weakly compact in  $L^1$ . Also, the first equation with  $A = (0, 1]$  shows that  $f \mapsto (\mu(B_n)^{1/p} f \circ \psi_n)$  is a lattice isometry from Weak  $L^p$  into  $(\sum \oplus L^1)_{\ell^\infty}$ . It follows from Corollary 1 that Weak  $L^p$  is lattice isometric to a closed sublattice of the Banach envelope of Weak  $L^1$ . Now let  $(S, \Sigma, \nu)$  be separable and  $\sigma$ -finite and let  $\alpha = \nu(S)$ . Then  $(S, \Sigma, \nu)$  is isomorphic to a  $\sigma$ -subalgebra  $\Sigma_1$  of the real interval  $(0, \alpha)$  with Lebesgue measure  $\mu$ . It is

clear that  $L(p, \infty, \nu)$  is lattice isomorphic to the sublattice of all  $\Sigma_1$ -measurable functions of  $L(p, \infty, \mu)$ . It follows from Lemma 8 of [6] that this isomorphism is an isometry.  $\square$

*Remark 2.* Let  $1 < p < \infty$  and let  $1/p + 1/q = 1$ . The inclusion map from  $L^p(0, 1]$  into Weak  $L^p$  on  $(0, 1]$  is a lattice isomorphism but not an isometry. For  $f \in L^p$ , define  $Tf$  by

$$(Tf)(s, t) = qf(t)/s^{1/p}, \quad (s, t) \in X = (0, 1] \times (0, 1].$$

Let  $\varphi : (0, 1] \rightarrow X$  be a measure-preserving transformation. Then  $f \mapsto (Tf) \circ \varphi$  is a lattice isometry from  $L^p$  into the Lorentz space  $L(p, \infty, \lambda)$ .

## 3

It follows from Theorem 2 that there is a lattice isometry from  $(\sum \oplus L^1)_{c_0}$  into the Banach envelope of Weak  $L^1$ . However, no such embedding factors through Weak  $L^1$  as we will see. This shows that the operator  $T$  in the proof of Theorem 2 is not a continuous operator into the quasinormed space Weak  $L^1$ .

**Lemma 4.** *Let  $0 < f \in \text{Weak } L^1$  with  $\|f\| > 1$ . Then for every  $\alpha > 1$  there exists  $p > 0$  such that*

$$\int_{p < f \leq \alpha p} f \, d\lambda \geq \ln \alpha.$$

*Proof.* Choose  $k \in \mathbb{N}$  such that  $\|f\| > (k+1)/k$ . Then there exist  $0 < r < q$  such that  $\alpha^k \leq q/r$  and  $(k+1)/k \ln(q/r) \leq \int_{r < f \leq q} f \, d\lambda$ . Let  $m = \lceil \ln(q/r)/\ln \alpha \rceil$ . Then  $m \geq k$  and

$$\begin{aligned} (m+1) \ln \alpha &\leq \left(1 + \frac{1}{k}\right) m \ln \alpha \leq \left(1 + \frac{1}{k}\right) \ln \frac{q}{r} \\ &\leq \int_{r < f \leq q} f \, d\lambda \leq \int_{r < f \leq \alpha^{m+1} r} f \, d\lambda = \sum_{i=0}^m \int_{\alpha^i r < f \leq \alpha^{i+1} r} f \, d\lambda \\ &\leq (m+1) \max \left\{ \int_{\alpha^i r < f \leq \alpha^{i+1} r} f \, d\lambda \right\}. \end{aligned}$$

Hence, for some  $j$ ,  $\ln \alpha \leq \int_{\alpha^j r < f \leq \alpha^{j+1} r} f \, d\lambda$ . Now let  $p = \alpha^j r$ .  $\square$

**Lemma 5.** *Let  $T$  be a lattice homomorphism from  $L^1$  into Weak  $L^1$  and let  $0 \leq f \in L^1$  and  $p < q$  be given. Suppose that  $\int_{p < Tf \leq q} Tf \, d\lambda > 0$ . Then for every  $\gamma \geq 1$  there exists  $0 \leq g \in L^1$  with  $\|g\| \leq \|f\|$  and*

$$\int_{\gamma p < Tg \leq \gamma q} Tg \, d\lambda = \int_{p < Tf \leq q} Tf \, d\lambda.$$

*Proof.* The assertion is trivial if  $\gamma = 1$ . So we assume that  $\gamma > 1$ . Let  $D = \{p < Tf \leq q\}$ . For every  $\lambda$ -measurable set  $A$  let  $\nu(A) = \int_D T(f\chi_A) \, d\lambda$ . Since  $T$  is a continuous map from  $L^1$  into Weak  $L^1$  (indeed,  $\|f_i\| \leq 2^{-2i}$  implies  $|Tf_n| \leq 2^{-n} T(\sum 2^i |f_i|)$ ),  $T(f\chi_A)$  tends to zero in measure as  $\lambda(A)$  tends to zero. By the dominated convergence theorem  $\int_D T(f_A) \, d\lambda$  tends to zero as  $\lambda(A)$  tends to zero. Hence  $\nu$  is a measure which is absolutely continuous with respect to  $\lambda$ .

Therefore, there exists  $0 \leq h_1 \in L^1$  with  $\nu(A) = \int_A h_1 \, d\lambda$ . It is clear that  $h_1$  vanishes  $\lambda$ -almost everywhere on the set  $\{f = 0\}$ . Let  $h(t) = h_1(t)/f(t)$  if  $f(t) > 0$

and  $h(t) = 0$  elsewhere. Then  $\nu(A) = \int_A hf d\lambda$  for every measurable set  $A$ . Let  $\beta = \inf\{\alpha : \int_{h>\alpha} hf d\lambda \leq \gamma^{-1}\|\nu\|\}$ . Then  $\int_{h>\beta} hf d\lambda \leq \gamma^{-1}\|\nu\| \leq \int_{h\geq\beta} hf d\lambda$ . There exists a measurable set  $B$  with  $\{h > \beta\} \subset B \subset \{h \geq \beta\}$  and  $\int_B hf d\lambda = \gamma^{-1}\|\nu\|$ . Then  $\beta \int_B f d\lambda \leq \gamma^{-1}\|\nu\|$  and  $(1 - \gamma^{-1})\|\nu\| \leq \beta(\|f\| - \int_B f d\lambda)$ , and so  $\int_B f d\lambda / (\|f\| - \int_B f d\lambda) \leq \gamma^{-1}/(1 - \gamma^{-1})$ . This easily implies that  $\int_B f d\lambda \leq \gamma^{-1}\|f\|$ . Let  $g = \gamma f \chi_B$ . Then  $\|g\| \leq \|f\|$  and

$$\int_{p < Tg \leq q} Tg d\lambda = \gamma \nu(B) = \gamma \int_B hf d\lambda = \|\nu\| = \int_{p < Tf \leq q} Tf d\lambda.$$

Since  $T$  is a lattice homomorphism,  $T(f\chi_B)$  and  $T(f\chi_{\mathbb{C}B})$  are disjoint. Hence

$$\int_{p < T(f\chi_B) \leq q} T(f\chi_B) d\lambda = \int_{p < Tf \leq q} T(f\chi_B) d\lambda.$$

Therefore

$$\int_{\gamma p < Tg \leq \gamma q} Tg d\lambda = \gamma \int_{p < Tf \leq q} T(f\chi_B) d\lambda = \int_{p < Tf \leq q} Tf d\lambda.$$

□

**Theorem 5.** *Let  $T$  be a lattice homomorphism from  $(\sum \oplus L^1)_{c_0}$  into Weak  $L^1$ . Then  $T$  as a map into the Banach envelope of Weak  $L^1$  factors through  $L^1$ .*

*Proof.* For  $i \in \mathbb{N}$  define  $\varphi_i : L^1 \rightarrow (\sum \oplus L^1)_{c_0}$  by  $\varphi_i(f) = (f_n)$ , where  $f_n = f$  if  $n = i$  and  $f_n = 0$  otherwise, and let  $T_i = t \circ \varphi_i$ . We may assume that  $\|T_i\| > 0$  for all  $i$ . Suppose that  $\sum \|T_i\| = \infty$ . Then there exist  $0 \leq f_i \in L^1$ ,  $\lim \|f_i\| = 0$ , with  $\sum \|T_i f_i\| = \infty$ . Let  $\{A_n\}$  be a partition of  $\mathbb{N}$  into finite subsets such that  $\sum_{i \in A_n} \|T_i f_i\| > n$  for every  $n$ .

By Lemma 4, for every  $i \in A_n$  we can choose  $p_i > 0$  such that  $\int_{p_i < T_i f_i \leq e^n p_i} T_i f_i > n \|T_i f_i\| (1 - n^{-1})$ . Let  $r_n = \max\{p_i : i \in A_n\}$ . By Lemma 5, for  $i \in A_n$  there exists  $0 \leq g_i \in L^1$ ,  $\|g_i\| \leq \|f_i\|$ , such that  $\int_{r_n < T_i g_i \leq e^n r_n} T_i g_i > n \|T_i f_i\| (1 - n^{-1})$ . Let  $g = (g_i)$ . Then  $g \in F$ ,  $Tg = \sum T_i g_i$ , and the functions  $T_i g_i$  are pairwise disjoint. Therefore,

$$\begin{aligned} \int_{r_n < Tg \leq e^n r_n} Tg d\mu &= \sum \int_{r_n < T_i g_i \leq e^n r_n} T_i g_i d\mu \\ &\geq \sum_{i \in A_n} \int_{r_n < T_i g_i \leq e^n r_n} T_i g_i d\mu \geq n(n-1). \end{aligned}$$

Obviously, this contradicts the fact that

$$\limsup_{q/p \rightarrow \infty} \int_{p < Tg \leq q} Tg d\mu / \ln \frac{q}{p} < \infty.$$

Thus,  $\sum \|T_i\| < \infty$ . Let  $G$  be the completion of  $F$  for the norm  $\|(f_i)\|_1 = \sum \|T_i\| \|f_i\|$ . Then  $G$  is isometric to  $L^1$  and  $T$  as a map into  $\mathbb{W}$  factors through  $G$ . □

4

A positive linear operator  $T$  from a Banach lattice  $E$  into a Banach lattice  $F$  is called an *order isometry* if  $Tx \geq 0$  implies  $x \geq 0$  and if  $\|Tx\| = \|x\|$  for all  $x \in E$ . It follows from Theorem 5 that no lattice isometry from  $(\sum \oplus L^1)_{\ell^\infty}$  into the Banach envelope of Weak  $L^1$  factors through Weak  $L^1$  by a lattice homomorphism.

However, there is an order isometry from  $(\sum \oplus L^1)_{\ell^\infty}$  into the Banach envelope of Weak  $L^1$ .

**Theorem 6.** *The Banach lattice  $(\sum \oplus L^1)_{\ell^\infty}$  is order isometric to a closed subspace of the Banach envelope of Weak  $L^1$ .*

*Proof.* For  $f \in L^1$  let  $P_n f$  be the conditional expectation of  $f$  with respect to the  $\sigma$ -subalgebra generated by the intervals  $((j-1)2^{-n}, j2^{-n}]$ . We may also consider  $P_n$  as a positive operator from  $L^1$  into  $\ell^1(2^n)$ . Since the  $P_n$ 's are positive contractions and  $P_n f$  converges to  $f$  in  $L^1$ , the map  $f \mapsto (P_n f)$  is an order isometry from  $L^1$  into  $(\sum \oplus L^1)_{\ell^\infty}$ . Then also  $T : (f_i) \mapsto (P_n f_i)$  is an order isometry from  $E = (\sum \oplus L^1)_{\ell^\infty}$  into  $F = (\sum \oplus \ell^1(k_{(i,n)}))_{\ell^\infty}$ , where  $k_{(i,n)} = 2^n$  for every  $(i, n) \in \mathbb{N} \times \mathbb{N}$ . By Corollary 2, there is a lattice isometry  $S$  from  $F$  into the Banach envelope of Weak  $L^1$ . Hence  $S \circ T$  is an order isometry from  $E$  into the Banach envelope of Weak  $L^1$ .  $\square$

**Corollary 5.** *Let  $E$  be a separable Banach lattice and let  $0 < x' \in E'$ . Then the band generated by  $x'$  in  $E'$  is lattice isometric to a closed sublattice of  $(\sum \oplus L^1)_{\ell^\infty}$ .*

*Proof.* A slight modification of the proofs of Lemma 3 and Theorem 3 shows that the band generated by  $x'$  is lattice isometric to a closed sublattice of  $(\sum \oplus L^1)_{\ell^\infty}$ .  $\square$

**Corollary 6.** *Let  $E$  be a separable Banach lattice. If  $E'$  has a weak order unit, then  $E'$  is order isometric to a closed subspace of the Banach envelope of Weak  $L^1$ .*

**Corollary 7.** *Let  $E$  be a separable Banach lattice. If  $E$  does not contain a closed subspace order isometric to  $C(\Delta)$ ,  $\Delta$  the Cantor set, then the dual  $E'$  is order isometric to a closed subspace of the Banach envelope of Weak  $L^1$ .*

*Proof.* Since  $E$  is separable, it follows easily from Theorem 2 of [8] that  $E'$  has a weak order unit if and only if  $E$  does not contain a closed subspace order isometric to  $C(\Delta)$ .  $\square$

*Remark 3.* It follows from Remark 1 that the spaces in Theorem 6 and Corollaries 5, 6, and 7 are order isomorphic (and thus topologically isomorphic) to closed subspaces of  $(\text{Weak } L^1, \|\cdot\|)$ .

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