

FUSION IN THE CHARACTER TABLE

GABRIEL NAVARRO

(Communicated by Ronald M. Solomon)

ABSTRACT. Suppose that P is a Sylow p -subgroup of a finite p -solvable group G . If $g \in P$, then the number of G -conjugates of g in P can be read off from the character table of G .

One of the classical problems in character theory is to find what kind of group theoretical information the character table determines.

Theorem A. *Let P be a Sylow p -subgroup of a finite p -solvable group G . If K is a conjugacy class of G , then $|K \cap P|$ can be read off from the character table of G .*

Although, it is possible to find non-isomorphic non- p -solvable groups with the same character table, it does not seem easy to construct examples where the conclusion of Theorem A fails. In fact, we shall prove Theorem A by using some of the deep facts in Isaacs π -theory. We should say that it is not clear at all how to find a direct proof without it.

Proof of Theorem A. Let K be a conjugacy class of a π -separable group G and let H be a Hall π -subgroup of G . We prove that the integer $|K \cap H|$ can be read off from the character table of G .

By a theorem of G. Higman ((8.21) of [3]), we know that the character table determines the set of primes that divide the common order of the elements in K . Therefore, we may assume that K consists of π -elements (otherwise, $|K \cap H|$ is zero). By Higman's Theorem, let K_1, \dots, K_h be the conjugacy classes of G consisting of π -elements, so that K is one of them. Write $G^0 = K_1 \cup \dots \cup K_h$, the set of π -elements of G . Also, let us denote by α^0 the restriction of a class function α of G to G^0 . Among the functions χ^0 for $\chi \in \text{Irr}(G)$, we consider the subset $I_\pi(G)$ consisting of those functions that cannot be written in the form $\nu^0 + \mu^0$ for nonzero characters ν and μ of G . By Theorem A of [1], the set $I_\pi(G)$ forms a basis for the vector space $\text{cf}(G^0)$ of class functions on G^0 . Hence, notice that, by using the character table, we may find $\{\chi_1, \dots, \chi_h\} \subseteq \text{Irr}(G)$ such that $I_\pi(G) = \{(\chi_1)^0, \dots, (\chi_h)^0\}$ (given $\chi \in \text{Irr}(G)$, since $\chi(1)$ is finite, there are only a finite number of characters ν, μ to check the equation $\chi^0 = \nu^0 + \mu^0$). We may certainly assume that $\chi_1 = 1_G$.

Now, by Corollary (2.5) of [2], we have that 1_H is a 'Fong character' associated to $(1_G)^0$ and therefore,

$$[(\chi_i)_H, 1_H] = 0 \quad \text{for } i \geq 2.$$

Received by the editors August 5, 1996.
1991 *Mathematics Subject Classification.* Primary 20C15.
Research partially supported by DGICYT.

If $x_j \in K_j$, then

$$[(\chi_i)_H, 1_H] = \frac{1}{|H|} \sum_{j=1}^h |K_j \cap H| \chi_i(x_j)$$

and thus, we may write

$$\begin{pmatrix} |H| \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_1(x_1) & \cdots & \chi_1(x_h) \\ \vdots & \ddots & \vdots \\ \chi_h(x_1) & \cdots & \chi_h(x_h) \end{pmatrix} \begin{pmatrix} |K_1 \cap H| \\ \vdots \\ |K_h \cap H| \end{pmatrix}.$$

By Theorem A of [1], we know that the matrix

$$\begin{pmatrix} \chi_1(x_1) & \cdots & \chi_1(x_h) \\ \vdots & \ddots & \vdots \\ \chi_h(x_1) & \cdots & \chi_h(x_h) \end{pmatrix}$$

is invertible. Since this matrix and $|H| = |G|_\pi$ are clearly determined by the character table, we have that

$$\begin{pmatrix} |K_1 \cap H| \\ \vdots \\ |K_h \cap H| \end{pmatrix}$$

is determined by the character table. \square

Although, in general, it is not clear how to calculate permutation characters from the character table, we may prove the following result.

Corollary B. *Let P be a Sylow p -subgroup of a finite p -solvable group G . Then the permutation character $(1_P)^G$ can be read off from the character table of G .*

Proof. Again, we assume that H is a Hall π -subgroup of a π -separable group G and we determine $(1_H)^G$ from the character table of G . If $\chi \in \text{Irr}(G)$, it is enough to show how to calculate $[\chi, (1_H)^G]$ from the character table of G . If K_1, \dots, K_h are the conjugacy classes of G consisting of π -elements and $x_j \in K_j$, we have that

$$[\chi, (1_H)^G] = [\chi_H, 1_H] = \frac{1}{|H|} \sum_{j=1}^h |K_j \cap H| \chi_i(x_j).$$

The result now follows from Theorem A. \square

REFERENCES

1. M. Isaacs, Characters of π -separable groups, J. Algebra, **86**, (1984), 98-128. MR **85h**:20012
2. M. Isaacs, Fong characters in π -separable groups, J. Algebra **99**, (1986), 89-107. MR **87e**:20015
3. M. Isaacs, Character Theory of Finite Groups, New York, Dover, 1994.

DEPARTAMENT D'ALGEBRA, FACULTAT DE MATEMÀTIQUES, UNIVERSITAT DE VALÈNCIA, 46100 BURJASSOT, VALÈNCIA, SPAIN

E-mail address: gabriel@uv.es