THE CLASSIFICATION OF COMPLETE LIE ALGEBRAS WITH COMMUTATIVE NILPOTENT RADICAL

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ABSTRACT. The work in this paper is a continuation of an earlier paper of the second author (Acta Math. 34 (1991), 191–202). We discuss the properties of finite-dimensional complete Lie algebras with abelian nilpotent radical over the complex field \( \mathbb{C} \). We solve the problems of isomorphism, classification and realization of complete Lie algebras with commutative nilpotent radical.

1. INTRODUCTION

A Lie algebra \( \mathcal{L} \) is called a complete Lie algebra if its centre \( C(\mathcal{L}) \) is zero and its derivations are all inner. The definition of complete Lie algebra was given by N. Jacobson in 1962 (cf. [8]). But the first important result—the derivation tower theorem—was obtained by E. V. Schenkman in 1951 (cf. [9]). In recent years, the theory of complete Lie algebras has been developing (see [1]–[7]). In [1], the properties of complete Lie algebras with commutative nilpotent radical have been discussed. The work in this paper is a continuation of [1].

Let \( \mathcal{L} \) be a finite-dimensional Lie algebra over a field of characteristic zero. Then \( \mathcal{L} \) has the Levi decomposition:

\[
\mathcal{L} = s + \mathfrak{r},
\]

where \( s \) is a maximal semisimple subalgebra of \( \mathcal{L} \) and is called the Levi subalgebra of \( \mathcal{L} \), and \( \mathfrak{r} \) is the maximal solvable ideal of \( \mathcal{L} \) and is called the radical of \( \mathcal{L} \). The ideal

\[
\mathfrak{n}_0 = [\mathcal{L}, \mathcal{L}] \cap \mathfrak{r} = [\mathcal{L}, \mathfrak{r}]
\]

is called the nilpotent radical of \( \mathcal{L} \).

Since \([s, \mathfrak{r}] \subseteq \mathfrak{r}\), \( \mathfrak{r} \) can be viewed as an \( s \)-module. The fact that \( s \) is semisimple implies that \( \mathfrak{r} \) can be decomposed into a direct sum of irreducible submodules. Let \( \mathfrak{r}_0 \) be the direct sum of trivial submodules, and \( \mathfrak{r}_n \), the direct sum of non-trivial irreducible submodules. Denote by \( C(\mathfrak{r}_0) \) the centre of \( \mathfrak{r}_0 \) and let \( C(\mathfrak{r}_0)(\mathfrak{r}_n) = \{ x \in \mathfrak{r}_0 | [x, \mathfrak{r}_n] = 0 \} \). It has been proved in [1] that \( \mathcal{L} \) can be decomposed into the direct sum of complete ideals as follows:

\[
\mathcal{L} = (s + C(\mathfrak{r}_0) + \mathfrak{r}_n) \oplus C(\mathfrak{r}_0)(\mathfrak{r}_n)
\]

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and $\mathcal{C}_0(r_n)$ is an extension of abelian Lie algebra by abelian Lie algebra, and if the base field of $\mathcal{L}$ is algebraically closed, then $\mathcal{C}_0(r_n)$ is a direct sum of 2-dimensional simple complete ideals. A complete Lie algebra is called a simple complete Lie algebra if it has no non-trivial complete ideals. By (1.3), if we study complete Lie algebras with commutative nilpotent radical, it is sufficient to discuss $s + C(r_0) + r_n$.

In this case, $n_0 = r_n$.

In [1], a complete Lie algebra $G = g + V + a$ over the complex field $\mathbb{C}$ was constructed in the following way. Let $g$ be a complex semisimple Lie algebra and $(\rho, V)$ a representation of $g$ which is decomposed into $V = V_1 + V_2 + \cdots + V_n$, where $V_i$ ($i = 1, 2, \ldots, n$) are irreducible invariant subspaces of $\rho$. Let $I_i$ ($i = 1, 2, \ldots, n$) be the linear transformations of $V$ such that 

$$I_i \left( \sum_{j \neq i} V_j \right) = 0, \quad I_i|_{V_i} = \text{id}|_{V_i}.$$

Let $a$ be the subalgebra of $\text{gl}(V)$ generated by $I_1, I_2, \ldots, I_n$. Then $a$ is an abelian Lie algebra. Set

$$G = g + V + a.$$

The bracket in $G$ is defined by 

$$[x_1 + v_1 + A_1, x_2 + v_2 + A_2] = [x_1, x_2] + \rho(x_1)v_2 - \rho(x_2)v_1 - A_2(v_1) + A_1(v_2),$$

where $x_1, x_2 \in g, v_1, v_2 \in V, A_1, A_2 \in a$. Then $G$ is a complete Lie algebra with commutative nilpotent radical.

In this paper, we discuss the properties of finite-dimensional complete Lie algebras with commutative nilpotent radical $n_0 = r_n$ over the complex field $\mathbb{C}$. We deduce that if $r_n$ is the direct sum of $t$ irreducible submodules, then the dimension of $r_n$ is $t$. We prove that $r_n$ can be decomposed properly so that the action of every element of $\text{ad}_{r_n} r_n$ on each irreducible submodule is scalar. Therefore, finite-dimensional complete Lie algebras with commutative nilpotent radical $n_0 = r_n$ over $\mathbb{C}$ are in fact the Lie algebras constructed above. Hence, all finite-dimensional complete Lie algebras with commutative nilpotent radical over $\mathbb{C}$ are known.

2. Some lemmas

Let

$$\mathcal{L} = s \oplus r = s + (t_0 + r_n)$$

be the Levi decomposition of $\mathcal{L}$. Then we have the following results.

Lemma 2.1 ([1]).

(2.2) $[s, t_0] = (0),$

(2.3) $[s, r_n] = r_n,$

(2.4) $[t_0, t_0] \subseteq r_0.$

Lemma 2.2.

(2.5) $[t_0, r_n] \subseteq r_n.$
Proof. By (2.3) and (2.2), we have
\[ [r_0, r_n] = [r_0, [s, r_n]] \subseteq [[r_0, s], r_n] + [[r_n, r_0], s] = [s, [r_0, r_n]] + [s, r] = [s, r_0 + r_n] = r_n. \]
The lemma holds.

Lemma 2.3. Let \( L \) be a Lie algebra with abelian nilpotent radical \( n_0 = r_n \). Then
\[ [r_0, r_0] = [r_n, r_n] = (0). \]

Proof. Since \( n_0 \) is commutative and \( n_0 = r_n \), we have
\[ n_0^{(1)} = r_n^{(1)} = [r_n, r_n] = (0) \]
and
\[ n_0 = [s + r_0 + r_n, r_0 + r_n] = r_n + [r_0, r_0]. \]
The lemma follows from (2.4) and Lemma 2.2.

Let \( a \) be an irreducible \( s \)-module. Then \( a \) is a highest weight \( s \)-module since \( s \) is semisimple and \( a \) is finite dimensional.

Let
\[ r_n = a_1 \oplus a_2 \oplus \cdots \oplus a_m \]
be the direct sum of submodules such that
\[ a_i = a_{i1} \oplus a_{i2} \oplus \cdots \oplus a_{in_i}, \quad (i = 1, 2, \ldots, m), \]
where \( a_{ik} (k = 1, 2, \ldots, n_i) \) are irreducible highest weight \( s \)-modules with highest weight \( \lambda_i \) and, if \( i \neq j \), then \( \lambda_i \neq \lambda_j \), \( i, j = 1, 2, \ldots, m \).

Denote by \( z_{ij} (j = 1, 2, \ldots, n_i) \) the highest weight vectors of \( a_{ij} \) \( (j = 1, 2, \ldots, n_i) \) respectively, and by \( V_i \) the linear space with basis \( \{ z_{i1}, \ldots, z_{in_i} \} \) \( (i = 1, 2, \ldots, m) \).

Then
\[ \dim V_i = n_i, \quad i = 1, 2, \ldots, m. \]

Let \( h_0 \) be a Cartan subalgebra of \( s \) and \( \Delta_0 \) the root system. Let \( \Pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \) be the simple root system and \( s = h_0 + \sum_{\alpha \in \Delta_0} s_\alpha \) the root space decomposition with respect to \( h_0 \).

Lemma 2.4. Let \( L \) be a Lie algebra with commutative nilpotent radical, \( D \in \text{Der}(s + r_n) \) be such that \( D(s) = (0) \) and \( D(r_n) \subseteq r_n \). Then
\[ D(V_i) \subseteq V_i, \quad i = 1, 2, \ldots, m, \]
and \( D \) is uniquely determined by \( D|_{V_i} \) \( (i = 1, 2, \ldots, m) \).

Proof. Since \( D \in \text{Der}(s + r_n) \) and \( D(s) = 0 \), for any \( h \in h_0 \) we have
\[ D[h, z_{ij}] = \lambda_i(h)Dz_{ij}. \]
Note that \( z_{ij} \) is a highest weight vector of \( a_{ij} \), therefore for \( \alpha \in \Delta_0^+ \) and \( e_\alpha \in s_\alpha \), we have
\[ [e_\alpha, z_{ij}] = 0. \]
Therefore
\[ [e_\alpha, Dz_{ij}] = D[e_\alpha, z_{ij}] = 0. \]
It is clear from (2.11) and (2.12) that $Dz_{i1}, Dz_{i2}, \ldots, Dz_{in_i}$ $(i = 1, 2, \ldots, m)$ are highest weight vectors associated to highest weight $\lambda_i$ $(i = 1, 2, \ldots, m)$. So $Dz_{ij} \in V_i$ $(j = 1, 2, \ldots, n_i, \ i = 1, 2, \ldots, m)$. On the other hand, for $z \in \mathfrak{a}_{ij}$, $z$ has the form:

$$z = [x_1, [x_2, \ldots, [x_q, z_{ij}]], \ldots],$$

where $x_i \in \mathfrak{s}$ $(i = 1, 2, \ldots, q)$. So

$$Dz = [x_1, [x_2, \ldots, [x_q, Dz_{ij}]], \ldots].$$

The lemma is proved. \[\square\]

**Lemma 2.5.** Let $\mathcal{L}$ be a Lie algebra with abelian nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. $D \in \text{Der}(\mathfrak{s} + \mathfrak{r}_n)$ is such that $D(\mathfrak{s}) = (0)$ and $D(\mathfrak{r}_n) \subseteq \mathfrak{r}_n$. Then

1) $D(\mathfrak{a}_i) \subseteq \mathfrak{a}_i, \quad i = 1, 2, \ldots, m.$

2) Set

$$\mathcal{L}_i = \{D \in \text{Der}(\mathfrak{s} + \mathfrak{r}_n) | D(\mathfrak{s}) = (0), \ D(\mathfrak{r}_n) \subseteq \mathfrak{r}_n \text{ and } D|_{\mathfrak{a}_j} = 0, \text{ if } j \neq i\}.$$

Then the Lie algebra $\mathcal{L}_i$ is isomorphic to the general linear Lie algebra $\text{gl}(V_i)$ which consists of all linear transformations of $V_i, \ i = 1, 2, \ldots, m$.

**Proof.** 1) follows from Lemma 2.4. Define

$$\varphi(D) = D|_{V_i}, \quad \text{for } D \in \mathcal{L}_i.$$

Then $\varphi$ is a linear mapping from $\mathcal{L}_i$ to $\text{gl}(V_i)$. For $D_1, D_2 \in \mathcal{L}_i$, if $D_1 \neq D_2$, then from Lemma 2.4 we know $\varphi(D_1) \neq \varphi(D_2)$. Let $A \in \text{gl}(V_i)$. Define the linear transformation $D$ of $\mathfrak{s} + \mathfrak{r}_n$ by

$$D(\mathfrak{s}) = 0, \ D|_{\mathfrak{a}_j} = 0, \quad \text{if } j \neq i,$$

$$D[x_1, [x_2, \ldots, [x_q, z_{ij}]], \ldots] = [x_1, [x_2, \ldots, [x_q, A z_{ij}]], \ldots] \quad (j = 1, 2, \ldots, n_i),$$

where $x_1, x_2, \ldots, x_q \in \mathfrak{s}$. Then $D \in \text{Der}(\mathfrak{s} + \mathfrak{r}_n)$. So $\varphi$ is a bijection.

For $D_1, D_2 \in \mathcal{L}_i$, we have

$$\varphi[D_1, D_2] = [D_1, D_2]|_{V_i} = D_1 D_2|_{V_i} - D_2 D_1|_{V_i} = D_1|_{V_i} D_2|_{V_i} - D_2|_{V_i} D_1|_{V_i} = [\varphi(D_1), \varphi(D_2)].$$

Hence $\varphi$ is a homomorphism from the Lie algebra $\mathcal{L}_i$ to the Lie algebra $\text{gl}(V_i)$. The lemma holds. \[\square\]

3. **The structure of radical $\mathfrak{r}$**

**Lemma 3.1.** Let $D$ be an inner derivation of $\mathcal{L}$ and $D(\mathfrak{s}) = (0)$. Then there exists an element $y \in \mathfrak{r}_0$ such that

$$D = \text{ad} \ y.$$

**Proof.** Since $D$ is an inner derivation of $\mathcal{L}$, there exist $x \in \mathfrak{s}, \ y \in \mathfrak{r}_0, \ z \in \mathfrak{r}_n$ such that

$$D = \text{ad}(x + y + z).$$

$D(\mathfrak{s}) = (0)$ implies that

$$[x + y + z, \mathfrak{s}] = [x, \mathfrak{s}] + [z, \mathfrak{s}] = (0).$$
From the fact that \([x, s] \subseteq s, [z, s] \subseteq \tau_n\), we have
\[
[x, s] = (0), \quad [z, s] = (0).
\]
But \(s\) is semisimple and \(\tau_n\) is the direct sum of non-trivial submodules. Therefore
\[
x = z = 0.
\]

\begin{lemma}
Let \(\mathcal{L}\) be a Lie algebra with trivial centre and commutative nilpotent radical \(n_0 = \tau_n\). Then
1) \(\mathcal{L}_0\) is isomorphic to \(\text{ad}_{\tau_n} \mathcal{L}_0\).
2) \(\text{ad}_{\tau_n} x(a_i) \subseteq a_i, \quad \text{ad}_{\tau_n} x|_{V_i} \in \text{gl}(V_i) \quad (i = 1, 2, \ldots, m)\).
\end{lemma}

\begin{lemma}
Let \(\mathcal{L}\) be a complete Lie algebra with commutative nilpotent radical \(n_0 = \tau_n\). For \(x \in \mathcal{L}_0\), define the linear transformations \(D_i (i = 1, 2, \ldots, m)\) of \(\mathcal{L}\) by
\[
D_i|_{a_0} = 0, \quad D_i|_{a_i} = \text{ad} x|_{a_i}, \quad D_i|_{a_i} = 0 \quad (j = 1, \ldots, i - 1, i + 1, \ldots, m).
\]
Then there exist \(y_1, y_2, \ldots, y_m \in \mathcal{L}_0\) such that
\[
D_i = \text{ad} y_i \quad (i = 1, 2, \ldots, m).
\]
\end{lemma}

\begin{proof}
For \(s_1, s_2 \in s, x_1, x_2 \in \mathcal{L}_0, y_1, y_2 \in a_i, z_1, z_2 \in a_1 + \cdots + a_{i-1} + a_i + a_{i+1} + \cdots + a_m\), by 2) of Lemma 3.2, we have \([x_1, y_2], [x_2, y_1] \in a_i, \ [x_1, z_2], [x_2, z_1] \in a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_m\). So by (2.2) and (2.6) we deduce that
\[
D_i[s_1 + x_1 + y_1 + z_1, s_2 + x_2 + y_2 + z_2]
= \text{ad} x([s_1, y_2] + [x_1, y_2] + [y_1, s_2] + [y_1, x_2])
= [s_1, \text{ad} x(y_2)] + [x_1, \text{ad} x(y_2)] + [\text{ad} x(y_1), s_2] + [\text{ad} x(y_1), x_2],
\]
\[
[D_i(s_1 + x_1 + y_1 + z_1), s_2 + x_2 + y_2 + z_2]
+ [s_1 + x_1 + y_1 + z_1, D_i(s_2 + x_2 + y_2 + z_2)]
= [\text{ad} x(y_1), s_2] + [\text{ad} x(y_1), x_2] + [s_1 + x_1 + y_1 + z_1, \text{ad} x(y_2)]
= [\text{ad} x(y_1), x_2] + [s_1, \text{ad} x(y_2)] + [x_1, \text{ad} x(y_2)].
\]
Therefore \(D_i \in \text{Der} \mathcal{L} \ (i = 1, 2, \ldots, m)\). Since \(\mathcal{L}\) is a complete Lie algebra, there exist \(y_1, y_2, \ldots, y_m \in \mathcal{L}\) such that
\[
D_i = \text{ad} y_i \quad (i = 1, 2, \ldots, m).
\]
By Lemma 3.1, \(y_i \in \mathcal{L}_0 \ (i = 1, 2, \ldots, m)\).
\end{proof}

\begin{theorem}
Let \(\mathcal{L}\) be a complete Lie algebra with commutative nilpotent radical \(n_0 = \tau_n\). Set
\[
(3.1) \quad h_i = \{ \text{ad} x|_{\tau_0} \text{ and } \text{ad} x|_{a_i} = 0 \ (j = 1, \ldots, i - 1, i + 1, \ldots, m)\},
\]
\[
(3.2) \quad H_i = h_i|_{V_i} = \varphi(h_i|_{s+\tau_n}), \quad i = 1, 2, \ldots, m.
\]
Then
1) \(h_i|_{s+\tau_n}\) is a commutative subalgebra of \(\mathcal{L}_i\) and
\[
(3.3) \quad \text{ad} x|_{\tau_0} = h_1 \oplus h_2 \oplus \cdots \oplus h_m \simeq h_1|_{s+\tau_n} \oplus \cdots \oplus h_m|_{s+\tau_n}.
\]
2) \(h_i\) is isomorphic to \(H_i\) which is an abelian subalgebra of \(\text{gl}(V_i), i = 1, 2, \ldots, m\).
\end{theorem}
Proof. The proof follows from Lemma 3.3 and Lemma 2.5. □

Lemma 3.4. Let \( \mathcal{L} \) be a Lie algebra with commutative nilpotent radical \( n_0 = r_\mathcal{L} \). Let \( D \in \text{Der}(s + r_\mathcal{L}) \) be such that \( D(s) = 0 \) and \( D(r_\mathcal{L}) \subseteq r_\mathcal{L} \). Extend \( D \) to the linear transformation of \( \mathcal{L} \) by

\[
D|_{r_\mathcal{L}} = 0.
\]

Then \( D \) is a derivation of \( \mathcal{L} \) if and only if

\[
[D, \text{ad}_{s + r_\mathcal{L}} v_0] = 0.
\]

Proof. \( D \in \text{Der} \mathcal{L} \) if and only if the formula

\[
D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] = [D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)]
\]

holds, for any \( x_1, x_2 \in s, y_1, y_2 \in r_\mathcal{L}, z_1, z_2 \in r_\mathcal{L} \).

Note that

\[
D[x_1 + y_1 + z_1, x_2 + y_2 + z_2] = D[z_1, x_1] + D[y_1, x_2] + D[x_1, z_2] + D[y_1, z_2] + D[z_1, z_2],
\]

and \( D \in \text{Der}(s + r_\mathcal{L}) \) is such that \( D(s) = 0 \). Therefore \( D \in \text{Der} \mathcal{L} \) if and only if

\[
D[y, z] = [y, Dz], \quad \text{for any } y \in r_\mathcal{L}, z \in r_\mathcal{L},
\]

i.e.,

\[
[D, \text{ad}_{s + r_\mathcal{L}} v_0] = 0.
\]

□

Theorem 3.2. Let \( \mathcal{L} \) be a complete Lie algebra with abelian nilpotent radical \( n_0 = r_\mathcal{L} \), and let \( h_i, H_i, \mathcal{L}_i \) \( (i = 1, 2, \ldots, m) \) be as above. Then

\[
C_{\mathcal{L}_i}(h_i|_{s + r_\mathcal{L}}) = h_i|_{s + r_\mathcal{L}}, \quad i = 1, 2, \ldots, m,
\]

\[
C_{\text{gl}(V_i)}(H_i) = H_i, \quad i = 1, 2, \ldots, m.
\]

Proof. Since \( r_0 \) is commutative, we have

\[
h_i|_{s + r_\mathcal{L}} \subseteq C_{\mathcal{L}_i}(h_i|_{s + r_\mathcal{L}}).
\]

Let \( D_i \in \mathcal{L}_i \) be such that \( [D_i, h_i|_{s + r_\mathcal{L}}] = 0 \). Then by (3.3) we have

\[
[D_i, \text{ad}_{s + r_\mathcal{L}} v_0] = 0.
\]

Extend \( D_i \) to the linear transformation of \( \mathcal{L} \) by

\[
D_i|_{r_\mathcal{L}} = 0.
\]

Then by Lemma 3.4, \( D_i \) is a derivation of \( \mathcal{L} \). Note that \( \mathcal{L} \) is a complete Lie algebra, therefore

\[
D_i = \text{ad} z_i \in h_i, \quad z_i \in r_\mathcal{L} \quad (i = 1, 2, \ldots, m).
\]

Hence \( D_i|_{s + r_\mathcal{L}} \in h_i|_{s + r_\mathcal{L}} \) \( (i = 1, 2, \ldots, m) \). This proves (3.4), (3.5) follows from the fact that \( \mathcal{L}_i \) is isomorphic to \( \text{gl}(V_i) \) and \( h_i|_{s + r_\mathcal{L}} \) is isomorphic to \( H_i \). □
Corollary 3.1. There exist elements $\text{ad} x_i \in \mathfrak{h}_i$ $(i = 1, 2, \ldots, m)$ such that
\[
\text{ad} x_i |_{\mathfrak{a}_i} = \text{id} |_{\mathfrak{a}_i}, \quad (i = 1, 2, \ldots, m).
\]
Thus $H_i$ contains identical transformations $I_i = \text{id} |_{\mathfrak{v}_i}$, $i = 1, 2, \ldots, m$.

From Corollary 3.1, we have
\[
(3.6) \quad H_i = H'_i \oplus C I_i
\]
and $H'_i \subseteq \text{sl}(V_i)$ $(i = 1, 2, \ldots, m)$, where $\text{sl}(V_i)$ is the special linear Lie algebra on $V_i$. Since $\mathfrak{h}_i$ is isomorphic to $H_i$, we have
\[
(3.7) \quad \mathfrak{h}_i = \mathfrak{h}'_i \oplus C \text{ad} x_i \quad (i = 1, 2, \ldots, m),
\]
where $\mathfrak{h}'_i$ is isomorphic to $H'_i$ and $\text{ad} x_i$ is the same as in Corollary 3.1. In fact, $\mathfrak{h}'_i |_{\mathfrak{v}_i} = H'_i$, $\text{ad} x_i |_{\mathfrak{v}_i} = I_i$ $(i = 1, 2, \ldots, m)$.

We will show that $H'_i$ is a maximal torus subalgebra of $\text{sl}(V_i)$, $i = 1, \ldots, m$.

Lemma 3.5. Let $\mathcal{L}$ be a Lie algebra with trivial centre and nilpotent radical $\mathfrak{n}_0 = \mathfrak{r}_n$. Let $D_1 \in \text{Der}(\mathfrak{s} + \mathfrak{r}_n)$ be such that $D_1(\mathfrak{s}) = (0)$ and $D_1(\mathfrak{r}_n) \subseteq \mathfrak{r}_n$, and
\[
(3.8) \quad [D_1, \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{r}_0] \subseteq \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{r}_0.
\]
Then there exists $D \in \text{Der} \mathcal{L}$ such that
\[
D|_{\mathfrak{s} + \mathfrak{r}_n} = D_1,
\]
and if $[D_1, \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{r}_0] \neq 0$, then $D$ is an outer derivation of $\mathcal{L}$.

Proof. From (3.8), for any $x \in \mathfrak{r}_0$ there exists $y \in \mathfrak{r}_0$ such that
\[
[D_1, \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} x] = \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} y.
\]
Define the transformation $D_2$ of $\mathfrak{r}_0$ by
\[
D_2(x) = y.
\]
By Lemma 3.2 there is no ambiguity in the definition of $D_2$. It is clear that $D_2$ is a linear transformation of $\mathfrak{r}_0$. $D_2$ is a derivation of $\mathfrak{r}_0$ since $\mathfrak{r}_0$ is commutative. From (3.8), for $x \in \mathfrak{r}_0$ and $z \in \mathfrak{r}_n$ we have
\[
(3.9) \quad D_1[x, z] = [x, D_1 z] + [D_2 x, z].
\]
Define the linear transformation $D$ of $\mathcal{L}$ by
\[
D|_{\mathfrak{s} + \mathfrak{r}_n} = D_1, \quad D|_{\mathfrak{r}_0} = D_2.
\]
Then for any $x_1, x_2 \in \mathfrak{s}$, $y_1, y_2 \in \mathfrak{r}_0$, $z_1, z_2 \in \mathfrak{r}_n$, we have
\[
D[x_1 + y_1 + z_1, x_2 + y_2 + z_2]
= [D_1 z_1, x_2] + D_1[z_1, y_2] + [x_1, D_1 z_2] + D_1[y_1, z_2] + D_1[z_1, z_2],
\]
\[
[D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)]
= [D_1 z_1, x_2] + [D_1 z_1, y_2] + [x_1, D_1 z_2] + [y_1, D_1 z_2] + [D_1 z_1, z_2] + [z_1, D_2 y_2].
\]
Since $D_1 \in \text{Der}(\mathfrak{s} + \mathfrak{r}_n)$ is such that $D_1(\mathfrak{s}) = (0)$ and (3.9) holds, we have
\[
D[x_1 + y_1 + z_1, x_2 + y_2 + z_2]
= [D(x_1 + y_1 + z_1), x_2 + y_2 + z_2] + [x_1 + y_1 + z_1, D(x_2 + y_2 + z_2)].
\]
Hence $D \in \text{Der} \mathcal{L}$. If $D \in \text{ad} \mathcal{L}$, then by Lemma 3.1, we have
\[
D_1 \in \text{ad}_{\mathfrak{s} + \mathfrak{r}_n} \mathfrak{r}_0.
\]
But \( r_0 \) is commutative. Therefore,
\[
[D_1, \text{ad}_{s + r_0} r_0] = 0.
\]
Thus we have completed the proof. \( \square \)

**Theorem 3.3.** Let \( \mathcal{L} \) be a complete Lie algebra with commutative nilpotent radical \( n_0 = r_0 \). Then \( H'_i \) is a maximal torus subalgebra of \( \text{sl}(V_i) \), \( i = 1, 2, \ldots, m \).

**Proof.** Let \( T_i \in \text{sl}(V_i) \) be such that \( [T_i, H_i] \subseteq H_i \). Then \( D_i = \varphi^{-1}(T_i) \in \mathcal{L}_i \), and \( D_i \) satisfies the conditions of Lemma 3.5. Thus there exists \( D'_i \in \text{Der} \mathcal{L} \) such that
\[
D'_i|_{s + r_0} = D_i, \quad i = 1, 2, \ldots, m.
\]
Note that \( \mathcal{L} \) is a complete Lie algebra, so by Lemma 3.1, we have
\[
D'_i \in \text{ad}_{\mathcal{L}} r_0, \quad i = 1, 2, \ldots, m.
\]
The fact that \( D'_i|_{s_j} = 0 \) (when \( j \neq i \)) implies \( D_i \in h_i \) (\( i = 1, 2, \ldots, m \)). Thus we have
\[
T_i = \varphi(D'_i|_{s + r_0}) = \varphi(D_i) \in H'_i, \quad i = 1, 2, \ldots, m.
\]
From this we know that \( H'_i \) is a self-normal subalgebra of \( \text{sl}(V_i) \). Since \( H'_i \) is commutative, we deduce that \( H'_i \) is a Cartan subalgebra of \( \text{sl}(V_i) \). But \( \text{sl}(V_i) \) is a simple Lie algebra, therefore \( H'_i \) is a maximal torus subalgebra of \( \text{sl}(V_i) \). \( \square \)

**Corollary 3.2.** Let \( \mathcal{L} \) be a complete Lie algebra with abelian nilpotent radical \( n_0 = r_0 \). Then \( H_i = H'_i + C_{L_i} \) is a maximal torus subalgebra of \( \text{gl}(V_i) \), \( i = 1, 2, \ldots, m \).

**Corollary 3.3.** Let \( \mathcal{L} \) be a complete Lie algebra with abelian nilpotent radical \( n_0 = r_0 \). Then
\[
\dim h_i = n_i, \quad i = 1, 2, \ldots, m.
\]
Thus we have
\[
\dim r_0 = n_1 + n_2 + \cdots + n_m.
\]
Since \( H_i \) is a maximal torus subalgebra of \( \text{gl}(V_i) \), we can choose a basis \( \{y_{i1}, y_{i2}, \ldots, y_{im_i}\} \) of \( V_i \) such that the matrix of every element of \( H_i \) relative to the basis is diagonal, \( i = 1, 2, \ldots, m \). On the other hand, \( y_{i1}, y_{i2}, \ldots, y_{im_i} \) are highest weight vectors associated to highest weight \( \lambda_i \). So the highest weight \( s \)-module \( a'_{ij} \) whose highest weight vector is \( y_{ij} \) is irreducible and
\[
a_i = a'_{i1} \oplus a'_{i2} \oplus \cdots \oplus a'_{in_i}, \quad i = 1, 2, \ldots, m.
\]
From this we deduce that

**Theorem 3.4.** Let \( \mathcal{L} \) be a complete Lie algebra with abelian nilpotent radical \( n_0 = r_0 \). Let \( r_n \) be the direct sum of \( t \) irreducible submodules. Then
\[
\dim r_0 = t
\]
and \( r_n \) can be decomposed properly into the direct sum of irreducible submodules:
\[
r_n = m_1 \oplus m_2 \oplus \cdots \oplus m_t
\]
so that
\[
r_0 \simeq \text{ad}_{r_n} r_0 = CI_1 \oplus CI_2 \oplus \cdots \oplus CI_t,
\]
where $I_i$ is the linear transformation of $\tau_n$ such that

$$I_i \left( \sum_{j \neq i} m_j \right) = (0), \quad I_i |_{m_i} = \text{id} |_{m_i}, \quad (i = 1, 2, \ldots, t).$$

So $\mathcal{L}$ is in fact the complete Lie algebra constructed in section 1.

**Theorem 3.5.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete Lie algebras. Let $\mathfrak{n}_i$ be the nilpotent radical of $\mathcal{L}_i$ and $\mathfrak{s}_i$ be the Levi subalgebra of $\mathcal{L}_i$ ($i = 1, 2$). Then $\mathcal{L}_1$ is isomorphic to $\mathcal{L}_2$ if and only if the Lie algebra $\mathfrak{s}_1$ is isomorphic to $\mathfrak{s}_2$ and the $\mathfrak{s}_1$-module $\mathfrak{n}_1$ is isomorphic to the $\mathfrak{s}_2$-module $\mathfrak{n}_2$.

**Theorem 3.6.** Let $\mathfrak{s}$ be a semisimple Lie algebra and $\mathfrak{n}$ an $\mathfrak{s}$-module. Define

$$(3.12) \quad [s_1 + x_1, s_2 + x_2] = [s_1, s_2] + s_1(x_2) - s_2(x_1),$$

where $s_1, s_2 \in \mathfrak{s}$, $x_1, x_2 \in \mathfrak{n}$. Then there is a unique up to isomorphism complete Lie algebra $\mathcal{L}$ such that $\mathfrak{n}$ is its nilpotent radical and $\mathfrak{s}$ is its Levi subalgebra and its bracket satisfies (3.12).

**References**


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