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SPECTRAL MULTIPLICITY OF SOME STOCHASTIC PROCESSES

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Abstract. In this paper we consider the connection between the canonical and the weak-canonical representations for the given second-order stochastic process in a separable Hilbert space and we extend a well-known theorem of H. Cramer concerning sufficient conditions for a process to be of multiplicity one.

Let $x(t)$, $t \in (a, b) \subset \mathbf{R}$, be a second-order real-valued process with $Ex(t) = 0$ for each t. Let $H(x, t)$ be the linear closure generated by $x(s)$, $s \in (a, t]$, in the Hilbert space H of all random variables with finite variance ($Ex^2(t) < \infty$). We will suppose that $x(t)$, $t \in (a, b)$, is continuous left and purely nondeterministic (i.e. $\bigcap_{t>a} H(x, t) = 0$). It is well known (see [1]) that there is a representation

$$x(t) = \sum_{n=1}^{N} \int_a^t g_n(t, u) dz_n(u), \qquad u \leq t, \ t \in (a, b), \tag{1}$$

where:

1. The processes $z_n(u)$, $n = 1, \dots, N$, are mutually orthogonal with orthogonal increments such that $Ez_n(u) = 0$ and $Ez_n^2(u) = F_n(u)$, where $F_n(u)$, $n = 1, \dots, N$, are nondecreasing functions left continuous everywhere on (a, b).
2. The nonrandom functions $g_n(t, u)$, $u \leq t$, are such that:

$$Ex^2(t) = \sum_{n=1}^{N} \int_a^t g_n^2(t, u) dF_n(u) < \infty, \quad \text{for each } t \in (a, b).$$

3. $dF_1 > dF_2 > \cdots > dF_n$, where the relation $>$ means absolute continuity between measures.
4. $H(x, t) = \sum_{n=1}^{N} \bigoplus H(z_n, t)$, $t \in (a, b)$.

The expansion (1) satisfying the conditions 1, 2, 3 and 4 is the *canonical representation* or Cramer representation for the process $x(t)$. The number N (finite or infinite) is called the *multiplicity* of $x(t)$, and N is uniquely determined by the process $x(t)$. But, the processes $z_n(u)$ and the functions $g_n(t, u)$ are not uniquely determined.

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For finite N, the representation (1) is canonical if and only if the family $\{g_n(t,u)\}_{n=1,\dots,N}$ is complete in the space $L^2(dF(u))$, $dF = \{dF_n\}_{n=1,\dots,N}$ (see Lemma 3.1 of Cramer [1]). If condition 4 in the representation (1) is replaced by the weaker condition

$$P_{H(x,s)}x(t) = \sum_{n=1}^{N} \int_a^s g_n(t,u)dz_n(u), \quad \text{for all } s \le t, s, t \in (a,b),$$

where $P_{H(x,s)}$ is the projection operator on $H(x,s)$, then (1) is said to be a *weak-canonical representation* of $x(t)$.

The kernel $\{g_n(t,u)\}_{n=1,\dots,N}$ of the weak-canonical representation need not be complete in the space $L^2(dF(u))$. Every canonical representation is the weak-canonical one (see [1], page 10). The converse need not hold. This fact is shown in the next simple example.

Example 1. If we have two mutually orthogonal stationary processes given by canonical representations:

$$x_1(t) = \int_{-\infty}^{t} g_1(t-u)dz_1(u),$$
$$x_2(t) = \int_{-\infty}^{t} g_2(t-u)dz_2(u), \qquad u \le t, \ u, t \in (-\infty, \infty),$$

then the representation of their sum, $x(t) = x_1(t) + x_2(t)$, is weak-canonical if and only if $f_1(u) = a \cdot f_2(u)$, where $f_1(u), f_2(u)$ are spectral densities, $a = \text{const.}$, but it is not canonical (see [4]).

Main result. One of the problems here is to determine the class of processes with multiplicity $N = 1$. Cramer stated in Theorem 5.1 in [1] that the *regularity conditions* ensure a multiplicity of unity for a process which has a canonical expansion. Here the same result is proved for a process which has only a weak-canonical representation.

Theorem. *Let X be the class of all processes $x(t)$ admitting a weak-canonical expansion* (1)*, with N finite and (a,b) a finite subinterval of R, so that the following regularity conditions are satisfied:*

R_1. The functions $g_n(t,u)$ and $\partial g_n(t,u)/\partial t$ are bounded and continuous for $u, t \in (a,b)$, $u \le t$.

R_2. $g_n(t,t) = 1$, $n = 1, \dots, N$, for all $t \in (a,b)$.

R_3. The function $F_n(u) = Ez_n^2(u)$ is absolutely continuous and not identically constant with $f_n(u) = \partial F_n(u)/\partial u$, $n = 1, \dots, N$, having at most finitely many discontinuity points in any finite subinterval of (a,b).

Then, every $x(t) \in X$ has multiplicity $N = 1$.

Proof. Let us suppose that the multiplicity M of $x(t)$ is > 1. For example let $M = 2$. Then, there exists a canonical representation of $x(t)$ of the form:

$$x(t) = \sum_{k=1}^{2} \int_a^t G_k(t,u)dw_k(u), \qquad u \le t, t \in (a,b), \tag{2}$$

where the family $\{G_k(t,u)\}_{k=1,2}$ is complete in the space $L^2(d\Phi(u))$, $\Phi(u) = (\Phi_1(u), \Phi_2(u))$, $\Phi_k = Ew_k^2$, $k = 1, 2$. Without loss of generality, we can assume that the functions Φ_k are absolutely continuous. Then, from that and the condition

3 (for a canonical representation), we can find a finite subinterval $(a_1, b_1) \subset (a, b)$ where both $\Phi'_k = \varphi_k$, $k = 1, 2$, are different from zero for all $u \in (a_1, b_1)$ (see [1]).

The main idea of the proof is to show that there exists $y(t)$ from the space $H(w_1, t) \oplus H(w_2, t)$, such that $0 < Ey^2 < \infty$, and y is orthogonal to $x(s)$ for all $a < s \le t$. It means the representation (2) is not canonical and then the multiplicity is not two.

From the hypotheses of the Theorem, it follows that $x(t)$ admits a weak-canonical representation (1), which satisfies the regularity conditions. Our first step is to find a connection between a canonical and a weak-canonical representation. Both representations (1) and (2) are weak-canonical and hence for all $s < v < t$:

$$\sum_{n=1}^{N} \int_s^v g_n(t, u) dz_n(u) = \sum_{k=1}^{2} \int_s^v G_k(t, u) dw_k(u),$$

where $z_n(u)$ and $w_k(u)$, $n = 1, \ldots, N$, $k = 1, 2$, are the processes with mutually orthogonal increments on disjoint intervals. Let us construct a structural measure γ_{nk} as follows:

$$E(z_n(s) - z_n(t)) \cdot (w_k(s') - w_k(t')) = \gamma_{nk}([s, t) \cap [s', t')).$$

The finite measure with sign $d\gamma_{nk}$ is absolutely continuous with respect to the measures dF_n and $d\Phi_k$, so we may write $d\gamma_{nk}(u) = a_{nk}(u) d\Phi_k(u)$, $n = 1, \ldots, N$, $k = 1, 2$. Using the scalar product of the previous form with $w_k(u)$ we first obtain for all $s < v < t$, $k = 1, 2$:

$$\sum_{n=1}^{N} \int_s^v g_n(t, u) d\gamma_{nk}(u) = \int_s^v G_k(t, u) d\Phi_k(u),$$

and hence, $G_k(t, u) = \sum_{n=1}^{N} g_n(t, u) a_{nk}(u)$, almost everywhere with respect to the measure $d\Phi_k$, $k = 1, 2$, $u \le t$, $t \in (a, b)$. So, we may write a canonical representation (2) of $x(t)$ in the following form:

$$x(t) = \sum_{k=1}^{2} \int_a^t \left[\sum_{n=1}^{N} a_{nk}(u) g_n(t, u) \right] dw_k(u), \qquad u \le t, t \in (a, b). \tag{3}$$

Let us consider the functions $\sum_{n=1}^{N} a_{nk}(u)$, $u \in (a, b)$, $k = 1, 2$. The second step in the proof is to find a set where both $\sum a_{n1}(u)$ and $\sum a_{n2}(u)$ are different from zero. If there are no points $u \in (a_1, b_1)$ such that on the interval $(u - \delta, u)$, $\delta \ne 0$, both $\sum a_{n1}(u)$ and $\sum a_{n2}(u)$ are different from zero, then, according to assumptions about $\varphi_k \ne 0$ and conditions for $g_n(t, u)$, the process x receives the impulse $M(u)$ successively from w_1 or w_2 during the interval (a_1, b_1) (see [2]). According to [2] this means multiplicity is one: $M = \sup_{u \in (a_1, b_1)} M(u) = \sup_{u \in (a, b)} M(u) = 1$. So, let (a_2, b_2) be a finite subinterval of (a_1, b_1), such that $\sum_n a_{nk}(u) \ne 0$, for $u \in (a_2, b_2)$, $k = 1, 2$, and $0 \notin (a_2, b_2)$.

Arguing as in [1], let t be any point in (a_2, b_2) and let $h(u) = (h_1(u), h_2(u))$ be a function in $L^2(d\Phi(u))$, such that:

$$\sum_{k=1}^{2} \int_{a_2}^{s} \left[\sum_{n=1}^{N} a_{nk}(u) g_n(s, u) \right] h_k(u) \varphi_k(u) du = 0, \quad \text{for all } s \le t.$$

We will show that such $h(u) \neq 0$ exists. By conditions R_1 and R_2 this relation may be differentiated with respect to s :

$$\sum_{k=1}^{2}\left\{\int_{a_2}^{s}\left[\sum_{n=1}^{N} a_{nk}(u)\partial g_n(s,u)/\partial s\right] h_k(u)\varphi_k(u)du + \sum_{n=1}^{N} a_{nk}(s)h_k(s)\varphi_k(s)\right\} = 0.$$

This equation is satisfied if for example:

$$\int_{a_2}^{s}\left[\sum_{n=1}^{N} a_{n1}(u)\partial g_n(s,u)/\partial s\right] h_1(u)\varphi_1(u)du + \sum_{n=1}^{N} a_{n1}(s)h_1(s)\varphi_1(s) = 1 \tag{4}$$

and

$$\int_{a_2}^{s}\left[\sum_{n=1}^{N} a_{n2}(u)\partial g_n(s,u)/\partial s\right] h_2(u)\varphi_2(u)du + \sum_{n=1}^{N} a_{n_2}(s)h_2(s)\varphi_2(s) = -1. \tag{5}$$

These are the nonhomogeneous Volterra integral equations of the second kind with unknown functions $h_k(s)\varphi_k(s)$, $s \in (a_2, t]$, $k = 1, 2$. Let us consider the first of them. By the restriction imposed on $\partial g_n(s,u)/\partial s$, $n = 1, \ldots, N$, there exists a solution $h_1(s)\varphi_1(s)$, $s \in (a_2, b_2)$, not equal to zero almost everywhere if the following conditions hold:

$$\int_{a_2}^{b_2}\left[\sum_{n=1}^{N} a_{n1}(u)\right]^2 du < \infty \quad \text{and} \quad \int_{a_2}^{b_2}\left[\sum_{n=1}^{N} a_{n1}(s)\right]^{-2} ds < \infty$$

(see [3]).

By condition R_1 for $g_n(t,u)$ and the fact that $G_1(t,u) \in L^2(\varphi_1(u)du)$, it is easy to see that $\sum a_{n1}(u) \in L^2(\varphi_1(u)du)$. As $\varphi_1 > 0$ and $\sum a_{n1} \neq 0$ on (a_2, b_2), it follows that $(\sum a_{n1})^2 \leq (\sum a_{n1})^2\varphi_1$, for $\varphi_1 \geq 1$, or $\varepsilon \cdot (\sum a_{n1})^2 \leq (\sum a_{n1})^2\varphi_1$, for $0 < \varepsilon \leq \varphi_1 \leq 1$. Hence, it is clear that $\sum a_{n1}(u) \in L^2(du)$, and that $[\sum a_{n1}(u)]^{-1} \in L^2(du)$, on the finite subinterval (a_2, b_2), which does not contain 0. So, a solution $h_1(s)\varphi_1(s)$, $s \in (a_2, b_2)$, of the integral equation (4) exists.

The same holds for the integral equation (5). Since $\varphi_k \neq 0$, $k = 1, 2$, on (a_2, b_2), it follows that:

$$\int_{a_2}^{t} h_1^2(u)d\Phi_1(u) + \int_{a_2}^{t} h_2^2(u)d\Phi_2(u) > 0, \quad \text{for all } t \in (a_2, b_2).$$

This means that the family $\{G_k(t,u)\}_{k=1,2}$ is not complete in the space $L^2(d\Phi(u))$, and multiplicity of $x(t)$ is not 2. Using similar arguments we see that the multiplicity cannot be any natural number > 1. The proof is completed. □

Note. The statement of the Theorem is valid even if we assume that (a, b) is an infinite subinterval of R.

Example 2. Let $x(t) = \int_{-\infty}^{t} e^{-c(t-u)}dz_1(u) + \int_{-\infty}^{t} d \cdot e^{-c(t-u)}dz_2(u)$, $u \leq t, u, t \in R$, be a process, where $z_1(u)$ and $z_2(u)$ are the mutually orthogonal processes with orthogonal increments such that $Ez_n(u) = 0$, $Ez_n^2(u) = f_n(u)du$, $n = 1, 2, d =$ const., $f_1(u) = 2c$, $f_2(u) = 2cd^2$. Clearly, $x(t)$ has a weak-canonical representation and since it satisfies the regularity conditions, it has multiplicity one.

Example 3. Let $x(t)$, $0 \leq t \leq \tau$, be represented by $x(t) = c_1z_1(t) + c_2z_2(t) + \cdots + c_Nz_N(t)$, where $z_n(t)$ are independent Wiener processes, $c_n =$ const., $n = 1, \ldots, N$. This representation is weak-canonical because $x(t) - x(s)$ is orthogonal to $x(s)$ for all $s < t$, and $P_{H(x,s)}x(t) = x(s)$. Since the regularity conditions are satisfied, $x(t)$ has multiplicity one.

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