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THE CLASSIFICATION OF COMPLETE LIE ALGEBRAS WITH COMMUTATIVE NILPOTENT RADICAL

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Abstract. The work in this paper is a continuation of an earlier paper of the second author (Acta Math. **34** (1991), 191–202). We discuss the properties of finite-dimensional complete Lie algebras with abelian nilpotent radical over the complex field **C**. We solve the problems of isomorphism, classification and realization of complete Lie algebras with commutative nilpotent radical.

1. Introduction

A Lie algebra $\mathcal{L}$ is called a complete Lie algebra if its centre $C(\mathcal{L})$ is zero and its derivations are all inner. The definition of complete Lie algebra was given by N. Jacobson in 1962 (cf. [8]). But the first important result—the derivation tower theorem—was obtained by E. V. Schenkman in 1951 (cf. [9]). In recent years, the theory of complete Lie algebras has been developing (see [1]–[7]). In [1], the properties of complete Lie algebras with commutative nilpotent radical have been discussed. The work in this paper is a continuation of [1].

Let $\mathcal{L}$ be a finite-dimensional Lie algebra over a field of characteristic zero. Then $\mathcal{L}$ has the Levi decomposition:

$$\mathcal{L} = \mathfrak{s} \dot{+} \mathfrak{r}, \tag{1.1}$$

where $\mathfrak{s}$ is a maximal semisimple subalgebra of $\mathcal{L}$ and is called the Levi subalgebra of $\mathcal{L}$, and $\mathfrak{r}$ is the maximal solvable ideal of $\mathcal{L}$ and is called the radical of $\mathcal{L}$. The ideal

$$\mathfrak{n}_0 = [\mathcal{L}, \mathcal{L}] \cap \mathfrak{r} = [\mathcal{L}, \mathfrak{r}] \tag{1.2}$$

is called the nilpotent radical of $\mathcal{L}$.

Since $[\mathfrak{s}, \mathfrak{r}] \subseteq \mathfrak{r}$, $\mathfrak{r}$ can be viewed as an $\mathfrak{s}$-module. The fact that $\mathfrak{s}$ is semisimple implies that $\mathfrak{r}$ can be decomposed into a direct sum of irreducible submodules. Let $\mathfrak{r}_0$ be the direct sum of trivial submodules, and $\mathfrak{r}_n$ the direct sum of non-trivial irreducible submodules. Denote by $C(\mathfrak{r}_0)$ the centre of $\mathfrak{r}_0$ and let $C_{\mathfrak{r}_0}(\mathfrak{r}_n) = \{x \in \mathfrak{r}_0 \mid [x, \mathfrak{r}_n] = 0\}$. It has been proved in [1] that $\mathcal{L}$ can be decomposed into the direct sum of complete ideals as follows:

$$\mathcal{L} = (\mathfrak{s} + C(\mathfrak{r}_0) + \mathfrak{r}_n) \oplus C_{\mathfrak{r}_0}(\mathfrak{r}_n) \tag{1.3}$$

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and $C_{\mathfrak{r}_0}(\mathfrak{r}_n)$ is an extension of abelian Lie algebra by abelian Lie algebra, and if the base field of $\mathcal{L}$ is algebraically closed, then $C_{\mathfrak{r}_0}(\mathfrak{r}_n)$ is a direct sum of 2-dimensional simple complete ideals. A complete Lie algebra is called a simple complete Lie algebra if it has no non-trivial complete ideals. By (1.3), if we study complete Lie algebras with commutative nilpotent radical, it is sufficient to discuss $\mathfrak{s}+C(\mathfrak{r}_0)+\mathfrak{r}_n$. In this case, $\mathfrak{n}_0 = \mathfrak{r}_n$.

In [1], a complete Lie algebra $G = \mathfrak{g} \dotplus V \dotplus \mathfrak{a}$ over the complex field $\mathbf{C}$ was constructed in the following way. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and (ρ, V) a representation of $\mathfrak{g}$ which is decomposed into

$$V = V_1 \dotplus V_2 \dotplus \cdots \dotplus V_n,$$

where V_i $(i = 1, 2, \ldots, n)$ are irreducible invariant subspaces of ρ. Let I_i $(i = 1, 2, \ldots, n)$ be the linear transformations of V such that

$$I_i\left(\sum_{j\neq i} V_j\right) = 0, \qquad I_i|_{V_i} = \mathrm{id}\,|_{V_i}.$$

Let $\mathfrak{a}$ be the subalgebra of $\mathrm{gl}(V)$ generated by $I_1, I_2, \ldots, I_n$. Then $\mathfrak{a}$ is an abelian Lie algebra. Set

$$G = \mathfrak{g} \dotplus V \dotplus \mathfrak{a}.$$

The bracket in G is defined by

$$[x_1 + \upsilon_1 + A_1, x_2 + \upsilon_2 + A_2] = [x_1, x_2] + \rho(x_1)\upsilon_2 - \rho(x_2)\upsilon_1 - A_2(\upsilon_1) + A_1(\upsilon_2),$$

where $x_1, x_2 \in \mathfrak{g}$, $\upsilon_1, \upsilon_2 \in V$, $A_1, A_2 \in \mathfrak{a}$. Then G is a complete Lie algebra with commutative nilpotent radical.

In this paper, we discuss the properties of finite-dimensional complete Lie algebras with commutative nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$ over the complex field $\mathbf{C}$. We deduce that if $\mathfrak{r}_n$ is the direct sum of t irreducible submodules, then the dimension of $\mathfrak{r}_0$ is t. We prove that $\mathfrak{r}_n$ can be decomposed properly so that the action of every element of $\mathrm{ad}_{\mathfrak{r}_n} \mathfrak{r}_0$ on each irreducible submodule is scalar. Therefore, finite-dimensional complete Lie algebras with commutative nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$ over $\mathbf{C}$ are in fact the Lie algebras constructed above. Hence, all finite-dimensional complete Lie algebras with commutative nilpotent radical over $\mathbf{C}$ are known.

2. Some lemmas

Let

$$\mathcal{L} = \mathfrak{s} + \mathfrak{r} = \mathfrak{s} + (\mathfrak{r}_0 + \mathfrak{r}_n) \tag{2.1}$$

be the Levi decomposition of $\mathcal{L}$. Then we have the following results.

Lemma 2.1 ([1]).

$$[\mathfrak{s}, \mathfrak{r}_0] = (0), \tag{2.2}$$

$$[\mathfrak{s}, \mathfrak{r}_n] = \mathfrak{r}_n, \tag{2.3}$$

$$[\mathfrak{r}_0, \mathfrak{r}_0] \subseteq \mathfrak{r}_0. \tag{2.4}$$

Lemma 2.2.

$$[\mathfrak{r}_0, \mathfrak{r}_n] \subseteq \mathfrak{r}_n. \tag{2.5}$$

Proof. By (2.3) and (2.2), we have

$$\begin{aligned}[\mathfrak{r}_0, \mathfrak{r}_n] &= [\mathfrak{r}_0, [\mathfrak{s}, \mathfrak{r}_n]] \subseteq [[\mathfrak{r}_0, \mathfrak{s}], \mathfrak{r}_n] + [[\mathfrak{r}_n, \mathfrak{r}_0], \mathfrak{s}] \\ &= [\mathfrak{s}, [\mathfrak{r}_0, \mathfrak{r}_n]] \subseteq [\mathfrak{s}, \mathfrak{r}] = [\mathfrak{s}, \mathfrak{r}_0 + \mathfrak{r}_n] = \mathfrak{r}_n.\end{aligned}$$

The lemma holds. □

Lemma 2.3. *Let $\mathcal{L}$ be a Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Then*

$$[\mathfrak{r}_0, \mathfrak{r}_0] = [\mathfrak{r}_n, \mathfrak{r}_n] = (0). \tag{2.6}$$

Proof. Since $\mathfrak{n}_0$ is commutative and $\mathfrak{n}_0 = \mathfrak{r}_n$, we have

$$\mathfrak{n}_0^{(1)} = \mathfrak{r}_n^{(1)} = [\mathfrak{r}_n, \mathfrak{r}_n] = (0)$$

and

$$\mathfrak{n}_0 = [\mathfrak{s} + \mathfrak{r}_0 + \mathfrak{r}_n, \mathfrak{r}_0 + \mathfrak{r}_n] = \mathfrak{r}_n + [\mathfrak{r}_0, \mathfrak{r}_0].$$

The lemma follows from (2.4) and Lemma 2.2. □

Let $\mathfrak{a}$ be an irreducible $\mathfrak{s}$-module. Then $\mathfrak{a}$ is a highest weight $\mathfrak{s}$-module since $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ is finite dimensional.

Let

$$\mathfrak{r}_n = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_m \tag{2.7}$$

be the direct sum of submodules such that

$$\mathfrak{a}_i = \mathfrak{a}_{i1} \oplus \mathfrak{a}_{i2} \oplus \cdots \oplus \mathfrak{a}_{in_i} \qquad (i = 1, 2, \ldots, m), \tag{2.8}$$

where $\mathfrak{a}_{ik}$ ($k = 1, 2, \ldots, n_i$) are irreducible highest weight $\mathfrak{s}$-modules with highest weight λ_i and, if $i \neq j$, then $\lambda_i \neq \lambda_j$, $i, j = 1, 2, \ldots, m$.

Denote by z_{ij} ($j = 1, 2, \ldots, n_i$) the highest weight vectors of $\mathfrak{a}_{ij}$ ($j = 1, 2, \ldots, n_i$) respectively, and by V_i the linear space with basis $\{z_{i1}, \ldots, z_{in_i}\}$ ($i = 1, 2, \ldots, m$). Then

$$\dim V_i = n_i, \qquad i = 1, 2, \ldots, m. \tag{2.9}$$

Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{s}$ and Δ_0 the root system. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the simple root system and $\mathfrak{s} = \mathfrak{h}_0 + \sum_{\alpha \in \Delta_0} \mathfrak{s}_\alpha$ the root space decomposition with respect to $\mathfrak{h}_0$.

Lemma 2.4. *Let $\mathcal{L}$ be a Lie algebra with commutative nilpotent radical, $D \in$* Der$(\mathfrak{s} + \mathfrak{r}_n)$ *be such that $D(\mathfrak{s}) = (0)$ and $D(\mathfrak{r}_n) \subseteq \mathfrak{r}_n$. Then*

$$D(V_i) \subseteq V_i, \qquad i = 1, 2, \ldots, m, \tag{2.10}$$

and D is uniquely determined by $D|_{V_i}$ ($i = 1, 2, \ldots, m$).

Proof. Since $D \in \mathrm{Der}(\mathfrak{s} + \mathfrak{r}_n)$ and $D(\mathfrak{s}) = 0$, for any $h \in \mathfrak{h}_0$ we have

$$D[h, z_{ij}] = \lambda_i(h) D z_{ij}. \tag{2.11}$$

Note that z_{ij} is a highest weight vector of $\mathfrak{a}_{ij}$, therefore for $\alpha \in \Delta_0^+$ and $e_\alpha \in \mathfrak{s}_\alpha$, we have

$$[e_\alpha, z_{ij}] = 0.$$

Therefore

$$[e_\alpha, D z_{ij}] = D[e_\alpha, z_{ij}] = 0. \tag{2.12}$$

It is clear from (2.11) and (2.12) that $Dz_{i1}, Dz_{i2}, \dots, Dz_{in_i}$ $(i = 1, 2, \dots, m)$ are highest weight vectors associated to highest weight λ_i $(i = 1, 2, \dots, m)$. So $Dz_{ij} \in V_i$ $(j = 1, 2, \dots, n_i,\ \ i = 1, 2, \dots, m)$. On the other hand, for $z \in \mathfrak{a}_{ij}$, z has the form:

$$z = [x_1, [x_2, [\dots, [x_q, z_{ij}]] \dots]],$$

where $x_i \in \mathfrak{s}$ $(i = 1, 2, \dots, q)$. So

$$Dz = [x_1, [x_2, [\dots, [x_q, Dz_{ij}]] \dots]].$$

The lemma is proved. □

Lemma 2.5. *Let $\mathcal{L}$ be a Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. $D \in \mathrm{Der}(\mathfrak{s} + \mathfrak{r}_n)$ is such that $D(\mathfrak{s}) = (0)$ and $D(\mathfrak{r}_n) \subseteq \mathfrak{r}_n$. Then*

1)

$$D(\mathfrak{a}_i) \subseteq \mathfrak{a}_i, \qquad i = 1, 2, \dots, m. \tag{2.13}$$

2) *Set*

$$\mathcal{L}_i = \{D \in \mathrm{Der}(\mathfrak{s} + \mathfrak{r}_n) | D(\mathfrak{s}) = (0),\ D(\mathfrak{r}_n) \subseteq \mathfrak{r}_n \text{ and } D|_{\mathfrak{a}_j} = 0,\ \text{if } j \neq i\}.$$

Then the Lie algebra $\mathcal{L}_i$ is isomorphic to the general linear Lie algebra $\mathrm{gl}(V_i)$ which consists of all linear transformations of V_i, $i = 1, 2, \dots, m$.

Proof. 1) follows from Lemma 2.4. Define

$$\varphi(D) = D|_{V_i}, \qquad \text{for } D \in \mathcal{L}_i.$$

Then φ is a linear mapping from $\mathcal{L}_i$ to $\mathrm{gl}(V_i)$. For $D_1, D_2 \in \mathcal{L}_i$, if $D_1 \neq D_2$, then from Lemma 2.4 we know $\varphi(D_1) \neq \varphi(D_2)$. Let $A \in \mathrm{gl}(V_i)$. Define the linear transformation D of $\mathfrak{s} + \mathfrak{r}_n$ by

$$D(\mathfrak{s}) = 0,\ D|_{\mathfrak{a}_j} = 0, \qquad \text{if } j \neq i,$$

$$D[x_1, [x_2, \dots, [x_q, z_{ij}]] \dots] = [x_1, [x_2, \dots, [x_q, Az_{ij}]] \dots] \qquad (j = 1, 2, \dots, n_i),$$

where $x_1, x_2, \dots, x_q \in \mathfrak{s}$. Then $D \in \mathrm{Der}(\mathfrak{s} + \mathfrak{r}_n)$. So φ is a bijection.

For $D_1, D_2 \in \mathcal{L}_i$, we have

$$\begin{aligned} \varphi[D_1, D_2] &= [D_1, D_2]|_{V_i} \\ &= D_1 D_2|_{V_i} - D_2 D_1|_{V_i} = D_1|_{V_i} D_2|_{V_i} - D_2|_{V_i} D_1|_{V_i} \\ &= [\varphi(D_1), \varphi(D_2)]. \end{aligned}$$

Hence φ is a homomorphism from the Lie algebra $\mathcal{L}_i$ to the Lie algebra $\mathrm{gl}(V_i)$. The lemma holds. □

3. The structure of radical $\mathfrak{r}$

Lemma 3.1. *Let D be an inner derivation of $\mathcal{L}$ and $D(\mathfrak{s}) = (0)$. Then there exists an element $y \in \mathfrak{r}_0$ such that*

$$D = \mathrm{ad}\, y.$$

Proof. Since D is an inner derivation of $\mathcal{L}$, there exist $x \in \mathfrak{s}$, $y \in \mathfrak{r}_0$, $z \in \mathfrak{r}_n$ such that

$$D = \mathrm{ad}(x + y + z).$$

$D(\mathfrak{s}) = (0)$ implies that

$$[x + y + z, \mathfrak{s}] = [x, \mathfrak{s}] + [z, \mathfrak{s}] = (0).$$

From the fact that $[x, \mathfrak{s}] \subseteq \mathfrak{s}$, $[z, \mathfrak{s}] \subseteq \mathfrak{r}_n$, we have

$$[x, \mathfrak{s}] = (0), \qquad [z, \mathfrak{s}] = (0).$$

But $\mathfrak{s}$ is semisimple and $\mathfrak{r}_n$ is the direct sum of non-trivial submodules. Therefore

$$x = z = 0.$$

□

Lemma 3.2. *Let $\mathcal{L}$ be a Lie algebra with trivial centre and commutative nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Then*

1) $\mathfrak{r}_0$ *is isomorphic to* $\mathrm{ad}_{\mathfrak{r}_n} \mathfrak{r}_0$.
2) *For $x \in \mathfrak{r}_0$, we have*

$$\mathrm{ad}_{\mathfrak{r}_n} x(\mathfrak{a}_i) \subseteq \mathfrak{a}_i, \qquad \mathrm{ad}_{\mathfrak{r}_n} x|_{V_i} \in \mathrm{gl}(V_i) \qquad (i = 1, 2, \ldots, m).$$

Lemma 3.3. *Let $\mathcal{L}$ be a complete Lie algebra with commutative nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. For $x \in \mathfrak{r}_0$, define the linear transformations D_i $(i = 1, 2, \ldots, m)$ of $\mathcal{L}$ by*

$$D_i|_{\mathfrak{s}+\mathfrak{r}_0} = 0, \quad D_i|_{\mathfrak{a}_i} = \mathrm{ad}\, x|_{\mathfrak{a}_i}, \quad D_i|_{\mathfrak{a}_j} = 0 \qquad (j = 1, \ldots, i-1, i+1, \ldots, m).$$

Then there exist $y_1, y_2, \ldots, y_m \in \mathfrak{r}_0$ such that

$$D_i = \mathrm{ad}\, y_i \qquad (i = 1, 2, \ldots, m).$$

Proof. For $s_1, s_2 \in \mathfrak{s}$, $x_1, x_2 \in \mathfrak{r}_0$, $y_1, y_2 \in \mathfrak{a}_i$, $z_1, z_2 \in \mathfrak{a}_1 + \cdots + \mathfrak{a}_{i-1} + \mathfrak{a}_{i+1} + \cdots + \mathfrak{a}_m$, by 2) of Lemma 3.2, we have $[x_1, y_2], [x_2, y_1] \in \mathfrak{a}_i$, $[x_1, z_2], [x_2, z_1] \in \mathfrak{a}_1 + \cdots + \mathfrak{a}_{i-1} + \mathfrak{a}_{i+1} + \cdots + \mathfrak{a}_m$. So by (2.2) and (2.6) we deduce that

$$\begin{aligned} &D_i[s_1 + x_1 + y_1 + z_1, s_2 + x_2 + y_2 + z_2] \\ &\quad = \mathrm{ad}\, x([s_1, y_2] + [x_1, y_2] + [y_1, s_2] + [y_1, x_2]) \\ &\quad = [s_1, \mathrm{ad}\, x(y_2)] + [x_1, \mathrm{ad}\, x(y_2)] + [\mathrm{ad}\, x(y_1), s_2] + [\mathrm{ad}\, x(y_1), x_2], \end{aligned}$$

$$\begin{aligned} &[D_i(s_1 + x_1 + y_1 + z_1), s_2 + x_2 + y_2 + z_2] \\ &\qquad + [s_1 + x_1 + y_1 + z_1, D_i(s_2 + x_2 + y_2 + z_2)] \\ &\quad = [\mathrm{ad}\, x(y_1), s_2 + x_2 + y_2 + z_2] + [s_1 + x_1 + y_1 + z_1, \mathrm{ad}\, x(y_2)] \\ &\quad = [\mathrm{ad}\, x(y_1), s_2] + [\mathrm{ad}\, x(y_1), x_2] + [s_1, \mathrm{ad}\, x(y_2)] + [x_1, \mathrm{ad}\, x(y_2)]. \end{aligned}$$

Therefore $D_i \in \mathrm{Der}\, \mathcal{L}$ $(i = 1, 2, \ldots, m)$. Since $\mathcal{L}$ is a complete Lie algebra, there exist $y_1, y_2, \ldots, y_m \in \mathcal{L}$ such that

$$D_i = \mathrm{ad}\, y_i \qquad (i = 1, 2, \ldots, m).$$

By Lemma 3.1, $y_i \in \mathfrak{r}_0$ $(i = 1, 2, \ldots, m)$. □

Theorem 3.1. *Let $\mathcal{L}$ be a complete Lie algebra with commutative nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Set*

$$\mathfrak{h}_i = \{\mathrm{ad}\, x | x \in \mathfrak{r}_0 \text{ and } \mathrm{ad}\, x|_{\mathfrak{a}_j} = 0 \ (j = 1, \ldots, i-1, i+1, \ldots, m)\}, \qquad i = 1, 2, \ldots, m, \tag{3.1}$$

$$H_i = \mathfrak{h}_i|_{V_i} = \varphi(\mathfrak{h}_i|_{\mathfrak{s}+\mathfrak{r}_n}), \qquad i = 1, 2, \ldots, m. \tag{3.2}$$

Then

1) $\mathfrak{h}_i|_{\mathfrak{s}+\mathfrak{r}_n}$ *is a commutative subalgebra of* $\mathcal{L}_i$ *and*

$$\mathrm{ad}\, \mathfrak{r}_0 = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_m \simeq \mathfrak{h}_1|_{\mathfrak{s}+\mathfrak{r}_n} \oplus \cdots \oplus \mathfrak{h}_m|_{\mathfrak{s}+\mathfrak{r}_n}. \tag{3.3}$$

2) $\mathfrak{h}_i$ *is isomorphic to H_i which is an abelian subalgebra of* $\mathrm{gl}(V_i)$, $i = 1, 2, \ldots, m$.

Proof. The proof follows from Lemma 3.3 and Lemma 2.5. □

Lemma 3.4. *Let $\mathcal{L}$ be a Lie algebra with commutative nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Let $D \in \mathrm{Der}(\mathfrak{s}+\mathfrak{r}_n)$ be such that $D(\mathfrak{s}) = 0$ and $D(\mathfrak{r}_n) \subseteq \mathfrak{r}_n$. Extend D to the linear transformation of $\mathcal{L}$ by*

$$D|_{\mathfrak{r}_0} = 0.$$

Then D is a derivation of $\mathcal{L}$ if and only if

$$[D, \mathrm{ad}_{\mathfrak{s}+\mathfrak{r}_n} \mathfrak{r}_0] = 0.$$

Proof. $D \in \mathrm{Der}\,\mathcal{L}$ if and only if the formula

$$\begin{aligned} &D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] \\ &\quad = [D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)] \end{aligned}$$

holds, for any $x_1, x_2 \in \mathfrak{s}$, $y_1, y_2 \in \mathfrak{r}_0$, $z_1, z_2 \in \mathfrak{r}_n$.

Note that

$$\begin{aligned} &D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] \\ &\quad = D[z_1, x_1] + D[z_1, y_2] + D[x_1, z_2] + D[y_1, z_2] + D[z_1, z_2], \end{aligned}$$

$$\begin{aligned} &[D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)] \\ &\quad = [Dz_1, x_2] + [Dz_1, y_2] + [x_1, Dz_2] + [y_1, Dz_2] + [Dz_1, z_2] + [z_1, Dz_2] \end{aligned}$$

and $D \in \mathrm{Der}(\mathfrak{s} + \mathfrak{r}_n)$ is such that $D(\mathfrak{s}) = 0$. Therefore $D \in \mathrm{Der}\,\mathcal{L}$ if and only if

$$D[y, z] = [y, Dz], \qquad \text{for any } y \in \mathfrak{r}_0,\ z \in \mathfrak{r}_n,$$

i.e.,

$$[D, \mathrm{ad}_{\mathfrak{s}+\mathfrak{r}_n} \mathfrak{r}_0] = 0.$$

□

Theorem 3.2. *Let $\mathcal{L}$ be a complete Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$, and let $\mathfrak{h}_i$, H_i, $\mathcal{L}_i$ $(i = 1, 2, \ldots, m)$ be as above. Then*

$$C_{\mathcal{L}_i}(\mathfrak{h}_i|_{\mathfrak{s}+\mathfrak{r}_n}) = \mathfrak{h}_i|_{\mathfrak{s}+\mathfrak{r}_n}, \qquad i = 1, 2, \ldots, m, \tag{3.4}$$

$$C_{\mathrm{gl}(V_i)}(H_i) = H_i, \qquad i = 1, 2, \ldots, m. \tag{3.5}$$

Proof. Since $\mathfrak{r}_0$ is commutative, we have

$$\mathfrak{h}_i|_{\mathfrak{s}+\mathfrak{r}_n} \subseteq C_{\mathcal{L}_i}(\mathfrak{h}_i|_{\mathfrak{s}+\mathfrak{r}_n}).$$

Let $D_i \in \mathcal{L}_i$ be such that $[D_i, \mathfrak{h}_i|_{\mathfrak{s}+\mathfrak{r}_n}] = 0$. Then by (3.3) we have

$$[D_i, \mathrm{ad}_{\mathfrak{s}+\mathfrak{r}_n} \mathfrak{r}_0] = 0.$$

Extend D_i to the linear transformation of $\mathcal{L}$ by

$$D_i|_{\mathfrak{r}_0} = 0.$$

Then by Lemma 3.4, D_i is a derivation of $\mathcal{L}$. Note that $\mathcal{L}$ is a complete Lie algebra, therefore

$$D_i = \mathrm{ad}\, z_i \in \mathfrak{h}_i, \qquad z_i \in \mathfrak{r}_0 \qquad (i = 1, 2, \ldots, m).$$

Hence $D_i|_{\mathfrak{s}+\mathfrak{r}_n} \in \mathfrak{h}_i|_{\mathfrak{s}+\mathfrak{r}_n}$ $(i = 1, 2, \ldots, m)$. This proves (3.4). (3.5) follows from the fact that $\mathcal{L}_i$ is isomorphic to $\mathrm{gl}(V_i)$ and $\mathfrak{h}_i|_{\mathfrak{s}+\mathfrak{r}_n}$ is isomorphic to H_i. □

Corollary 3.1. *There exist elements* $\operatorname{ad} x_i \in \mathfrak{h}_i$ $(i = 1, 2, \ldots, m)$ *such that*

$$\operatorname{ad} x_i|_{\mathfrak{a}_i} = \operatorname{id}|_{\mathfrak{a}_i} \qquad (i = 1, 2, \ldots, m).$$

Thus H_i *contains identical transformations* $I_i = \operatorname{id}|_{V_i}$, $i = 1, 2, \ldots, m$.

From Corollary 3.1, we have

$$H_i = H_i' \oplus \mathbf{C} I_i \tag{3.6}$$

and $H_i' \subseteq \operatorname{sl}(V_i)$ $(i = 1, 2, \ldots, m)$, where $\operatorname{sl}(V_i)$ is the special linear Lie algebra on V_i. Since $\mathfrak{h}_i$ is isomorphic to H_i, we have

$$\mathfrak{h}_i = \mathfrak{h}_i' \oplus \mathbf{C} \operatorname{ad} x_i \qquad (i = 1, 2, \ldots, m), \tag{3.7}$$

where $\mathfrak{h}_i'$ is isomorphic to H_i' and $\operatorname{ad} x_i$ is the same as in Corollary 3.1. In fact, $\mathfrak{h}_i'|_{V_i} = H_i'$, $\operatorname{ad} x_i|_{V_i} = I_i$ $(i = 1, 2, \ldots, m)$.

We will show that H_i' is a maximal torus subalgebra of $\operatorname{sl}(V_i)$, $i = 1, \ldots, m$.

Lemma 3.5. *Let* $\mathcal{L}$ *be a Lie algebra with trivial centre and nilpotent radical* $\mathfrak{n}_0 = \mathfrak{r}_n$. *Let* $D_1 \in \operatorname{Der}(\mathfrak{s} + \mathfrak{r}_n)$ *be such that* $D_1(\mathfrak{s}) = (0)$ *and* $D_1(\mathfrak{r}_n) \subseteq \mathfrak{r}_n$, *and*

$$[D_1, \operatorname{ad}_{\mathfrak{s}+\mathfrak{r}_n} \mathfrak{r}_0] \subseteq \operatorname{ad}_{\mathfrak{s}+\mathfrak{r}_n} \mathfrak{r}_0. \tag{3.8}$$

Then there exists $D \in \operatorname{Der} \mathcal{L}$ *such that*

$$D|_{\mathfrak{s}+\mathfrak{r}_n} = D_1,$$

and if $[D_1, \operatorname{ad}_{\mathfrak{s}+\mathfrak{r}_n} \mathfrak{r}_0] \neq 0$, *then* D *is an outer derivation of* $\mathcal{L}$.

Proof. From (3.8), for any $x \in \mathfrak{r}_0$ there exists $y \in \mathfrak{r}_0$ such that

$$[D_1, \operatorname{ad}_{\mathfrak{s}+\mathfrak{r}_n} x] = \operatorname{ad}_{\mathfrak{s}+\mathfrak{r}_n} y.$$

Define the transformation D_2 of $\mathfrak{r}_0$ by

$$D_2(x) = y.$$

By Lemma 3.2 there is no ambiguity in the definition of D_2. It is clear that D_2 is a linear transformation of $\mathfrak{r}_0$. D_2 is a derivation of $\mathfrak{r}_0$ since $\mathfrak{r}_0$ is commutative. From (3.8), for $x \in \mathfrak{r}_0$ and $z \in \mathfrak{r}_n$ we have

$$D_1[x, z] = [x, D_1 z] + [D_2 x, z]. \tag{3.9}$$

Define the linear transformation D of $\mathcal{L}$ by

$$D|_{\mathfrak{s}+\mathfrak{r}_n} = D_1, \qquad D|_{\mathfrak{r}_0} = D_2.$$

Then for any $x_1, x_2 \in \mathfrak{s}$, $y_1, y_2 \in \mathfrak{r}_0$, $z_1, z_2 \in \mathfrak{r}_n$, we have

$$\begin{aligned} &D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] \\ &\quad = [D_1 z_1, x_2] + D_1[z_1, y_2] + [x_1, D_1 z_2] + D_1[y_1, z_2] + D_1[z_1, z_2], \end{aligned}$$

$$\begin{aligned} &[D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)] \\ &\quad = [D_1 z_1, x_2] + [D_1 z_1, y_2] + [x_1, D_1 z_2] + [y_1, D_1 z_2] + [D_1 z_1, z_2] \\ &\qquad + [z_1, D_1 z_2] + [D_2 y_1, z_2] + [z_1, D_2 y_2]. \end{aligned}$$

Since $D_1 \in \operatorname{Der}(\mathfrak{s} + \mathfrak{r}_n)$ is such that $D_1(\mathfrak{s}) = (0)$ and (3.9) holds, we have

$$\begin{aligned} &D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] \\ &\quad = [D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)]. \end{aligned}$$

Hence $D \in \operatorname{Der} \mathcal{L}$. If $D \in \operatorname{ad} \mathcal{L}$, then by Lemma 3.1, we have

$$D_1 \in \operatorname{ad}_{\mathfrak{s}+\mathfrak{r}_n} \mathfrak{r}_0.$$

But $\mathfrak{r}_0$ is commutative. Therefore,

$$[D_1, \mathrm{ad}_{\mathfrak{s}+\mathfrak{r}_n} \mathfrak{r}_0] = 0.$$

Thus we have completed the proof. □

Theorem 3.3. *Let $\mathcal{L}$ be a complete Lie algebra with commutative nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Then H_i' is a maximal torus subalgebra of $\mathrm{sl}(V_i)$, $i = 1, 2, \ldots, m$.*

Proof. Let $T_i \in \mathrm{sl}(V_i)$ be such that $[T_i, H_i] \subseteq H_i$. Then $D_i = \varphi^{-1}(T_i) \in \mathcal{L}_i$, and D_i satisfies the conditions of Lemma 3.5. Thus there exists $D_i' \in \mathrm{Der}\,\mathcal{L}$ such that

$$D_i'|_{\mathfrak{s}+\mathfrak{r}_n} = D_i, \qquad i = 1, 2, \ldots, m.$$

Note that $\mathcal{L}$ is a complete Lie algebra, so by Lemma 3.1, we have

$$D_i' \in \mathrm{ad}_{\mathcal{L}} \mathfrak{r}_0, \qquad i = 1, 2, \ldots, m.$$

The fact that $D_i|_{\mathfrak{a}_j} = 0$ (when $j \neq i$) implies $D_i \in \mathfrak{h}_i$ ($i = 1, 2, \ldots, m$). Thus we have

$$T_i = \varphi(D_i'|_{\mathfrak{s}+\mathfrak{r}_n}) = \varphi(D_i) \in H_i', \qquad i = 1, 2, \ldots, m.$$

From this we know that H_i' is a self-normal subalgebra of $\mathrm{sl}(V_i)$. Since H_i' is commutative, we deduce that H_i' is a Cartan subalgebra of $\mathrm{sl}(V_i)$. But $\mathrm{sl}(V_i)$ is a simple Lie algebra, therefore H_i' is a maximal torus subalgebra of $\mathrm{sl}(V_i)$. □

Corollary 3.2. *Let $\mathcal{L}$ be a complete Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Then $H_i = H_i' + \mathbf{C}I_i$ is a maximal torus subalgebra of $\mathrm{gl}(V_i)$, $i = 1, 2, \ldots, m$.*

Corollary 3.3. *Let $\mathcal{L}$ be a complete Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Then*

$$\dim \mathfrak{h}_i = n_i, \qquad i = 1, 2, \ldots, m. \tag{3.10}$$

Thus we have

$$\dim \mathfrak{r}_0 = n_1 + n_2 + \cdots + n_m. \tag{3.11}$$

Since H_i is a maximal torus subalgebra of $\mathrm{gl}(V_i)$, we can choose a basis $\{y_{i1}, y_{i2}, \ldots, y_{in_i}\}$ of V_i such that the matrix of every element of H_i relative to the basis is diagonal, $i = 1, 2, \ldots, m$. On the other hand, $y_{i1}, y_{i2}, \ldots, y_{in_i}$ are highest weight vectors associated to highest weight λ_i. So the highest weight $\mathfrak{s}$-module $\mathfrak{a}_{ij}'$ whose highest weight vector is y_{ij} is irreducible and

$$\mathfrak{a}_i = \mathfrak{a}_{i1}' \oplus \mathfrak{a}_{i2}' \oplus \cdots \oplus \mathfrak{a}_{in_i}', \qquad i = 1, 2, \ldots, m.$$

From this we deduce that

Theorem 3.4. *Let $\mathcal{L}$ be a complete Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Let $\mathfrak{r}_n$ be the direct sum of t irreducible submodules. Then*

$$\dim \mathfrak{r}_0 = t$$

and $\mathfrak{r}_n$ can be decomposed properly into the direct sum of irreducible submodules:

$$\mathfrak{r}_n = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_t$$

so that

$$\mathfrak{r}_0 \simeq \mathrm{ad}_{\mathfrak{r}_n} \mathfrak{r}_0 = \mathbf{C}I_1 \oplus \mathbf{C}I_2 \oplus \cdots \oplus \mathbf{C}I_t,$$

where I_i is the linear transformation of $\mathfrak{r}_n$ such that

$$I_i\left(\sum_{j\neq i}\mathfrak{m}_j\right) = (0), \qquad I_i|_{\mathfrak{m}_i} = \mathrm{id}\,|_{\mathfrak{m}_i} \qquad (i = 1, 2, \ldots, t).$$

So $\mathcal{L}$ is in fact the complete Lie algebra constructed in section 1.

Theorem 3.5. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete Lie algebras. Let $\mathfrak{n}_i$ be the nilpotent radical of $\mathcal{L}_i$ and $\mathfrak{s}_i$ be the Levi subalgebra of $\mathcal{L}_i$ ($i = 1, 2$). Then $\mathcal{L}_1$ is isomorphic to $\mathcal{L}_2$ if and only if the Lie algebra $\mathfrak{s}_1$ is isomorphic to $\mathfrak{s}_2$ and the $\mathfrak{s}_1$-module $\mathfrak{n}_1$ is isomorphic to the $\mathfrak{s}_2$-module $\mathfrak{n}_2$.*

Theorem 3.6. *Let $\mathfrak{s}$ be a semisimple Lie algebra and $\mathfrak{n}$ an $\mathfrak{s}$-module. Define*

$$[s_1 + x_1, s_2 + x_2] = [s_1, s_2] + s_1(x_2) - s_2(x_1), \tag{3.12}$$

where $s_1, s_2 \in \mathfrak{s}$, $x_1, x_2 \in \mathfrak{n}$. Then there is a unique up to isomorphism complete Lie algebra $\mathcal{L}$ such that $\mathfrak{n}$ is its nilpotent radical and $\mathfrak{s}$ is its Levi subalgebra and its bracket satisfies (3.12).

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