OPTIMAL ESTIMATION OF SHELL THICKNESS
IN CUTLAND’S CONSTRUCTION OF WIENER MEASURE

BANG-HE LI AND YA-QING LI

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ABSTRACT. In Cutland’s construction of Wiener measure, he used the product of Gaussian measures on \(*R^N\), where \(N\) is an infinite integer. It is mentioned by Cutland and Ng that for the product measure \(\gamma\),

\[ \gamma(\{x : R_1 \leq \|x\| \leq R_2\}) \simeq 1, \]

where \(R_1 = 1 - (\log N)^2 N^{-\frac{1}{2}}\) and \(R_2 = 1 + MN^{-\frac{1}{2}}\) with \(M\) any positive infinite number. We prove here that \(R_1\) may be replaced by \(1 - mN^{-\frac{1}{2}}\) with \(m\) any positive infinite number. This is the optimal estimation for the shell thickness. It is also proved that \(\gamma(\{x : \|x\| < 1\}) \simeq \gamma(\{x : \|x\| > 1\}) \simeq \frac{1}{2}\). And for the *Lebesgue measure \(\mu\), \(\mu(\{x : \|x\| \leq r\})\) is finite and not infinitesimal iff \(r = (2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}} e^N\) with a finite, while for the *Lebesgue area of the sphere \(S^{N-1}(r)\), \(r\) should be \((2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}} e^N\).

N. Cutland constructed the Wiener measure in [1] via the internal measure \(\gamma\) in a nonstandard *-finite Euclidean space \(*R^N\), where \(N\) is an infinite positive integer, and

\[ \gamma(A) = (2\pi N^{-1})^{-\frac{1}{2}} N \int_A \exp \left( -\frac{1}{2} N \sum_{i=1}^{N} x_i^2 \right) dx_1 \cdots dx_N. \]

\(\gamma\) has a very interesting property that almost all points in \(*R^N\) are near the unit sphere. So it is interesting to estimate the exact thickness of the shell with almost all points.

In Remark 2 of [2] N. Cutland and S.-A. Ng mentioned the following:

Let \(R_1 = 1 - (\log N)^2 N^{-\frac{1}{2}}\) and \(R_2 = 1 + MN^{-\frac{1}{2}}\), where \(M\) is any positive infinite number. Then

\[ \gamma(\{x : R_1 \leq \|x\| \leq R_2\}) \simeq 1. \]

Actually they proved in the preprint of [2] that \(R_1\) or \(R_2\) cannot be replaced by \(R_1 = 1 - mN^{-\frac{1}{2}}\) or \(R_2 = 1 + mN^{-\frac{1}{2}}\), if \(m\) is finite. So their estimation for the outer one is already optimal. Is their result for the inner one optimal? They said in the preprint: “we are less sure”. This is indeed not optimal and the optimal estimation is shown by the following theorem.

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**Theorem 1.** Let $R_1 = 1 - MN^\frac{1}{2}$ and $R_2 = 1 + M'N^\frac{1}{2}$ where $M$ and $M'$ are any positive infinite numbers. Then

$$\gamma(\{x : R_1 \leq \|x\| \leq R_2\}) \simeq 1.$$ 

**Proof.** By the results of the preprint of [2], we need only prove that

$$\gamma(\{x : \|x\| \leq R_1\}) \simeq 0.$$ 

Also from the preprint of [2],

$$\gamma(\{x : \|x\| \leq a\}) = \alpha \pi^{-\frac{1}{2}} \int_0^a \beta(r) \, dr$$

where $\alpha \simeq 1$ and

$$\beta(r) = N^{\frac{1}{2}} \exp\left(\frac{1}{2} N(1 - r^2)\right) r^{N-1}.$$ 

We have for $r \neq 0$:

$$\beta(r) = N^{\frac{1}{2}} \exp\left(\frac{1}{2} N(1 - r^2) + (N - 1) \log r \right)$$

$$= N^{\frac{1}{2}} \exp\left(\frac{1}{2} N(1 - r^2) + (N - 1) \left( r - 1 - \frac{(r-1)^2}{2} + \varepsilon \right) \right)$$

where $\varepsilon = \frac{(r-1)^3}{2(1 + \xi)^3}$ for some $\xi$ in between $r - 1$ and 0 according to the Lagrange remainder theorem for the Taylor expansion. Simplifying this gives

$$\beta(r) = N^{\frac{1}{2}} \exp\left( - \frac{2N - 1}{2} (1 - r)^2 + \varepsilon' \right)$$

where $\varepsilon' = 1 - r + (N - 1)\varepsilon$.

Now consider $r = 1 - mN^{-\frac{1}{2}}$ with $m^3 N^{-\frac{1}{2}} \simeq 0$. Then

$$(N - 1)\varepsilon = (N - 1) \frac{(r - 1)^3}{3(1 + \xi)^3} = -\frac{(N - 1)}{3(1 + \xi)^3} m^3 N^{-\frac{1}{2}} \simeq 0$$

and

$$\varepsilon' \simeq 0.$$ 

Hence

$$\beta(r) = a(r) \exp\left(\frac{1}{2} \log N - \frac{2N - 1}{2} (1 - r)^2\right)$$

where $a(r) \simeq 1$ for such $r$. Notice that $m$ may be negative for (*) true. This will be used in the proof of Theorem 2. Let

$$m_0 = \left(\frac{N \log N}{2N - 1}\right)^{-\frac{1}{2}};$$

then $m_0$ is infinite and $m_0^3 N^{-\frac{1}{2}} \simeq 0$. So the formula (*) is used to give

$$\beta(r_1) = a(r_1) \simeq 1$$

where $r_1$ stands for $1 - m_0 N^{-\frac{1}{2}}$.

It is easy to check that $\beta(r) \simeq 0$ for $r > 0$ with $r < 1$. Then using the Robinson Lemma there is $r_0 \simeq 1$ such that $\beta(r_0) \simeq 0$ for all $0 \leq r \leq r_0$; and we may take
\( r_0 < r_1 \). Then, since \( \beta(r) \) is increasing for \( r < (1 - N^{-1})^{\frac{1}{2}} = r_2 \), say, and \( r_1 < r_2 \), we have \( \beta(r) \leq 2 \) for \( r \leq r_1 \) and so
\[
\int_{r_0}^{r_1} \beta(r) \, dr < 2(r_1 - r_0) \simeq 0
\]
and so
\[
\int_{0}^{r_1} \beta(r) \, dr \simeq 0.
\]

Now, letting \( m < m_0 \) be any positive infinite number, we have
\[
\left( \int_{1-mN^{-\frac{1}{2}}}^{1-m_0 N^{-\frac{1}{2}}} \beta(r) \, dr \right)^2 = \left( \int_{1-m_0 N^{-\frac{1}{2}}}^{1-mN^{-\frac{1}{2}}} a(r) N^{\frac{1}{2}} \exp \left( -\frac{2N - 1}{2} (1 - r^2) \right) \, dr \right)^2
\]
\[
\leq 4 \left( \int_{m_0 N^{-\frac{1}{2}}}^{mN^{-\frac{1}{2}}} N^{\frac{1}{2}} \exp \left( -\frac{2N - 1}{2} s^2 \right) \, ds \right)^2
\]
\[
= 4N \int_{m_0 N^{-\frac{1}{2}}}^{mN^{-\frac{1}{2}}} ds_2 \int_{m_0 N^{-\frac{1}{2}}}^{mN^{-\frac{1}{2}}} \exp \left( -\frac{2N - 1}{2} (s_1^2 + s_2^2) \right) \, ds_1
\]
\[
\leq 4N \int_{0}^{\frac{\pi}{2}} d\theta \int_{m_0 N^{-\frac{1}{2}}}^{\sqrt{2}m_0 N^{-\frac{1}{2}}} \exp \left( -\frac{2N - 1}{2} t^2 \right) t \, dt
\]
\[
= 2\pi N(2N - 1)^{-1} (\exp(-(2N - 1)N^{-1}m^2) - \exp(-(2N - 1)N^{-1}m_0^2)).
\]

Since \( m \) and \( m_0 \) are both infinite, we conclude that
\[
\int_{1-m_0 N^{-\frac{1}{2}}}^{1-mN^{-\frac{1}{2}}} \beta(r) \, dr \simeq 0;
\]
therefore
\[
\int_{0}^{1-mN^{-\frac{1}{2}}} \beta(r) \, dr \simeq 0
\]
and the theorem is proved.

It is interesting to see how much mass lies inside the unit sphere. We have

**Theorem 2.** \( \gamma(\{ x : \|x\| < 1 \}) \simeq \gamma(\{ x : \|x\| > 1 \}) \simeq \frac{1}{2} \).

**Proof.** Let \( m \) be any positive infinite number with \( m^3N^{-\frac{1}{2}} \simeq 0 \). We know from the preprint of [2] or Theorem 1 that
\[
\gamma(\{ x : \|x\| > 1 \}) \simeq \gamma(\{ x : 1 + mN^{-\frac{1}{2}} > \|x\| > 1 \}).
\]
Thus from the formula (\( * \)) above, we have for \( 1 < r < 1 + mN^{-\frac{1}{2}} \),
\[
\beta(r) = a(r) N^{\frac{1}{2}} \exp \left( -\frac{2N - 1}{2} (1 - r^2) \right)
\]
with \( a(r) \simeq 1 \). Let

\[
I = \int_{1}^{1+\frac{mN}{N^{-\frac{1}{2}}}} N^{\frac{1}{2}} \exp \left( -\frac{2N-1}{2} (1-r)^2 \right) \, dr \\
= \int_{0}^{\frac{mN}{N^{-\frac{1}{2}}}} N^{\frac{1}{2}} \exp \left( -\frac{2N-1}{2} s^2 \right) \, ds;
\]

then

\[
N \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{2}} \int_{0}^{\frac{mN^{-\frac{1}{2}}}{N^{-\frac{1}{2}}}} \exp \left( -\frac{2N-1}{2} t^2 \right) \, dt \leq I^2 \\
\leq N \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{2}} \int_{0}^{\frac{\sqrt{2mN^{-\frac{1}{2}}}}{N^{-\frac{1}{2}}}} \exp \left( -\frac{2N-1}{2} t^2 \right) \, dt.
\]

Now for any infinite positive number \( M \),

\[
N \pi \frac{M}{2} \int_{0}^{\frac{MN^{-\frac{1}{2}}}{N^{-\frac{1}{2}}}} \exp \left( -\frac{2N-1}{2} t^2 \right) \, dt \\
= \frac{\pi N}{2(2N-1)} \left( 1 - \exp \left( -\frac{2N-1}{2N-M^2} \right) \right) \simeq \frac{\pi}{4}.
\]

So

\[
I \simeq \sqrt{\frac{\pi}{2}}
\]

and

\[
\gamma(\{ x : 1 + mN^{-\frac{1}{2}} > \| x \| > 1 \}) = \alpha \pi^{-\frac{1}{2}} \int_{1}^{1+\frac{mN^{-\frac{1}{2}}}{N^{-\frac{1}{2}}}} \beta(r) \, dr \simeq \frac{1}{2}.
\]

This proves the theorem.

It seems strange that the Gaussian measure \( \mathcal{N}(0, N^{-1}) \) on any axis of \( \ast \mathbb{R}^N \) concentrates in the monad of the zero, hence in a set with \( \ast \)Lebesgue measure infinitesimal, while the product measure \( \gamma \) concentrates on a set with distance to the origin nearly 1. The following theorem may give a partial explanation of this phenomenon, since it tells us that there is a ball in \( \ast \mathbb{R}^N \) of infinite radius with \( \ast \)Lebesgue measure infinitesimal.

**Theorem 3.** For the \( \ast \)Lebesgue measure \( \mu \) on \( \ast \mathbb{R}^N \) and surface measure \( \mu_s \) on spheres, we have

\[
\mu(\{ x : \| x \| \leq r \}) \text{ is finite and } \neq 0 \\
\text{iff } \quad r = (2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}(1+\frac{1}{N})} e^\frac{a}{N} \text{ with a finite}
\]

and

\[
\mu_s(\{ x : \| x \| = r \}) \text{ is finite and } \neq 0 \\
\text{iff } \quad r = (2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}} e^\frac{a}{N} \text{ with a finite}.
\]
**Hint to the Proof.** Consider the case when \( N \) is even; then

\[
\mu(\{x : |x| \leq r\}) = \frac{\pi^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!} r^N \triangleq A
\]

and

\[
\ln r = \frac{1}{N} \ln \binom{N}{2}! - \frac{1}{2} \ln \pi + \frac{1}{N} \ln A.
\]

By Stirling's formula

\[
\binom{N}{2}! = (\pi N)^{\frac{N}{2}} e^{-\frac{1}{2} A} \text{ with } b \approx 0
\]

we have

\[
r = (2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}} \left(\frac{1}{2} e^{\frac{1}{2} A} + \frac{1}{2} \ln \pi + \frac{1}{2} \ln A\right).
\]

Let

\[
a = \ln A + b + \frac{1}{2} \ln \pi.
\]

It is easy to see that \( A > 0 \) is finite and \( \neq 0 \) iff \( a \) is finite. The proof for the other cases are similar.

We give some comments on the results. First, the set of \( r \)'s making the balls finite and \( \neq 0 \) is disjoint with that for the spheres. Secondly, any point in one set differs from any point in another one by an infinitesimal.

It is interesting to notice that the sets

\[
\{(2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}} e^{\frac{1}{2} A} \triangleq a \text{ finite}\}
\]

and

\[
\{(2\pi e)^{-\frac{1}{2}} N^{\frac{1}{2}} e^{\frac{1}{2} A} \triangleq a \text{ finite}\}
\]

are just two different \( N^{-\frac{1}{2}}O \)-equivalence classes according to the language in [3], and the set

\[
\{r : \gamma(\{\|x\| \leq r\}) \neq 0 \text{ or } 1\}
\]

is also an \( N^{-\frac{1}{2}}O \)-equivalence class.

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**References**


Institute of Systems Science, Academia Sinica, Beijing 100080, People’s Republic of China

E-mail address: libh@iss06.iss.ac.cn

E-mail address: yli@iss06.iss.ac.cn