

NORM ATTAINING FUNCTIONALS ON $C(T)$

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ABSTRACT. It is shown that for any infinite compact Hausdorff space T , the Bishop-Phelps set in $C(T)^*$ is of the first Baire category when $C(T)$ has the supremum norm.

For a Banach space X with closed unit ball $B(X)$, the *Bishop-Phelps set* is the set of all functionals in the dual X^* which attain their supremum on $B(X)$; that is, the set $\{\mu \in X^* : \text{there exists } x \in B(X) \text{ with } \mu(x) = \|\mu\|\}$. It is well known that the Bishop-Phelps set is always dense in the dual X^* . It is more difficult, however, to decide whether the Bishop-Phelps set is residual as this would require a concrete characterisation of those linear functionals in X^* which attain their supremum on $B(X)$. In general it is difficult to characterise such functionals, however it is possible when $X = C(T)$, for some compact T , with the supremum norm. The aim of this note is to show that in this case the Bishop-Phelps set is a first Baire category subset of $C(T)^*$.

A problem of considerable interest in differentiability theory is the following.

If the Bishop-Phelps set is a residual subset of the dual X^ , is the dual norm necessarily densely Fréchet differentiable?*

It is known, for example, that this is the case if the dual X^* is weak Asplund, [1, Corollary 1.6(i)], or if X has an equivalent weakly mid-point locally uniformly rotund norm and the weak topology on X is σ -fragmented by the norm, [2, Theorems 3.3 and 4.4]. It is also known that each dual norm on X^* is densely Fréchet differentiable on X^* provided that the Bishop-Phelps set of each equivalent norm on X is residual in X^* , [6]. No counterexamples are known, and a consequence of our result is that no space of the form $X = C(T)$ can be a counterexample to the problem above.

Let T be an infinite compact Hausdorff space and $C(T)$ be the space of all continuous real-valued functions on T with the supremum norm $\|x\| \equiv \sup\{|x(t)| : t \in T\}$. We will denote by B the unit ball of $C(T)$; that is, $B \equiv \{x \in C(T) : \|x\| \leq 1\}$.

For every pair of real-valued functions on T (not necessarily continuous) we write $x \leq y$ if and only if $x(t) \leq y(t)$ for every $t \in T$. We denote by 1_U the characteristic function of a subset U of T ; that is,

$$1_U(t) = \begin{cases} 1 & \text{if } t \in U, \\ 0 & \text{if } t \notin U. \end{cases}$$

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Given $\lambda \in C(T)^*$ and an open $U \subset T$ we define

$$\lambda_+^*(U) \equiv \sup\{\lambda(h) : h \in C(T), 0 \leq h \leq 1_U\}.$$

Clearly $\lambda_+^*(U) \geq 0$. Similarly we define

$$\begin{aligned} \lambda_-^*(U) &\equiv \sup\{\lambda(-h) : h \in C(T), 0 \leq h \leq 1_U\} \\ &= \sup\{(-\lambda)(h) : h \in C(T), 0 \leq h \leq 1_U\} \\ &= (-\lambda)_+^*(U). \end{aligned}$$

For our first result we will need to consider the following closed sets,

$$\begin{aligned} S_+^\lambda &\equiv \{t \in T : \lambda_+^*(U) > 0 \text{ for every open neighbourhood } U \text{ of } t\}, \\ S_-^\lambda &\equiv \{t \in T : \lambda_-^*(U) > 0 \text{ for every open neighbourhood } U \text{ of } t\}. \end{aligned}$$

Theorem 1. *A linear functional $\lambda \in C(T)^*$ attains its supremum on B at x_0 if and only if*

$$x_0(t) = \begin{cases} 1 & \text{for every } t \in S_+^\lambda, \\ -1 & \text{for every } t \in S_-^\lambda. \end{cases}$$

That is, λ is in the Bishop-Phelps set if and only if $S_+^\lambda \cap S_-^\lambda = \emptyset$.

Proof. Suppose λ attains its supremum over B at x_0 but there exists a $t_0 \in S_+^\lambda$ such that $x_0(t_0) < r < 1$. Then there is an open neighbourhood U_0 of t_0 such that $x_0(t) < r < 1$ for every $t \in U_0$, by the continuity of x_0 . Since $t_0 \in S_+^\lambda$ we have $\lambda_+^*(U_0) > 0$. Then there exists some $h \in C(T)$, $0 \leq h \leq 1_{U_0}$, with $\lambda(h) > 0$. Take some $0 < \epsilon < 1 - r$ and consider the function $x_0 + \epsilon h$. Then

$$(x_0 + \epsilon h)(t) = \begin{cases} x_0(t) + \epsilon h(t) < r + \epsilon < 1 & \text{for } t \in U_0, \\ x_0(t) & \text{for } t \notin U_0. \end{cases}$$

Remembering $h \geq 0$ we conclude that $x_0 + \epsilon h \in B$. On the other hand

$$\lambda(x_0 + \epsilon h) = \lambda(x_0) + \epsilon \lambda(h) > \lambda(x_0)$$

contradicting the assumption that λ attained its supremum on B at x_0 . Hence $x_0(t_0) = 1$ for every $t_0 \in S_+^\lambda$.

Similarly, or by noting that $\lambda_-^* = (-\lambda)_+^*$, we conclude that $x_0(t_0) = -1$ for every $t_0 \in S_-^\lambda$.

Conversely, suppose

$$x_0(t) = \begin{cases} 1 & \text{for every } t \in S_+^\lambda, \\ -1 & \text{for every } t \in S_-^\lambda, \end{cases}$$

and $y_0 \in B$. Set

$$(y_0 - x_0)^+(t) = \max\{(y_0 - x_0)(t), 0\}$$

and

$$(y_0 - x_0)^-(t) = \min\{(y_0 - x_0)(t), 0\},$$

so $(y_0 - x_0) = (y_0 - x_0)^+ + (y_0 - x_0)^-$. Consider $T_\epsilon = \{t \in T : (y_0 - x_0)^+(t) \geq \epsilon\}$, a compact subset of T disjoint from S_+^λ . For each $t \in T_\epsilon$ there is an open neighbourhood U of t for which $\lambda_+^*(U) = 0$. These form an open cover of T_ϵ , and so there is a finite subcover U_i of T_ϵ for $i = 1, 2, \dots, n$, which together with $U_0 \equiv (T_\epsilon)^c$ form an open cover of T . Let p_i be a partition of unity subordinate to this cover, [5, p. 171]. Thus

$$\begin{aligned} \lambda((y_0 - x_0)^+) &= \lambda\left(\sum_{i=0}^n p_i(y_0 - x_0)^+\right) \\ &\leq \lambda(p_0(y_0 - x_0)^+) \\ &\leq \|\lambda\| \|p_0((y_0 - x_0)^+)\| \\ &\leq \|\lambda\| \epsilon. \end{aligned}$$

Similarly $\lambda((y_0 - x_0)^-) \leq \|\lambda\| \epsilon$, and so we conclude that $\lambda(y_0 - x_0) \leq 2\|\lambda\| \epsilon$. Since this is true for any $\epsilon > 0$ we have $\lambda(y_0) \leq \lambda(x_0)$ for all $y_0 \in B$, so λ attains its supremum over B at x_0 . \square

We need one more definition to formulate our next proposition. Given $\lambda \in C(T)^*$ and $t \in T$, define

$$\begin{aligned} r_+^\lambda(t) &= \inf\{\lambda_+^*(U) : t \in U, U \text{ open}\} \text{ and} \\ r_-^\lambda(t) &= \inf\{\lambda_-^*(U) : t \in U, U \text{ open}\}. \end{aligned}$$

Theorem 2. *For every $\lambda \in C(T)^*$ the sets $\{t \in T : r_+^\lambda(t) > 0\}$ and $\{t \in T : r_-^\lambda(t) > 0\}$ are countable. Moreover, $\sum_{t \in T} r_+^\lambda(t) < \infty$ and $\sum_{t \in T} r_-^\lambda(t) < \infty$.*

Proof. Let $t_i \in T$ for $i = 1, 2, \dots, n$, be distinct points in T and let $\epsilon > 0$. There exist disjoint open sets U_i such that $t_i \in U_i$. Since $\lambda_+^*(U_i) \geq r_+^\lambda(t_i) > r_+^\lambda(t_i) - \epsilon/2^i$, there exists some $h_i \in C(T)$, with $0 \leq h_i \leq 1_{U_i}$, such that $\lambda(h_i) > r_+^\lambda(t_i) - \epsilon/2^i$. Then $h \equiv \sum_{i=1}^n h_i$ is in B since the open sets U_i are a disjoint family. Thus

$$\|\lambda\| \geq \lambda(h) = \sum_{i=1}^n \lambda(h_i) > \sum_{i=1}^n r_+^\lambda(t_i) - \epsilon \sum_{i=1}^n \frac{1}{2^i}.$$

Consequently $\|\lambda\| + \epsilon > \sum_{i=1}^n r_+^\lambda(t_i)$, which shows that $\sum_{t \in T} r_+^\lambda(t) < \infty$, and so the set $\{t \in T : r_+^\lambda(t) > 0\}$ must be countable. That $\sum_{t \in T} r_-^\lambda(t) < \infty$, and the set $\{t \in T : r_-^\lambda(t) > 0\}$ is countable, follow similarly. \square

We note that the points $t \in T$ where $r_+^\lambda(t) > 0$ ($r_-^\lambda(t) > 0$) are points with positive (negative) λ -measure. They are related to the “ l_1 -part” of the functional λ , an l_1 -weighted sum of point evaluations.

Lemma 3. *Given $\lambda \in C(T)^*$ and $t \in T$ where $r_+^\lambda(t) > 0$ ($r_-^\lambda(t) > 0$) then there exists an open neighbourhood A of λ such that $r_+^\mu(t) > 0$ ($r_-^\mu(t) > 0$) for each $\mu \in A$.*

Proof. We consider the case $r_+^\lambda(t) > 0$. Choose $\epsilon > 0$ such that $r_+^\lambda(t) > 3\epsilon$, and consider any $\mu \in A \equiv \{\mu \in C(T)^* : \|\lambda - \mu\| < \epsilon\}$. Let U be any open neighbourhood t , then $\lambda_+^*(U) > 3\epsilon$. Then there exists an $h \in B$, $0 \leq h \leq 1_U$, such that $\lambda(h) > 2\epsilon$. Since $\|\lambda - \mu\| < \epsilon$ we conclude that $\mu(h) > \epsilon$. Hence $r_+^\mu(t) \geq \epsilon > 0$.

The case $r_-^\lambda(t) > 0$ follows similarly. \square

All is in readiness for our main theorem.

Theorem 4. *For any infinite compact Hausdorff space T , the Bishop-Phelps set is of the first Baire category in $C(T)^*$.*

Proof. The proof splits into two cases, T is scattered or not.

Case 1. T scattered. In this case T contains a convergent sequence $\{t_i\}$ of distinct points with limit point $t_\infty \in T$, [3, Lemma 5.3]. Consider the sets

$$\begin{aligned} G_i &\equiv \{\mu \in C(T)^* : \text{for some } j > i, r_+^\mu(t_j) > 0\}, \\ H_i &\equiv \{\mu \in C(T)^* : \text{for some } j > i, r_-^\mu(t_j) > 0\}. \end{aligned}$$

Now these are open by Lemma 3. They are also dense since for any $\lambda \in C(T)^*$ and any $\epsilon > 0$ there exists a $j > i$ such that $r_-^\lambda(t_j) < \epsilon$ ($r_+^\lambda(t_j) < \epsilon$) and so for $\mu(x) \equiv \lambda(x) + \epsilon x(t_j)$ ($\mu(x) \equiv \lambda(x) - \epsilon x(t_j)$) we have $r_+^\mu(t_j) > 0$ ($r_-^\mu(t_j) > 0$), and so $\mu \in G_i$ ($\mu \in H_i$). But then for any $\lambda \in \bigcap_{i=1}^\infty (G_i \cap H_i)$, we have $t_\infty \in S_+^\lambda \cap S_-^\lambda$, and by Theorem 1, λ does not attain its supremum over B . That is, the Bishop-Phelps set lies in the complement of a dense G_δ set, and so is of the first Baire category in $C(T)^*$.

Case 2. T not scattered. In this case there exists a continuous function p mapping T onto $[0, 1]$ [4, §8.5.4, (i)⇒(ii)]. By Zorn's Lemma there exists a minimal compact subset T' of T such that $p(T') = [0, 1]$. Let $\{V_i\}$ be a countable open base for the interval $[0, 1]$ and define the sets

$$\begin{aligned} G_i &\equiv \{\mu \in C(T)^* : \text{for some } t \in p^{-1}(V_i) \cap T', r_+^\mu(t) > 0\}, \\ H_i &\equiv \{\mu \in C(T)^* : \text{for some } t \in p^{-1}(V_i) \cap T', r_-^\mu(t) > 0\}. \end{aligned}$$

Again by Lemma 3 these sets are open. Since each set $p^{-1}(V_i) \cap T'$ is uncountable, for any $\lambda \in C(T)^*$ there exists a $t \in p^{-1}(V_i) \cap T'$ such that $r_+^\lambda(t) = 0$ ($r_-^\lambda(t) = 0$). So for any $\epsilon > 0$, and $\mu(x) \equiv \lambda(x) + \epsilon x(t)$ ($\mu(x) \equiv \lambda(x) - \epsilon x(t)$), we have $r_+^\mu(t) > 0$ ($r_-^\mu(t) > 0$), and so $\mu \in G_i$ ($\mu \in H_i$). That is, G_i and H_i are dense.

Consider any $\lambda \in \bigcap_{i=1}^\infty (G_i \cap H_i)$. Let $t \in T'$ and N be an open neighbourhood of t . Since T' is minimal, $p(T' \setminus N) \neq [0, 1]$, and so we may choose $V_n \subset [0, 1] \setminus p(T' \setminus N)$. Thus $p^{-1}(V_n) \cap T' \subset N \cap T' \subset N$. As $\lambda \in G_n(H_n)$ there exists an element $t_n \in p^{-1}(V_n) \cap T' \subset N$ such that $r_+^\lambda(t_n) > 0$ ($r_-^\lambda(t_n) > 0$). In particular, $N \cap S_+^\lambda \neq \emptyset$ and $N \cap S_-^\lambda \neq \emptyset$. Now this holds for arbitrary open sets N , and the sets S_+^λ and S_-^λ are both closed. We conclude that $t \in S_+^\lambda$ and $t \in S_-^\lambda$. That is, $T' \subset S_+^\lambda \cap S_-^\lambda$, and so by Theorem 1, λ does not attain its supremum over B . That is, the Bishop-Phelps set lies in the complement of a dense G_δ set, and so is of the first Baire category in $C(T)^*$. \square

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