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OPTIMAL ESTIMATION OF SHELL THICKNESS IN CUTLAND'S CONSTRUCTION OF WIENER MEASURE

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Abstract. In Cutland's construction of Wiener measure, he used the product of Gaussian measures on ${}^*R^N$, where N is an infinite integer. It is mentioned by Cutland and Ng that for the product measure γ,

$$\gamma(\{x : R_1 \leq \|x\| \leq R_2\}) \simeq 1,$$

where $R_1 = 1 - (\log N)^{\frac{1}{2}} N^{-\frac{1}{2}}$ and $R_2 = 1 + MN^{-\frac{1}{2}}$ with M any positive infinite number. We prove here that R_1 may be replaced by $1 - mN^{-\frac{1}{2}}$ with m any positive infinite number. This is the optimal estimation for the shell thickness. It is also proved that $\gamma(\{x : \|x\| < 1\}) \simeq \gamma(\{x : \|x\| > 1\}) \simeq \frac{1}{2}$. And for the *Lebesgue measure μ, $\mu(\{x : \|x\| \leq r\})$ is finite and not infinitesimal iff $r = (2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}(1+\frac{1}{N})} e^{\frac{a}{N}}$ with a finite, while for the *Lebesgue area of the sphere $S^{N-1}(r)$, r should be $(2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}} e^{\frac{a}{N}}$.

N. Cutland constructed the Wiener measure in [1] via the internal measure γ in a nonstandard $*$-finite Euclidean space ${}^*R^N$, where N is an infinite positive integer, and

$$\gamma(A) = (2\pi N^{-1})^{-\frac{1}{2}N} \int_A \exp\left(-\frac{1}{2} N \sum_{i=1}^{N} x_i^2\right) dx_1 \cdots dx_N.$$

γ has a very interesting property that almost all points in ${}^*R^N$ are near the unit sphere. So it is interesting to estimate the exact thickness of the shell with almost all points.

In Remark 2 of [2] N. Cutland and S.-A. Ng mentioned the following:

Let $R_1 = 1 - (\log N)^{\frac{1}{2}} N^{-\frac{1}{2}}$ and $R_2 = 1 + MN^{-\frac{1}{2}}$, where M is any positive infinite number. Then

$$\gamma(\{x : R_1 \leq \|x\| \leq R_2\}) \simeq 1.$$

Actually they proved in the preprint of [2] that R_1 or R_2 cannot be replaced by $R_1 = 1 - mN^{-\frac{1}{2}}$ or $R_2 = 1 + mN^{-\frac{1}{2}}$, if m is finite. So their estimation for the outer one is already optimal. Is their result for the inner one optimal? They said in the preprint: "we are less sure". This is indeed not optimal and the optimal estimation is shown by the following theorem.

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Theorem 1. *Let $R_1 = 1 - MN^{\frac{1}{2}}$ and $R_2 = 1 + M'N^{\frac{1}{2}}$ where M and M' are any positive infinite numbers. Then*

$$\gamma(\{x : R_1 \le \|x\| \le R_2\}) \simeq 1.$$

Proof. By the results of the preprint of [2], we need only prove that

$$\gamma(\{x : \|x\| \le R_1\}) \simeq 0.$$

Also from the preprint of [2],

$$\gamma(\{x : \|x\| \le a\}) = \alpha\pi^{-\frac{1}{2}} \int_0^a \beta(r)\, dr$$

where $\alpha \simeq 1$ and

$$\beta(r) = N^{\frac{1}{2}} \exp\left(\frac{1}{2}N(1-r^2)\right) r^{N-1}.$$

We have for $r \neq 0$:

$$\begin{aligned}\beta(r) &= N^{\frac{1}{2}} \exp\left(\frac{1}{2}N(1-r^2) + (N-1)\log r\right)\\ &= N^{\frac{1}{2}} \exp\left(\frac{1}{2}N(1-r^2) + (N-1)\left(r - 1 - \frac{(r-1)^2}{2} + \varepsilon\right)\right)\end{aligned}$$

where $\varepsilon = \frac{(r-1)^3}{3(1+\xi)^3}$ for some ξ in between $r-1$ and 0 according to the Lagrange remainder theorem for the Taylor expansion. Simplifying this gives

$$\beta(r) = N^{\frac{1}{2}} \exp\left(-\frac{2N-1}{2}(1-r)^2 + \varepsilon'\right)$$

where $\varepsilon' = 1 - r + (N-1)\varepsilon$.

Now consider $r = 1 - mN^{-\frac{1}{2}}$ with $m^3N^{-\frac{1}{2}} \simeq 0$. Then

$$(N-1)\varepsilon = (N-1)\frac{(r-1)^3}{3(1+\xi)^3} = \frac{-(N-1)}{3(1+\xi)^3} m^3 N^{-\frac{3}{2}} \simeq 0$$

and

$$\varepsilon' \simeq 0.$$

Hence

$$(*) \qquad \beta(r) = a(r) \exp\left(\frac{1}{2}\log N - \frac{2N-1}{2}(1-r)^2\right)$$

where $a(r) \simeq 1$ for such r. Notice that m may be negative for $(*)$ true. This will be used in the proof of Theorem 2. Let

$$m_0 = \left(\frac{N\log N}{2N-1}\right)^{\frac{1}{2}};$$

then m_0 is infinite and $m_0^3 N^{-\frac{1}{2}} \simeq 0$. So the formula $(*)$ is used to give

$$(**) \qquad \beta(r_1) = a(r_1) \simeq 1$$

where r_1 stands for $1 - m_0 N^{-\frac{1}{2}}$.

It is easy to check that $\beta(r) \simeq 0$ for $r > 0$ with ${}^\circ r < 1$. Then using the Robinson Lemma there is $r_0 \simeq 1$ such that $\beta(r_0) \simeq 0$ for all $0 \le r \le r_0$; and we may take

$r_0 < r_1$. Then, since $\beta(r)$ is increasing for $r < (1 - N^{-1})^{\frac{1}{2}} = r_2$, say, and $r_1 < r_2$, we have $\beta(r) \leq 2$ for $r \leq r_1$ and so

$$\int_{r_0}^{r_1} \beta(r)\, dr < 2(r_1 - r_0) \simeq 0$$

and so

$$\int_0^{r_1} \beta(r)\, dr \simeq 0.$$

Now, letting $m < m_0$ be any positive infinite number, we have

$$\begin{aligned}
\left(\int_{1-m_0N^{-\frac{1}{2}}}^{1-mN^{-\frac{1}{2}}} \beta(r)\, dr\right)^2 &= \left(\int_{1-m_0N^{-\frac{1}{2}}}^{1-mN^{-\frac{1}{2}}} a(r)N^{\frac{1}{2}} \exp\left(-\frac{2N-1}{2}(1-r^2)\right)\, dr\right)^2 \\
&\leq 4\left(\int_{mN^{-\frac{1}{2}}}^{m_0N^{-\frac{1}{2}}} N^{\frac{1}{2}} \exp\left(-\frac{2N-1}{2}s^2\right)\, ds\right)^2 \\
&= 4N \int_{mN^{-\frac{1}{2}}}^{m_0N^{-\frac{1}{2}}} ds_2 \int_{mN^{-\frac{1}{2}}}^{m_0N^{-\frac{1}{2}}} \exp\left(-\frac{2N-1}{2}(s_1^2+s_2^2)\right)\, ds_1 \\
&\leq 4N \int_0^{\frac{\pi}{2}} d\theta \int_{mN^{-\frac{1}{2}}}^{\sqrt{2}m_0N^{-\frac{1}{2}}} \exp\left(-\frac{2N-1}{2}t^2\right) t\, dt \\
&= 2\pi N(2N-1)^{-1}(\exp(-(2N-1)N^{-1}m^2) - \exp(-(2N-1)N^{-1}m_0^2)).
\end{aligned}$$

Since m and m_0 are both infinite, we conclude that

$$\int_{1-m_0N^{-\frac{1}{2}}}^{1-mN^{-\frac{1}{2}}} \beta(r)\, dr \simeq 0;$$

therefore

$$\int_0^{1-mN^{-\frac{1}{2}}} \beta(r)\, dr \simeq 0$$

and the theorem is proved.

It is interesting to see how much mass lies inside the unit sphere. We have

Theorem 2. $\gamma(\{x : \|x\| < 1\}) \simeq \gamma(\{x : \|x\| > 1\}) \simeq \frac{1}{2}$.

Proof. Let m be any positive infinite number with $m^3N^{-\frac{1}{2}} \simeq 0$. We know from the preprint of [2] or Theorem 1 that

$$\gamma(\{x : \|x\| > 1\}) \simeq \gamma(\{x : 1 + mN^{-\frac{1}{2}} > \|x\| > 1\}).$$

Thus from the formula $(*)$ above, we have for $1 < r < 1 + mN^{-\frac{1}{2}}$,

$$\beta(r) = a(r)N^{\frac{1}{2}} \exp\left(-\frac{2N-1}{2}(1-r)^2\right)$$

with $a(r) \simeq 1$. Let

$$\begin{aligned} I &= \int_1^{1+mN^{-\frac{1}{2}}} N^{\frac{1}{2}} \exp\left(-\frac{2N-1}{2}(1-r)^2\right) dr \\ &= \int_0^{mN^{-\frac{1}{2}}} N^{\frac{1}{2}} \exp\left(-\frac{2N-1}{2}s^2\right) ds; \end{aligned}$$

then

$$\begin{aligned} N\int_0^{\frac{\pi}{2}} d\theta \int_0^{mN^{-\frac{1}{2}}} \exp\left(-\frac{2N-1}{2}t^2\right) t\,dt &\le I^2 \\ \le N\int_0^{\frac{\pi}{2}} d\theta \int_0^{\sqrt{2}mN^{-\frac{1}{2}}} \exp\left(-\frac{2N-1}{2}t^2\right) t\,dt. \end{aligned}$$

Now for any infinite positive number M,

$$\begin{aligned} N\frac{\pi}{2}\int_0^{MN^{-\frac{1}{2}}} \exp\left(-\frac{2N-1}{2}t^2\right) t\,dt \\ = \frac{\pi N}{2(2N-1)}\left(1-\exp\left(-\frac{2N-1}{2N}M^2\right)\right) \simeq \frac{\pi}{4}. \end{aligned}$$

So

$$I \simeq \frac{\sqrt{\pi}}{2}$$

and

$$\gamma(\{x : 1+mN^{-\frac{1}{2}} > \|x\| > 1\}) = \alpha\pi^{-\frac{1}{2}} \int_1^{1+mN^{-\frac{1}{2}}} \beta(r)\,dr \simeq \frac{1}{2}.$$

This proves the theorem.

It seems strange that the Gaussian measure $\mathcal{N}(0, N^{-1})$ on any axis of ${}^*R^N$ concentrates in the monad of the zero, hence in a set with *Lebesgue measure infinitesimal, while the product measure γ concentrates on a set with distance to the origin nearly 1. The following theorem may give a partial explanation of this phenomenon, since it tells us that there is a ball in ${}^*R^N$ of infinite radius with *Lebesgue measure infinitesimal.

Theorem 3. *For the* **Lebesgue measure* μ *on* ${}^*R^N$ *and surface measure* μ_s *on spheres, we have*

$$\mu(\{x : \|x\| \le r\}) \text{ is finite and } \not\simeq 0$$
$$\text{iff}$$
$$r = (2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}(1+\frac{1}{N})} e^{\frac{a}{N}} \text{ with } a \text{ finite}$$

and

$$\mu_s(\{x : \|x\| = r\}) \text{ is finite and } \not\simeq 0$$
$$\text{iff}$$
$$r = (2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}} e^{\frac{a}{N}} \text{ with } a \text{ finite.}$$

Hint to the Proof. Consider the case when N is even; then

$$\mu(\{x : |x| \le r\}) = \frac{\pi^{\frac{N}{2}} r^N}{(\frac{N}{2})!} \triangleq A$$

and

$$\ln r = \frac{1}{N} \ln \left(\frac{N}{2}\right)! - \frac{1}{2} \ln \pi + \frac{1}{N} \ln A.$$

By Stirling's formula

$$\left(\frac{N}{2}\right)! = (\pi N)^{\frac{1}{2}} \left(\frac{N}{2e}\right)^{\frac{N}{2}} e^b \text{ with } b \simeq 0$$

we have

$$r = (2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}(1+\frac{1}{N})} e^{\frac{1}{N}\ln A + \frac{b}{N} + \frac{1}{2N}\ln \pi}.$$

Let

$$a = \ln A + b + \frac{1}{2} \ln \pi.$$

It is easy to see that $A > 0$ is finite and $\not\simeq 0$ iff a is finite. The proof for the other cases are similar.

We give some comments on the results. First, the set of r's making the balls finite and $\not\simeq 0$ is disjoint with that for the spheres. Secondly, any point in one set differs from any point in another one by an infinitesimal.

It is interesting to notice that the sets

$$\{(2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}(1+\frac{1}{N})} e^{\frac{a}{N}} \text{ with } a \text{ finite}\}$$

and

$$\{(2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}} e^{\frac{a}{N}} \text{ with } a \text{ finite}\}$$

are just two different $N^{-\frac{1}{2}}O$-equivalence classes according to the language in [3], and the set

$$\{r : \gamma(\{\|x\| \le r\}) \not\simeq 0 \text{ or } 1\}$$

is also an $N^{-\frac{1}{2}}O$-equivalence class.

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