PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 1, January 1998, Pages 257–267
S 0002-9939(98)04407-4

QUASIPOSITIVE PLUMBING (CONSTRUCTIONS OF QUASIPOSITIVE KNOTS AND LINKS, V)

LEE RUDOLPH

(Communicated by Ronald Stern)

Dedicated to Professor Kunio Murasugi

Abstract. A Seifert surface $S \subset S^3 = \partial D^4$ is a *fiber surface* if a push-off $S \to S^3 \setminus S$ induces a homotopy equivalence; roughly, S is *quasipositive* if pushing $\operatorname{Int} S$ into $\operatorname{Int} D^4 \subset \mathbb{C}^2$ produces a piece of complex plane curve. A *Murasugi sum* (or *plumbing*) is a way to fit together two Seifert surfaces to build a new one. Gabai proved that a Murasugi sum is a fiber surface iff both its summands are; we prove the analogue for quasipositive Seifert surfaces.

The *slice* (or *Murasugi*) *genus* $g_s(L)$ of a link $L \subset S^3$ is the least genus of a smooth surface $S \subset D^4$ bounded by L. By the local Thom Conjecture, $g_s(\partial S) = g(S)$ if $S \subset S^3$ is quasipositive; we derive a lower bound for $g_s(\partial S)$ for any Seifert surface S, in terms of quasipositive subsurfaces of S.

1. Introduction

Murasugi [7], studying certain alternating links (retrospectively, those that fiber), introduced a construction of Seifert surfaces from simpler pieces (namely, fiber surfaces of 2-strand torus links $O\{2, k\}$). Stallings [17], as one of his "constructions of fibered knots and links" (actually, of fiber surfaces) generalized [7] and named the construction "plumbing". Gabai [3] renamed it "Murasugi sum" (reserving "plumbing" for a special case, rather different from that in [7], which had been investigated earlier, cf. [17], [2]), and put the operation in a broader geometric context (least-genus surfaces and "Reebless foliations"). As applied to fiber surfaces, one aspect of Gabai's slogan [5] that "Murasugi sum is a natural geometric operation" is his theorem that a Murasugi sum is a fiber surface iff the summands are fiber surfaces; in §4, I use *braided Seifert surfaces* to prove an analogue.

Theorem. *A Murasugi sum is quasipositive iff the summands are quasipositive.*

Murasugi [8] was also an early investigator of that "numerical invariant of link types" now called "slice genus" or "Murasugi genus". A theorem of Kronheimer and Mrowka [6] implies that, if S is a quasipositive Seifert surface, then $g_s(\partial S) = g(S)$; this implication was shown in [14], where it was used to derive a lower bound (the "slice-Bennequin inequality") for the slice genus of any link presented as a closed braid. In §5, I derive a more general (and often stronger) lower bound for $g_s(L)$ in

Received by the editors October 1, 1995.
1991 *Mathematics Subject Classification.* Primary 57M25; Secondary 32S55, 14H99.
Key words and phrases. Murasugi sum, plumbing, quasipositive, slice genus, Thom conjecture.
Partially supported by grants from CAICYT, NSF (DMS-8801915, DMS-9504832), and CNRS.

terms of any Seifert surface S with $L = \partial S$. As an application, I combine this with the Theorem to estimate the slice genus of certain plumbed links. For instance, let S be a Seifert surface plumbed from unknotted, twisted annuli according to a weighted tree each vertex of which has even (total) weight; many arborescent links (including many of the fibered ones, cf. [4]) have such Seifert surfaces.

Proposition. *Let ∂S have r components. If there are p vertices with positive weight and q with negative weight, then $g_s(\partial S) \geq (1 - r + |p - q|)/2$.*

Notation and general definitions are established in §2. A braid-free exposition of braided Seifert surfaces and quasipositivity is given in §3.

2. Preliminary notations and definitions

The notations $A := B$ and $B =: A$ both define A to mean B. For any cartesian product (e.g., $\mathbb{R}^3$), pr_i denotes projection on the ith factor; similarly for $\mathrm{pr}_{i,j}$.

2.1. Sets. Write $\mathrm{two}(Y)$ for the set of 2-element subsets of Y. If Y is (totally) ordered, such notations as $\{s,t\} \in \mathrm{two}(Y)$, $\{i,j\} : C \to \mathrm{two}(Y)$, etc., typically presuppose that $s < t$, $i(c) < j(c)$ for all $c \in C$, etc. If $\{s,t\} \in \mathrm{two}(Y)$ and there is no $u \in Y$ such that $s < u < t$, then t is *the successor of s in Y*. If $\{s,t\}, \{s',t'\} \in \mathrm{two}(Y), \{s,t\} \neq \{s',t'\}$, then $\{s,t\}$ and $\{s',t'\}$ *link* iff either $s < s' < t < t'$ or $s' < s < t' < t$, *unlink* iff either $s < s' < t' < t$ or $s' < s < t < t'$, and *touch at u* iff $\{s,t\} \cap \{s',t'\} = \{u\}$. Write $\mathbf{n} := \{1, \dots, n\}$ for $n \in \mathbb{N} := \{1, 2, \dots\}$, and $\mathbf{0} := \varnothing$. If $\mathrm{card}(Y) =: m < \infty$, then $\#_Y : \mathbf{m} \to Y$ denotes the increasing bijection; so, e.g., $\min Y = \#_Y(1)$, $\max Y = \#_Y(m)$. (Read $\#_Y$ as "count Y".)

2.2. Manifolds. Spaces, maps, etc., are piecewise smooth. Manifolds may have boundary and are always oriented; in particular, $\mathbb{R}$, $\mathbb{C}^n$, and $S^{2n-1} \subset \mathbb{C}^n$ have standard orientations, as does $\mathbb{R}^3$ which is identified with the complement of a point $\infty \in S^3$. If M is a manifold, then $-M$ denotes M with its orientation reversed (and, where notation requires it, $+M$ denotes M). A submanifold $P \subset M$ is *proper* (resp., *interior*; *boundary*) if $\partial P = P \cap \partial M$ (resp., $P \subset \mathrm{Int}\, M$; $P \subset \partial M$). For a suitable subset $Q \subset M$, $N_M(Q)$ denotes a closed regular neighborhood of Q in $(M, \partial M)$. For a suitable codimension-1 submanifold $Q \subset M$, a *collaring* of Q in M is an orientation-preserving embedding $\mathrm{col}_{Q \subset M} : Q \times [0,1] \to M$ such that $\mathrm{col}_{Q \subset M}(q, 0) = q$ for all q; a *collar* is the image of a collaring. The *push-off* of Q is the embedding $Q \to M \setminus Q : q \mapsto \mathrm{col}_{Q \subset M}(q, 1)$; let Q^+ denote the image of Q by the push-off, oriented so that the push-off preserves orientation (thus $Q^+ \subset \partial(\mathrm{col}_{Q \subset M}(Q \times [0,1]))$ has the conventional, "outward normal" orientation, whereas the inclusion $Q \subset \partial(\mathrm{col}_{Q \subset M}(Q \times [0,1]))$ reverses orientation).

An *arc* is a manifold diffeomorphic to $[0,1]$. A *surface* is a compact 2-manifold no component of which has empty boundary. A handle decomposition

$$S = \bigcup_{x \in X} h_x^{(0)} \cup \bigcup_{z \in Z} h_z^{(1)} \tag{2.2.1}$$

of a surface S is not necessarily ordered; it is understood that if $z_1 \neq z_2$ then the attaching regions $h_{z_t}^{(1)} \cap \partial\left(\bigcup_{x \in X} h_x^{(0)}\right)$ $(t = 1, 2)$ are disjoint. Write $S^{(0)} := \bigcup_{x \in X} h_x^{(0)}$, $S^{(1)} := \bigcup_{z \in Z} h_z^{(1)}$: thus, $S^{(0)} \cap S^{(1)}$ is the union of the attaching regions. For $p \in S^{(i)}$ (resp., $P \subset S^{(i)}$), write $h^{(i)}(p)$ (resp., $h^{(i)}(P)$) for the unique i-handle of (2.2.1) containing p (resp., P).

A *core* (resp., *transverse*) arc of a 1-handle $h^{(1)}$ is any proper arc which joins interior points of the two components of the attaching region of $h^{(1)}$ (resp., the complement in $\partial h^{(1)}$ of the attaching region of $h^{(1)}$). Note that, if S is a surface and $\alpha \subset S$ is an arc, then S has some handle decomposition (2.2.1) such that α is a core arc (resp., a transverse arc) of some $h_z^{(1)}$, $z \in Z$, iff α is interior (resp., proper).

2.3. Stars and patches. An arc τ contained in a surface S is *half-proper* if one endpoint of τ belongs to ∂S and the rest of τ is contained in $\operatorname{Int} S$. An *n-star* $\psi \subset S$ is a union of n half-proper arcs, the *rays* of ψ, which are pairwise disjoint except for a common endpoint in $\operatorname{Int} S$, the *center* of ψ; the *tip* of a ray $\tau \subset \psi$ is that endpoint $\operatorname{tip}(\tau)$ of τ which is not the center of ψ. An *n-patch* is the regular neighborhood $N_S(\psi)$ of an n-star. An n-patch is a 2-disk naturally endowed with the structure of a $2n$-gon of which the edges are alternately boundary arcs and proper arcs in S.

An n-star $\psi \subset S$ is *transverse* to the handle decomposition (2.2.1) of S if

(2.3.1) the center of ψ lies in $\operatorname{Int} S^{(0)}$,
(2.3.2) for each ray $\tau \subset \psi$, $\operatorname{tip}(\tau) \in \partial S^{(0)} \setminus S^{(1)}$, and
(2.3.3) each ray $\tau \subset \psi$ is transverse to $S^{(0)} \cap S^{(1)}$.

Let ψ be transverse to (2.2.1). Define $c(\psi) := \operatorname{card}(\psi \cap S^{(0)} \cap S^{(1)})$. A ray $\tau \subset \psi$ will be called *long* if $\tau \not\subset S^{(0)}$; so $c(\psi) = 0$ iff no ray $\tau \subset \psi$ is long iff $\psi \subset S^{(0)}$ iff $\psi \subset h_x^{(0)}$ for some $x \in X$. Let $\tau \subset \psi$ be a long ray. The *tail* of τ, denoted $\operatorname{tail}(\tau)$, is that component arc of $\tau \cap S^{(0)}$ which has $\operatorname{tip}(\tau)$ as one endpoint; the *coccyx* of τ, denoted $\operatorname{coccyx}(\tau)$, is the other endpoint of $\operatorname{tail}(\tau)$. Call τ *loose* if either of the two 2-disks into which $\operatorname{tail}(\tau)$ divides $h^{(0)}(\operatorname{tail}(\tau))$ has empty intersection with $S^{(1)} \setminus h^{(1)}(\operatorname{coccyx}(\tau))$. Call τ *slack* if τ contains an arc α with both endpoints on the same component of $S^{(0)} \cap S^{(1)}$. Call ψ *minimal* with respect to (2.2.1) if $c(\psi) \leq c(\psi')$ for every n-star $\psi' \subset S$ which is transverse to (2.2.1) and ambient isotopic to ψ on S. The following is easily proved.

2.3.4. Lemma. *If ψ is minimal, then no ray of ψ is either slack or loose.*

2.4. Seifert surfaces. A *Seifert surface* is a surface $S \subset S^3$. A *link* L is the boundary of a Seifert surface; a *knot* is a connected link. If K is a knot, then $A(K,n)$ denotes any Seifert surface diffeomorphic to an annulus such that $K \subset \partial A(K,n)$ and the linking number in S^3 of K and $K' := \partial A(K,n) \setminus K$ is $-n$ (that is, the Seifert matrix $\theta_{A(K,n)}$ is $[n]$). Since clearly K' and $-K$ are ambient isotopic, so are $A(K,n)$ and $A(-K,n)$; $A(K,n)$ and $-A(K,n)$ are also ambient isotopic.

A Seifert surface S is: (1) a *fiber surface* (and ∂S is a *fibered link*) if there exists a fibration $\varphi : S^3 \setminus \partial S \to S^1$ such that $\operatorname{Int} S$ is a fiber of φ and the closure of every fiber of φ is a Seifert surface with boundary ∂S; (2) *least-genus* if S maximizes Euler characteristic among all Seifert surfaces with boundary ∂S; (3) *incompressible* if, whenever $D^2 \subset S^3$ is a disk such that $D^2 \cap S = \partial D^2$, then ∂D^2 bounds a disk on S. The following facts are well known (cf., e.g., [17], [3]–[5]): (1) S is a fiber surface iff S is connected and a push-off induces an isomorphism $\pi_1(S) \to \pi_1(S^3 \setminus S)$; (2) a least-genus surface S is incompressible; (3) a fiber surface is least-genus, and up to ambient isotopy it is the unique incompressible surface with its boundary; (4) $A(K,n)$ is least-genus iff $(K,n) \neq (O,0)$; (5) $A(K,n)$ is a fiber surface iff $(K,n) = (O,-1)$ or $(K,n) = (O,1)$. The annulus $A(O,-1)$ (resp., $A(O,1)$) is called a *positive* (resp., *negative*) *Hopf annulus* (sometimes "Hopf band"); the sign convention reflects the linking number of the components of $\partial A(O,\mp 1)$.

2.5. Stallings plumbing; Murasugi sum. Let S be a Seifert surface. Let $B \subset S^3$ be a 3-ball. Let $S_1 := S \setminus \operatorname{Int} B$, $S_2 := S \cap B$, $N := S_1 \cap S_2 = S \cap \partial B$. Say that B *deplumbs* S into *plumbands* S_1 and S_2 if S_1 is a Seifert surface and $N \subset S_1$ is an n_1-patch (for some n_1); in this case, necessarily S_2 is also a Seifert surface, and $N \subset S_2$ is an n_2-patch, where, it should be noted, n_2 need not equal n_1.

If S_1 and S_2 are Seifert surfaces, $N_s \subset S_s$ is an n_s-patch, and $h : N_1 \to N_2$ is an orientation-preserving diffeomorphism with

$$h(N_1 \cap \partial S_1) \cup (N_2 \cap \partial S_2) = \partial N_2 \tag{2.5.1}$$

then S_2 is ambient isotopic to S_2' such that $S := S_1 \cup S_2'$ is a Seifert surface deplumbed by a 3-ball $\operatorname{col}_{S_1 \subset S^3}(N_1)$ into plumbands S_1 and S_2', and the isotopy carries h to the identity $N_1 \to N_2'$. Call S a *Stallings plumbing* of S_1 and S_2 *along* h (cf. [17]) and denote it by $S_1 *_h S_2$, or just $S_1 * S_2$ when it is safe to leave h inexplicit. (It is possible, and in a sense typical, that changing the isotopy class of h will change the ambient isotopy type of $S_1 *_h S_2$.) There are n_s-stars $\psi_1 \subset S_1$ and $\psi_2' \subset S_2'$ (corresponding to $\psi_2 \subset S_2$) such that $\psi_1 \cup \psi_2'$ is an $(n_1 + n_2)$-star on the disk $N_1 = N_2'$; the combinatorics of the interleaving of the rays of ψ_1 and ψ_2' in $N_1 = N_2'$ contains all the information needed to (re)construct $S_1 *_h S_2$.

On its face, Stallings plumbing is a strict generalization of *Murasugi sum* (cf. [7], [3], [4]), its seemingly special case in which $n_1 = n_2$ and (2.5.1) is supplemented by

$$h(N_1 \cap \partial S_1) \cap (N_2 \cap \partial S_2) = \partial(N_2 \cap \partial S_2). \tag{2.5.2}$$

In fact, however, it is easy to see that (up to ambient isotopy) every Stallings plumbing is a Murasugi sum of the same plumbands. The distinction is nonetheless useful and will be maintained here.

Stallings [17] showed that any Stallings plumbing of fiber surfaces is a fiber surface. Gabai [3] showed that any Murasugi sum (*viz.,* Stallings plumbing) of least-genus surfaces is least-genus, and, further, that if $S_1 *_h S_2$ is a fiber (resp., least-genus) surface, then S_1 and S_2 are fiber (resp., least-genus) surfaces.

3. Braided Seifert surfaces and quasipositivity

3.1. Braided Seifert surfaces. Let $\mathbb{R}_{\geq \xi}$ (resp., $\mathbb{R}_{\leq \xi}$) denote $\{t \in \mathbb{R} : t \geq \xi\}$ (resp., $\{t \in \mathbb{R} : t \leq \xi\}$). A Seifert surface $S \subset \mathbb{R}^3 = S^3 \setminus \{\infty\}$ is *braided* if it has a handle decomposition (2.2.1) with $X, Z \subset \mathbb{R}$, which satisfies the following conditions:

(3.1.1) for $x \in X$, $S \cap \{x\} \times \mathbb{R}_{\geq 0} \times \mathbb{R} = h_x^{(0)}$, and $h_x^{(0)}$ induces the same orientation on $\{x\} \times \{0\} \times \mathbb{R}$ as $\operatorname{pr}_3 |\{x\} \times \{0\} \times \mathbb{R} : \{x\} \times \{0\} \times \mathbb{R} \to \mathbb{R}$;

(3.1.2) for $z \in Z$, $\operatorname{pr}_3 |h_z^{(1)}$ is Morse with exactly one (interior) critical point, of index 1 (so $S \cap \mathbb{R} \times \mathbb{R}_{\leq 0} \times \{z\}$ is the union of a core arc and a transverse arc of $h_z^{(1)}$ meeting transversely there).

Combinatorial data characterizing S is readily extracted from (3.1.1-2), to wit:

(3.1.3) $\{i, j\} : Z \to \operatorname{two}(X)$ such that $h_z^{(1)}$ is attached to $h_{i(z)}^{(0)}$ and $h_{j(z)}^{(0)}$,

(3.1.4) $\varepsilon : Z \to \{+, -\}$ such that the positive normal vector to S at the critical point of $\operatorname{pr}_3 |h_z^{(1)}$ is a positive multiple of $\varepsilon(t) D \operatorname{pr}_3$.

Then S is determined, up to isotopy through braided Seifert surfaces with X and Z fixed, by $(\{i, j\}, \varepsilon)$. Denote any braided Seifert surface with data $(\{i, j\}, \varepsilon)$ by $S[i, j, \varepsilon]$. Call S *standardized* if $X = \mathbf{n}$, $Z = \mathbf{k}$. Clearly, $S[i, j, \varepsilon]$ is isotopic (through braided Seifert surfaces) to its *standardization* $S[i', j', \varepsilon']$, $i' := (\#_X)^{-1} \circ i \circ \#_Z$, $j' :=$

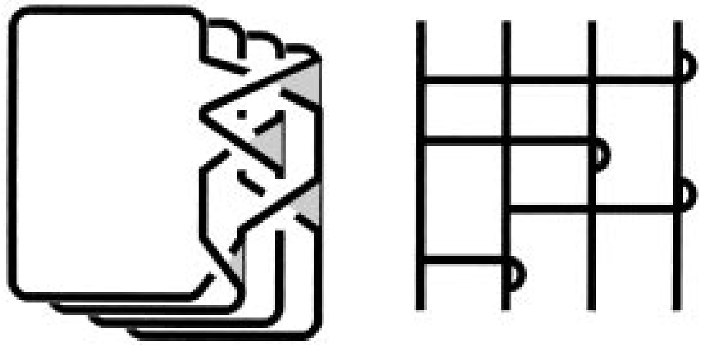

FIGURE 1. $S[i,j,\varepsilon]$, for $(i,j,\varepsilon) : 1 \mapsto (1,2,-), 2 \mapsto (2,4,+), 3 \mapsto (1,3,-), 4 \mapsto (1,4,+)$; the corresponding charged fence diagram.

$(\#_X)^{-1} \circ j \circ \#_Z, \varepsilon' := \varepsilon \circ \#_Z$. It is, however, convenient not to be limited to standardized braided Seifert surfaces.

Let S be a braided Seifert surface. Then the inverse $-S$ is surely not braided; however, $-S$ is isotopic to the braided surface $-R(S)$, where $R : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \mapsto (-x,y,-z)$. On the other hand, the mirror image $\mathrm{Mir}(S)$ is braided, where $\mathrm{Mir} : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \mapsto (x,y,-z)$. Let $\rho_m : \mathbf{m} \to \mathbf{m} : i \mapsto m+1-i$. If $S[i,j,\varepsilon]$ is standardized then $-S[i,j,\varepsilon]$ is isotopic to $S[\rho_n \circ j \circ \rho_k, \rho_n \circ i \circ \rho_k, \varepsilon \circ \rho_k]$, and $\mathrm{Mir}(S[i,j,\varepsilon])$ is isotopic to $S[i \circ \rho_k, j \circ \rho_k, -\varepsilon \circ \rho_k]$.

Remark on the figures. Perhaps the most convenient graphic portrayal of a braided Seifert surface, one which conveys all its combinatorics at a glance, is a *charged fence diagram* (cf. [13]). The fence diagram corresponding to $S = S[i,j,\varepsilon]$ is the union (in the (x,z)-plane) of the set $\mathrm{pr}_{1,3}(S^{(0)})$ of *posts* and the set $\mathrm{pr}_{1,3}(S^{(1)} \cap \mathbb{R} \times \mathbb{R}_{\leq 0} \times Z)$ of *wires*; the corresponding *charge*, which formally is the map induced by ε from the set of wires to $\{+,-\}$, is conveniently depicted by adding a "hook" of the correct handedness to the end of each wire. Figure 1 shows a braided Seifert surface and a corresponding charged fence diagram.

3.2. Braided Stallings plumbing. Let $S_s := S[i_s, j_s, \varepsilon_s]$ $(s = 1,2)$ be braided Seifert surfaces. If

(3.2.1) $\max X_2 = \min X_1$, $Z_1 \cap Z_2 = \varnothing$

and we set $X = X_1 \cup X_2$, $Z = Z_1 \cup Z_2$, and define $(\{i,j\},\varepsilon)$ by $(\{i,j\},\varepsilon)|Z_s = (\{i_s,j_s\},\varepsilon_s)$, then $S[i,j,\varepsilon]$ is a Stallings plumbing $S_1 *_h S_2$ (deplumbed by the 3-ball $(\{\min X_1\} \times \mathbb{R} \times \mathbb{R}) \cup \{\infty\} \subset \mathbb{R}^3 \cup \{\infty\} = S^3$) which satisfies a further constraint:

(3.2.2) $N_1 \subset S_1$ (resp., $N_2 \subset S_2$) lies in the leftmost (resp., rightmost) 0-handle of the handle decomposition (2.2.1) of S_1 (resp., S_2).

Call such a Stallings plumbing *braided*. (The plumbings used by Stallings to prove Theorem 2 of [17] are, inexplicitly, braided.)

Given S_1, S_2', and an increasing injection $H : Z_2' \to \mathbb{R} \setminus Z_1$, if $S_2 := S[i_2, j_2, \varepsilon_2]$ is a braided Seifert surface with the same standardization as S_2' such that $X_2 := X_2' + (\min X_1 - \max X_2')$, $Z_2 := H(Z_2')$, then (3.2.1) and (3.2.2) are satisfied. The isotopy class of h for the resulting braided Stallings plumbing $S_1 *_h S_2$ is determined by H (actually, by the homotopy class of $H| : {j_2'}^{-1}(\max X_2') \to \mathbb{R} \setminus i_1^{-1}(\min X_1)$).

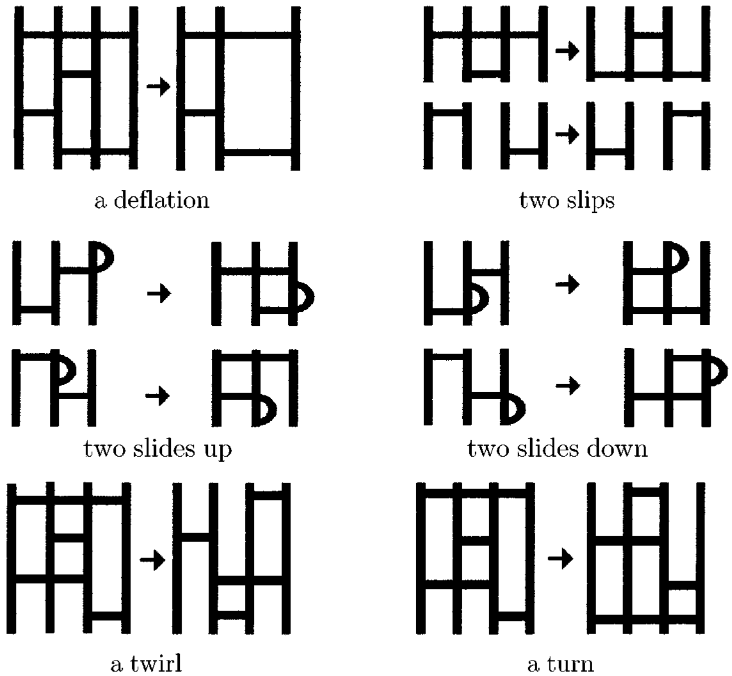

FIGURE 2

3.3. Inflations, deflations, slips, slides, twirls, and turns. Each of the following operations replaces a braided Seifert surface $S := S[i, j, \varepsilon]$ with an ambient isotopic braided Seifert surface $S' := S[i', j', \varepsilon']$ having a different standardization. The operations are illustrated with fence diagrams in Figure 2.

(3.3.1) Suppose that for some $x_0 \in X$, there exists a unique $z_0 \in Z$ with $x_0 \in \{i, j\}(z_0)$; let $X' := X \setminus \{x_0\}, Z' := Z \setminus \{z_0\}, (\{i', j'\}, \varepsilon')(z) := (\{i, j\}, \varepsilon)(z)$ for $z \in Z'$. Say that S' is obtained from S by a *deflation*, and that S is obtained from S' by an *inflation* of sign $\varepsilon(z_0)$.

(3.3.2) Suppose that z_2 is the successor of z_1 in Z, and $\{i, j\}(z_1)$ and $\{i, j\}(z_2)$ unlink; let $(\{i', j'\}, \varepsilon')(z_s) := (\{i, j\}, \varepsilon)(z_{3-s})$ $(s = 1, 2)$, $(\{i', j'\}, \varepsilon')(z) := (\{i, j\}, \varepsilon)(z)$ for $z \in Z \setminus \{z_1, z_2\}$. Say S' is obtained from S by a *slip*.

(3.3.3) Suppose that z_2 is the successor of z_1 in Z, and $\{i, j\}(z_1)$ and $\{i, j\}(z_2)$ touch at $j(z_s) = i(z_{3-s})$. (A) Suppose either $s = 1$ and $\varepsilon(z_2) = +$, or $s = 2$ and $\varepsilon(z_2) = -$; let $(\{i', j'\}, \varepsilon')(z_1) := (\{i, j\}, \varepsilon)(z_2)$ and $(\{i', j'\}, \varepsilon')(z_2) := (\{i(z_s), j(z_{3-s})\}, \varepsilon(z_1))$. (B) Suppose either $s = 1$ and $\varepsilon(z_1) = +$, or $s = 2$ and $\varepsilon(z_1) = -$; let $(\{i', j'\}, \varepsilon')(z_1) := (\{i(z_s), j(z_{3-s})\}, \varepsilon(z_2))$ and $(\{i', j'\}, \varepsilon')(z_2) := (\{i, j\}, \varepsilon)(z_1)$. In each case, let $(\{i', j'\}, \varepsilon')(z) := (\{i, j\}, \varepsilon)(z)$ for $z \in Z \setminus \{z_1, z_2\}$. Say S' is obtained from S by a *slide up* (resp., *slide down*) in cases (A) (resp., (B)).

(3.3.4) Let $x' > \max X$ and $X' := X \setminus \{\min X\} \cup \{x'\}$; let $f : X \to X'$ be defined by $f \mid (X \cap X') = \mathrm{id}_{X \cap X'}, f(\min X) = x'$. Let $(\{i', j'\}, \varepsilon') = (\{f \circ i, f \circ j\}, \varepsilon)$. Say S' is obtained from S by a *twirl*.

(3.3.5) Let $z' > \max Z, Z' := Z \setminus \{\min Z\} \cup \{z'\}$; let $g : Z' \to Z$ be defined by $g \mid (Z \cap Z') = \mathrm{id}_{Z \cap Z'}, g(z') = \min Z$. Let $(\{i', j'\}, \varepsilon') = (\{i, j\}, \varepsilon) \circ g$. Say S' is obtained from S by a *turn*.

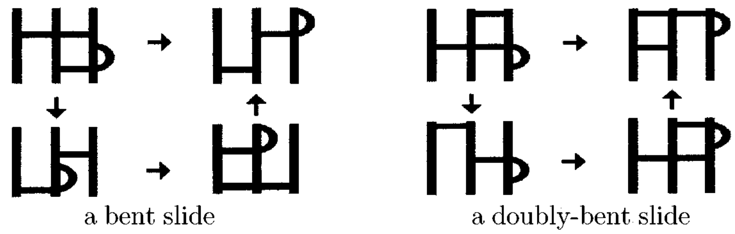

FIGURE 3

Remark. Other operations preserving isotopy type can be obtained by composing slides, twirls, and turns: the inverse (up to standardization) of a twirl (resp., turn) is the composition of twirls (resp., turns); the conjugate of a slide by twirls is either a *bent slide* (the inverse of a slide) or a *doubly-bent slide*, cf. Figure 3. Question: if $S[i,j,\varepsilon]$ and $S[i',j',\varepsilon']$ are isotopic (as Seifert surfaces), then do they differ (up to standardization) only by inflations, deflations, slips, slides, twirls, and turns?

3.4. Quasipositivity. A Seifert surface $S \subset S^3$ is *quasipositive* if it is ambient isotopic to some $S[i,j,+]$ (i.e., ε is the constant function $+$). Note that S is quasipositive iff $-S$ is.

Every Seifert surface is ambient isotopic to a braided Seifert surface [9], but not every Seifert surface is quasipositive [11]. A quasipositive Seifert surface is incompressible [10]. In fact, something much stronger is true: if S is quasipositive, then the proper surface $S' = \mathrm{col}_{S^3 \subset D^4}(\partial S \times [0,1] \cup S \times \{1\}) \subset D^4$ obtained by pushing $\operatorname{Int} S$ into $\operatorname{Int} D^4$ is ambient isotopic to $D^4 \cap \Gamma$ for some smooth complex algebraic curve $\Gamma := f^{-1}(0)$, $f(z,w) \in \mathbb{C}[z,w]$, cf. [9], [10] (by no means do all such "pieces of complex plane curve" so arise, but many interesting examples, e.g., the Milnor fiber of a singularity, do), and it follows [14] from work of Kronheimer and Mrowka [6] that S maximizes Euler characteristic among all surfaces smoothly embedded in D^4 sharing its boundary—in particular, S is least-genus.

3.5. Characterization of quasipositive Seifert surfaces. A subset W of a surface F is *full* if no component of $F \setminus W$ is contractible.

3.5.1. Theorem [12]. *A Seifert surface S is quasipositive iff, for some $n > 0$, S is ambient isotopic to a full subsurface of a fiber surface of the torus link $O\{n,n\}$, i.e., the link of the singularity at the origin of the complex plane curve $z^n + w^n = 0$.*

3.5.2. Corollary [12]. *A full subsurface of a quasipositive Seifert surface is quasipositive.*

4. MURASUGI SUM PRESERVES QUASIPOSITIVITY

4.1. Stars on braided Seifert surfaces. Let $S := S[i,j,\varepsilon]$ be a braided Seifert surface with handle decomposition (2.2.1) satisfying 3.1.1-4. Define an n-star $\psi \subset S$ to be *braided* if it satisfies

(4.1.1) the center of ψ is an interior point of $S^{(0)}$,
(4.1.2) for each ray $\tau \subset \psi$, $\operatorname{tip}(\tau) \subset X \times \{0\} \times \mathbb{R}$,
(4.1.3) ψ is transverse (as in 2.3) to (2.2.1).

Any n-star on S is isotopic to a braided n-star which is minimal (as in 2.3). Let ψ be minimal and braided. Let the ray $\tau \subset \psi$ be long. Then $\operatorname{tip}(\tau)$ and $\operatorname{coccyx}(\tau)$ are the endpoints of a line segment on $\partial h^{(0)}(\operatorname{tail}(\tau)) \cap \mathbb{R} \times \{0\} \times \mathbb{R}$; denote by $D(\tau)$ the subdisk of $h^{(0)}(\operatorname{tail}(\tau))$ bounded by the union of that segment and $\operatorname{tail}(\tau)$. Say that $\operatorname{tail}(\tau)$ is *innermost* if $\psi \cap \operatorname{Int} D(\tau) = \varnothing$.

4.1.4. Lemma. *Let $\psi \subset S = S[i,j,\varepsilon]$ be a minimal braided n-star with $c(\psi) > 0$. Then (S,ψ) is isotopic to (S',ψ'), where $S' = S[i',j',\varepsilon']$ is a braided surface with one more 0-handle and one more 1-handle than S, and ψ' is a minimal braided n-star with $c(\psi') < c(\psi)$. If $\varepsilon = +$ then we may take $\varepsilon' = +$.*

Proof. Since $c(\psi) > 0$, there is a long ray $\tau \subset \psi$. Let τ be such that $\operatorname{tail}(\tau)$ is innermost. Let $h^{(0)}(\operatorname{tail}(\tau)) =: h^{(0)}_{x_0}$, $h^{(1)}(\operatorname{coccyx}(\tau)) =: h^{(1)}_{z_0}$. There are eight cases, according as x_0 is $i(z_0)$ or $j(z_0)$, $\varepsilon(z_0)$ is $+$ or $-$, and $\operatorname{pr}_3(\operatorname{tip}(\tau))$ is larger or smaller than z_0. By a sequence of twirls, a case in which $x_0 = j(z_0)$ may be reduced to the otherwise similar case in which $x_0 = i(z_0)$; by considering $\operatorname{Mir} S$, a case in which $\varepsilon(z_0) = -$ may be reduced to the otherwise similar case in which $\varepsilon(z_0) = +$. Thus we may assume that $\varepsilon(z_0) = +$ and $x_0 = i(z_0)$. The proof when $\operatorname{pr}_3(\operatorname{tip}(\tau)) - z_0$ is positive (resp., negative) is indicated in Figure 4 (resp., Figure 5).

Here is a detailed description of what is going on in Figure 4 (Figure 5 is similar but simpler). (A): A neighborhood of $D(\tau)$ on $h^{(0)}_{x_0}$ and a region on $h^{(0)}_{j(z_0)}$ are shown, joined by $h^{(1)}_{z_0}$. The innermost $\operatorname{tail}(\tau)$ (the thin solid line) crosses $h^{(1)}_{z_0}$, as do, possibly, other arcs of ψ (collectively the thin shaded line). By 2.3.4, τ is not loose, so $z_0 < z_1 := \max\{[\operatorname{pr}_3(\operatorname{coccyx}(\tau)), \operatorname{pr}_3(\operatorname{tip}(\tau))] \cap (i^{-1}(x_0) \cup j^{-1}(x_0))\}$; part of $h^{(1)}_{z_1}$ is shown, and $\{x_0\} \times \{0\} \times (Z \cap]z_0, z_1[)$ is indicated with heavy dots. (B): An inflation of sign $+$ introduces $h^{(1)}_{z_2}$, where $z_1 < z_2 < \operatorname{pr}_3(\operatorname{tip}(\tau))$, $i'(z_2) = x_0$, and $j'(z_2)$ is the successor of x_0 in X'. (C): A slide up moves $h^{(1)}_{z_1}$ past $h^{(1)}_{z_2}$; then, in order, each $h^{(1)}_{z}$, $z \in Z \cap]z_0, z_1[$, moves past $h^{(1)}_{z_2}$ by a slide up or slip up, as appropriate (these moves are unobstructed because $j'(z_2)$ is the successor of x_0 in X'). (D): $h^{(1)}_{z_0}$ slides up over $h^{(1)}_{z_2}$, changing ψ by isotopy to ψ_1 with $c(\psi_1) = c(\psi) + 2$. (E): By an isotopy of $\operatorname{tail}(\tau_1)$, ψ_1 is replaced by ψ_2 with $c(\psi_1) = c(\psi) + 1$. (F): After a final slide up, $(S[i,j,\varepsilon], \psi)$ has been replaced by $(S[i',j',\varepsilon'], \psi_3)$ where $c(\psi_3) = c(\psi)$ and ψ_3 is loose. A final isotopy (not illustrated) replaces ψ_3 by ψ' with $c(\psi') < c(\psi)$. □

4.1.5. Corollary. *If ψ is an n-star on a Seifert surface S, then there is a braided Seifert surface $S' = S[i,j,\varepsilon]$ and an isotopy carrying (S,ψ) to (S',ψ'), where $\psi' \subset S'^{(0)}$ is a minimal braided n-star. If S is quasipositive, then we may take $\varepsilon = +$.*

4.2. Proposition. *Every Stallings plumbing is realizable as a braided Stallings plumbing; if the plumbands are quasipositive then so is the plumbing.*

Proof. Let S_1 and S_2 be Seifert surfaces, $\psi_s \subset S_s$ an n_s-star. By 4.1.5, we may assume that $S_s = S[i_s, j_s, \varepsilon_s]$ is braided (and, if S_s is quasipositive, that $\varepsilon_s = +$). Up to twirls of S_1 and S_2, we may further assume that $\psi_1 \subset h^{(0)}_{\min X_1}$, $\psi_2 \subset h^{(0)}_{\max X_2}$. Finally, given h, up to turns of S_1 (or S_2) we may assume that there is an increasing injection $H : Z_2 \to \mathbb{R} \setminus Z_1$ which determines the isotopy class of h as in 3.2. □

4.3. Theorem. *A Stallings plumbing $S_1 *_h S_2$ is quasipositive iff both S_1 and S_2 are quasipositive.*

Proof. One direction follows from 3.5, the other from 4.2. □

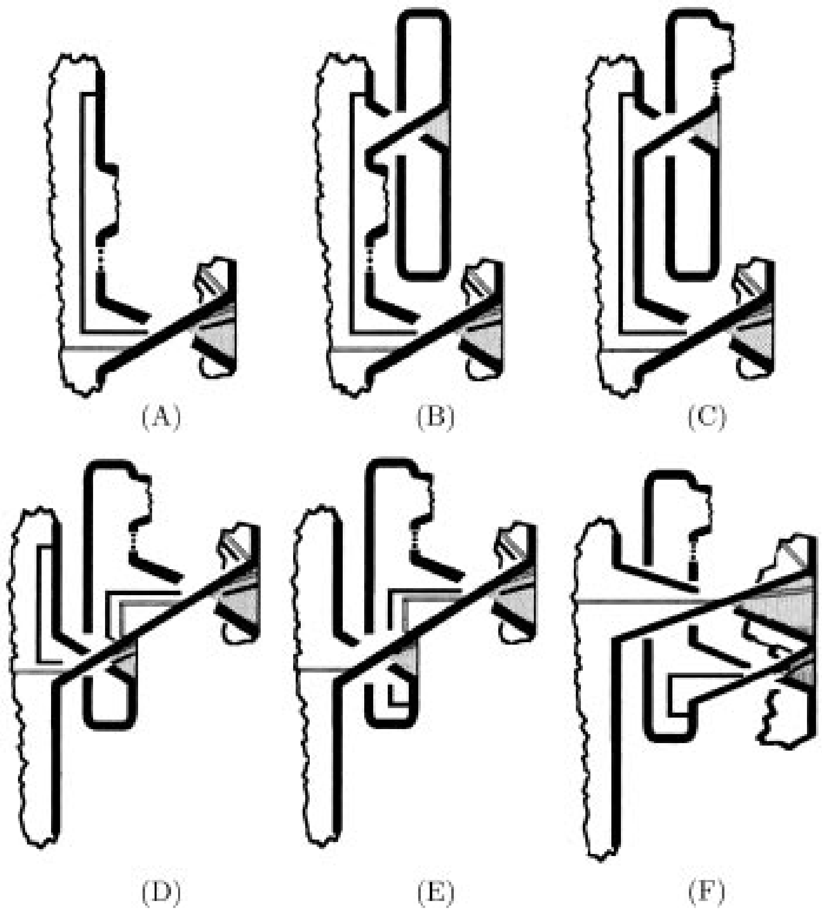

FIGURE 4

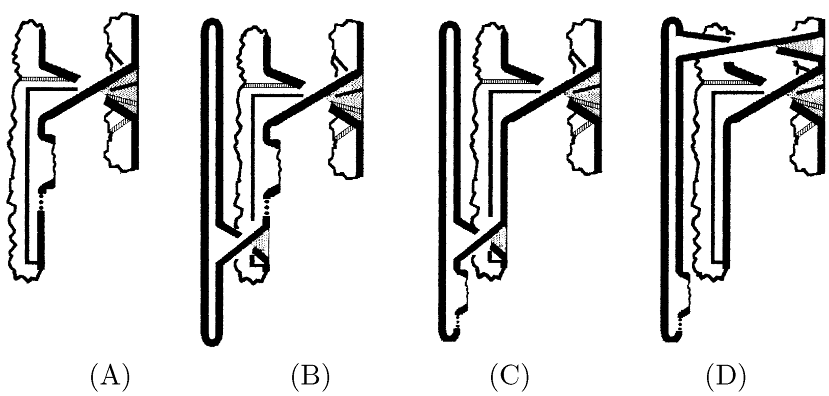

FIGURE 5

5. Applications

5.1. Baskets. A Seifert surface

$$S = \varepsilon_\mu((\dots\varepsilon_2((D^2 * A(O,n_1)) * A(O,n_2))\dots) * A(O,n_\mu)), \tag{5.1.1}$$

where $A(O,n_s)$ is plumbed to $(\dots((D^2 * A(O,n_1)) * A(O,n_2))\dots) * A(O,n_{s-1})$ along $N_s \subset D^2$, $\varepsilon_s \in \{+,-\}$, and $N_1,\dots,N_\mu$ are 2-patches such that $N_i \cap N_j \cap S^1 = \varnothing$ for $i \neq j$, will be called a *basket*, specifically a μ-basket. (Note that, up to isotopy, the link ∂S does not depend on the signs ε_s, but S might.)

Examples. (1) Let Γ be a planar tree, $\mathbf{e} : V(\Gamma) \to 2\mathbb{Z}$ an even weighting of its vertices, $P(\Gamma,\mathbf{e}) \subset S^3$ the corresponding plumbed surface [4], [16]. Induction on $\mu := \operatorname{card}(V(\Gamma))$ shows that $P(\Gamma,\mathbf{e})$ is isotopic to a μ-basket, [1], [15]. The link $\partial P(\Gamma,\mathbf{e})$ is *arborescent* in the sense of [4] and [16] (but not every arborescent link bounds a basket). By [4, 1.19], $P(\Gamma,\mathbf{e})$ is a fiber surface iff $\mathbf{e}(V(\Gamma)) \subset \{2,-2\}$ (but there are fibered arborescent links with fiber surfaces that are not baskets).

(2) If $\beta \in B_n$ is $\mathcal{T}$-*homogeneous* (where $\mathcal{T}$ is a suitable tree with $V(\mathcal{T}) = \mathbf{n}$, cf. [15]), then the closed braid $\widehat{\beta}$ is fibered and its fiber surface is isotopic to a basket; the "homogeneous braids" in [17] are $\mathcal{I}_n$-homogeneous, where $\mathcal{I}_n$ has edges $\{k,k+1\}$, $1 \le k \le n-1$. A fibered arborescent link $\partial P(\Gamma,\mathbf{e})$, $\mathbf{e}(V(\Gamma)) \subset \{2,-2\}$, is isotopic ([1],[15]) to a closed $\mathcal{Y}_{\mu+1}$-homogeneous braid, where $\mathcal{Y}_n$ has edges $\{1,k\}$, $2 \le k \le n$.

(3) The link consisting of $N > 0$ identically oriented fibers of the Hopf fibration $S^3 \to S^2$ is a closed positive (hence $\mathcal{I}_N$-homogeneous) N-braid; its fiber surface (the Milnor fiber of $z^N + w^N$ at $(0,0)$) is, in effect, presented as a basket in [12].

5.1.2. Proposition. *The μ-basket in (5.1.1) is quasipositive iff $n_s < 0$ for all s.*

Proof. For $\mu = 1$, this is proved in [13]. The result follows by induction, using 4.3. □

5.2. An inequality for the slice genus of a link. Let L be a link of r components. Let $\chi_s(L)$ be the greatest Euler characteristic $\chi(F)$ of a surface F in D^4 with $L = \partial F$, so that the slice genus $g_s(L)$ equals $(2 - r - \chi_s(L))/2$.

5.2.1. Lemma [14]. *If S is quasipositive, then $\chi_s(\partial S) = \chi(S)$.*

5.2.2. Corollary. *If S is any Seifert surface, and $Q \subset S$ is quasipositive, then $\chi_s(\partial S) \le 2\chi(Q) - \chi(S)$.*

Proof. Without loss of generality, $Q \subset \operatorname{Int} S$. Let $F \subset D^4$ be a surface with $\partial F = \partial S$ and $\chi(F) = \chi_s(\partial S)$. Let $Q' := (S \setminus \operatorname{Int} Q) \cup F$, so $\partial Q' = \partial Q$. By 5.2.1, $\chi(Q) \ge \chi(Q') = \chi(S) - \chi(Q) + \chi(F)$, so $\chi_s(\partial S) = \chi(F) \le 2\chi(Q) - \chi(S)$. □

Remark. For any given link L presented as the closure $\widehat{\beta}$ of a braid $\beta \in B_n$, 5.2.2. implies the *slice-Bennequin inequality* for β [14]; and there are links L (e.g., the fibered knot 6_2, which is the closure of the homogeneous braid $\sigma_1^3\sigma_2^{-1}\sigma_1\sigma_2^{-1} \in B_3$) such that the bound on $\chi_s(L)$ given by some Seifert surface for L is sharper than the slice-Bennequin inequality for any braid β with $\widehat{\beta} = L$. In that sense 5.2.2 is stronger than the slice-Bennequin inequality. In another sense, they are equivalent, for (as observed in [14]) from the knowledge of the slice-Bennequin inequality for all braids, one can conclude 5.2.1, and thus 5.2.2, for all Seifert surfaces.

5.3. The slice genus of the boundary of a basket. Let S be a μ-basket presented by (5.1.1). Let $p := \mathrm{card}\{s : n_s < 0\}$ and $q := \mathrm{card}\{s : n_s > 0\}$ (so $p + q \le \mu$); let ∂S have r components.

5.3.1. Proposition. $g_s(\partial S) \ge (1 - r + |p - q|)/2$.

Proof. The p-basket S_+ with presentation derived from (5.1.1) by simply omitting plumbands $A(O, n_s)$ with $n_s \ge 0$ is quasipositive. By 5.2.2, $\chi_s(\partial S) \le 2\chi(S_+) - \chi(S) = 2(1 - p) - (1 - \mu) \le 1 - p + q$, so $g_s(\partial S) \ge (1 - r + p - q)/2$. By applying the argument to $\mathrm{Mir}(S)$, we also get $g_s(\partial S) \ge (1 - r + q - p)/2$. So $g_s(\partial S) \ge (1 - r + |p - q|)/2$. □

References

1. Michel Boileau and Lee Rudolph, *Stein fillings via branched covers and plumbing*, in preparation, 1995.
2. J. Conway, *An enumeration of knots and links and some of their algebraic properties*, Computational Problems in Abstract Algebra, Proc. Conf. Oxford, 1967, Pergamon Press, 1970, pp. 329–358. MR **41:**2661
3. David Gabai, *Foliations and the topology of* 3*-manifolds*, J. Diff. Geom. **18** (1983), 445–503. MR **86a:**57009
4. ———, *Genera of the arborescent links*, Memoirs A.M.S. **339** (1986). MR **87h:**57010
5. ———, *The Murasugi sum is a natural geometric operation*, Contemp. Math. **20** (1983), 131–144. MR **85d:**57003
6. P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces.* I, Topology **32** (1993), 773–826. MR **94k:**57048
7. K. Murasugi, *On a certain subgroup of the group of an alternating link*, Amer. J. Math. **85** (1963), 544–550. MR **28:**609
8. ———, *On a certain numerical invariant of link types*, Trans. A.M.S. **117** (1965), 387–422. MR **30:**1506
9. Lee Rudolph, *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helvetici **58** (1983), 1–37. MR **84j:**57006
10. ———, *Quasipositivity and new knot invariants*, Rev. Mat. Univ. Complut. Madrid **2** (1989), 85–109. MR **90k:**57009
11. ———, *A congruence between link polynomials*, Math. Proc. Camb. Phil. Soc. **107** (1990), 319–327. MR **90k:**57010
12. ———, *A characterization of quasipositive Seifert surfaces* (*Constructions of quasipositive knots and links,* III), Topology **31** (1992), 231–237. MR **93g:**57014
13. ———, *Quasipositive annuli* (*Constructions of quasipositive knots and links,* IV), J. Knot Theory Ramif. **1** (1993), 451-466. MR **94c:**57017
14. ———, *Quasipositivity as an obstruction to sliceness*, Bull. A.M.S. **29** (1993), 51–59. MR **94d:**57028
15. ———, *Baskets, Hopf plumbing, $\mathcal{T}$-homogeneous braids, and arborescent surfaces*, in preparation, 1997.
16. L. Siebenmann, *Exercices sur les nœuds rationnels*, mimeographed notes, Orsay, 1975.
17. John R. Stallings, *Constructions of fibred knots and links*, Algebraic and Geometric Topology (R. James Milgram, ed.), Proc. Sympos. Pure Math., vol. XXXII, Part 2, Amer. Math. Soc., Providence, RI, 1978, pp. 55–60. MR **80e:**57004

Department of Mathematics and Computer Science, Clark University, Worcester, Massachusetts 01610

E-mail address: `lrudolph@black.clarku.edu`