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FUSION IN THE CHARACTER TABLE

GABRIEL NAVARRO

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ABSTRACT. Suppose that P is a Sylow p-subgroup of a finite p-solvable group G. If $g \in P$, then the number of G-conjugates of g in P can be read off from the character table of G.

One of the classical problems in character theory is to find what kind of group theoretical information the character table determines.

Theorem A. *Let P be a Sylow p-subgroup of a finite p-solvable group G. If K is a conjugacy class of G, then $|K \cap P|$ can be read off from the character table of G.*

Although, it is possible to find non-isomorphic non-p-solvable groups with the same character table, it does not seem easy to construct examples where the conclusion of Theorem A fails. In fact, we shall prove Theorem A by using some of the deep facts in Isaacs π-theory. We should say that it is not clear at all how to find a direct proof without it.

Proof of Theorem A. Let K be a conjugacy class of a π-separable group G and let H be a Hall π-subgroup of G. We prove that the integer $|K \cap H|$ can be read off from the character table of G.

By a theorem of G. Higman ((8.21) of [3]), we know that the character table determines the set of primes that divide the common order of the elements in K. Therefore, we may assume that K consists of π-elements (otherwise, $|K \cap H|$ is zero). By Higman's Theorem, let $K_1, \dots, K_h$ be the conjugacy classes of G consisting of π-elements, so that K is one of them. Write $G^0 = K_1 \cup \ldots \cup K_h$, the set of π-elements of G. Also, let us denote by α^0 the restriction of a class function α of G to G^0. Among the functions χ^0 for $\chi \in \mathrm{Irr}(G)$, we consider the subset $\mathrm{I}_\pi(G)$ consisting of those functions that cannot be written in the form $\nu^0 + \mu^0$ for nonzero characters ν and μ of G. By Theorem A of [1], the set $\mathrm{I}_\pi(G)$ forms a basis for the vector space $\mathrm{cf}(G^0)$ of class functions on G^0. Hence, notice that, by using the character table, we may find $\{\chi_1, \dots, \chi_h\} \subseteq \mathrm{Irr}(G)$ such that $I_\pi(G) = \{(\chi_1)^0, \dots, (\chi_h)^0\}$ (given $\chi \in \mathrm{Irr}(G)$, since $\chi(1)$ is finite, there are only a finite number of characters ν, μ to check the equation $\chi^0 = \nu^0 + \mu^0$). We may certainly assume that $\chi_1 = 1_G$.

Now, by Corollary (2.5) of [2], we have that 1_H is a 'Fong character' associated to $(1_G)^0$ and therefore,

$$[(\chi_i)_H, 1_H] = 0 \quad \text{for } i \geq 2\,.$$

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If $x_j \in K_j$, then

$$[(\chi_i)_H, 1_H] = \frac{1}{|H|} \sum_{j=1}^{h} |K_j \cap H| \chi_i(x_j)$$

and thus, we may write

$$\begin{pmatrix} |H| \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_1(x_1) & \dots & \chi_1(x_h) \\ \vdots & \ddots & \vdots \\ \chi_h(x_1) & \dots & \chi_h(x_h) \end{pmatrix} \begin{pmatrix} |K_1 \cap H| \\ \vdots \\ |K_h \cap H| \end{pmatrix}.$$

By Theorem A of [1], we know that the matrix

$$\begin{pmatrix} \chi_1(x_1) & \dots & \chi_1(x_h) \\ \vdots & \ddots & \vdots \\ \chi_h(x_1) & \dots & \chi_h(x_h) \end{pmatrix}$$

is invertible. Since this matrix and $|H| = |G|_\pi$ are clearly determined by the character table, we have that

$$\begin{pmatrix} |K_1 \cap H| \\ \vdots \\ |K_h \cap H| \end{pmatrix}$$

is determined by the character table. $\square$

Although, in general, it is not clear how to calculate permutation characters from the character table, we may prove the following result.

Corollary B. *Let P be a Sylow p-subgroup of a finite p-solvable group G. Then the permutation character $(1_P)^G$ can be read off from the character table of G.*

Proof. Again, we assume that H is a Hall π-subgroup of a π-separable group G and we determine $(1_H)^G$ from the character table of G. If $\chi \in \mathrm{Irr}(G)$, it is enough to show how to calculate $[\chi, (1_H)^G]$ from the character table of G. If $K_1, \dots, K_h$ are the conjugacy classes of G consisting of π-elements and $x_j \in K_j$, we have that

$$[\chi, (1_H)^G] = [\chi_H, 1_H] = \frac{1}{|H|} \sum_{j=1}^{h} |K_j \cap H| \chi_i(x_j)\,.$$

The result now follows from Theorem A. $\square$

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Departament d'Algebra, Facultat de Matematiques, Universitat de Valencia, 46100 Burjassot, Valencia, Spain

E-mail address: gabriel@uv.es