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CENTROAFFINE SURFACES IN $\mathbb{R}^4$ WITH PLANAR ∇-GEODESICS

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Abstract. For (positive) definite surfaces in $\mathbb{R}^4$ there is a canonical choice of a centroaffine normal plane bundle, which induces a centroaffine invariant Ricci-symmetric connection ∇. We classify all surfaces in $\mathbb{R}^4$ with planar ∇-geodesics. It turns out that the resulting class of surfaces is umbilical with projectively flat induced connection and flat normal plane bundle.

1. Introduction

This paper is part of an investigation of the centroaffine invariants of surfaces in $\mathbb{R}^4$ which were introduced in [Sch3]. As in every submanifold theory, the choice of a transversal plane bundle leads to an induced connection ∇ on the surface. In the equiaffine theory of surfaces in $\mathbb{R}^4$ the assumption that the ∇-geodesics are plane curves is very restrictive. It was proved in [Vr] that the complex paraboloid is the only definite equiaffine surface with planar ∇-geodesics (if one chooses one of the three equiaffine transversal plane bundles studied in [N-Vr]). Here we study the centroaffine theory for (positive) definite oriented surfaces in $\mathbb{R}^4$, i.e. oriented surfaces with definite affine semi-conformal structure. We recall the basic theory and notions in Section 2. Note, that for the complex paraboloid the centroaffine and the equiaffine transversal plane bundle cannot be the same, since the equiaffine one is parallel while the centroaffine one contains the position vector field of the immersed surface. Thus the induced connections are not the same, and we have to expect a different class of surfaces with planar ∇-geodesics for the centroaffine induced connection.

In Section 3 we prove that a (positive) definite oriented surface with planar ∇-geodesics is centroaffinely equivalent to a surface parametrized by

$$x(s,t) = \frac{1}{\epsilon - \frac{h}{2}(s^2-t^2) - st}(s,t,s^2-t^2,1), \quad h \in \mathbb{R} \text{ and } \epsilon \in \{0, \pm 1\}.$$

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Furthermore we study the properties of this class of surfaces, showing that they are all umbilical and that for all of them ∇ is projectively flat and the normal bundle is flat.

2. Preliminaries

We will recall the basic centroaffine theory of (positive) definite oriented surfaces in $\mathbb{R}^4$. More details can be found in [Sch2] and [Sch3]. Let U be a connected open subset of a two-dimensional oriented manifold M and let $x\colon U \to \mathbb{R}^4 \setminus \{0\}$ be a smooth immersion. As usual we assume that the position vector is nowhere tangential to the surface, i.e., for all $u \in U$, $x(u) \notin x_*(T_uM)$. We define a *first order frame* at $u \in U$ to be an element $F = \{v_1, v_2, \xi, u\} \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times U$ such that $(v_1, v_2, \xi, x(u)) \in Gl(4, \mathbb{R})$, $\operatorname{span}(v_1, v_2) = x_*(T_uM)$, and the preimages $x_*^{-1}v_1, x_*^{-1}v_2$ are positive oriented. If no confusion is possible, we will identify U with its image $x(U)$ and T_uM with $x_*(T_uM)$. We denote the canonical flat connection on $\mathbb{R}^4$ by D.

With respect to a first order frame $F = (v_1, v_2, \xi, x)$ we can write the structure equations for arbitrary tangential vector fields $X, Y \in \mathfrak{X}(U)$ in the form:

$$D_Y X = \nabla_Y X + h^3(X,Y)\xi + h^4(X,Y)x, \tag{2.1}$$

$$D_X \xi = -S(X) + \tau^3(X)\xi + \tau^4(X)x. \tag{2.2}$$

A standard proof shows that ∇ is a torsionfree affine connection, h^3 and h^4 are symmetric bilinear forms, S is a $(1,1)$-tensor field and τ^3 and τ^4 are 1-forms defined on U. Furthermore, an affine (transversal) connection is defined by $\nabla^\perp_X \xi = \tau^3(X)\xi + \tau^4(X)x, \nabla^\perp_X x = 0$. Whereas h^3 depends on the choice of first order frame, it is well known that the corresponding semiconformal structure $\{f \cdot h^3 : f$ is a nowhere vanishing function on $U\}$ is a centroaffine invariant. Another (centro)affine second order invariant is the affine semiconformal structure, generated by the quadratic form:

$$\phi = \frac{[v_1, v_2, Dv_1, Dv_2]}{[v_1, v_2, \xi, x]},$$

where $[\ ,\ ,\ ,\ ]$ is some determinant form on $\mathbb{R}^4$.

For a nondegenerate and h^3-nondegenerate surface (i.e., both induced structures are nondegenerate) we can define an invariant nondegenerate symmetric bilinear form g on U, the *centroaffine metric* induced by x, as follows:

We choose an orientation F_0 on the first order frame bundle, i.e. a first order frame F is positive (negative) oriented, if and only if $\det(F_0F^{-1})$ is positive (negative). Further we denote the determinant of h^3 with respect to $\{v_1, v_2\}$ by $\det_{\{v_1,v_2\}} h^3 := h^3(v_1, v_1)h^3(v_2, v_2) - (h^3(v_1, v_2))^2$. Then g is defined by

$$g = \epsilon |\det_{\{v_1,v_2\}} h^3|^{-\frac{1}{2}} \phi, \qquad \epsilon = \begin{cases} +1, & \text{if } F \text{ is positive oriented,} \\ -1, & \text{if } F \text{ is negative oriented.} \end{cases}$$

For a definite surface we always can choose an orientation such that g is positive definite, and we can show that h^3 and h^4 are indefinite (for every first order frame). Furthermore, an adaption of the frame bundle to the surface geometry by means of the method of moving frames developed by E. Cartan leads to a uniquely determined centroaffine frame:

Theorem ([Sch3, Thm. 2]). *If $x\colon U \to \mathbb{R}^4 \setminus \{0\}$ is a (positive) definite oriented surface, then there exists a uniquely determined* centroaffine frame *field $F = (v_1, v_2, n, x)$ along x (up to a rotation of 180° of $\{v_1, v_2\}$) such that*

$$h^3(v_i, v_j) = (-1)^{i+1}\delta_{ij}, \tag{2.3}$$

$$h^4(v_i, v_j) = 1 - \delta_{ij}, \tag{2.4}$$

$$\tau^3 = 0. \tag{2.5}$$

Using a centroaffine frame field, we define by (2.1) and (2.2) the induced connection ∇ (which is Ricci-symmetric), the second fundamental forms h^3 and h^4, the shape operator S and the (normal connection) 1-form $\tau := \tau^4$. Note that $\{v_1, v_2\}$ are orthonormal with respect to g and h^3 and normal light-like for h^4. If we denote by R^∇ the curvature tensor of ∇, i.e., for arbitrary tangent vector fields $X, Y, Z \in \mathfrak{X}(U)$, $R^\nabla(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, then the centroaffine integrability conditions are

$$R^\nabla(X,Y)Z = h^3(Y,Z)SX - h^4(Y,Z)X - h^3(X,Z)SY + h^4(X,Z)Y, \tag{2.6}$$

$$(\nabla_X h^3)(Y,Z) \text{ is totally symmetric,} \tag{2.7}$$

$$(\nabla_X h^4)(Y,Z) + \tau(X)h^3(Y,Z) \text{ is totally symmetric,} \tag{2.8}$$

$$(\nabla_X S)(Y) - (\nabla_Y S)(X) = \tau(Y)X - \tau(X)Y, \tag{2.9}$$

$$0 = h^3(X, SY) - h^3(Y, SX), \tag{2.10}$$

$$d\tau(X,Y) = h^4(X, SY) - h^4(Y, SX). \tag{2.11}$$

We see that we can define two totally symmetric trilinear forms, the *cubic forms* C^3 and C^4, by

$$\begin{aligned} C^3(X,Y,Z) &= (\nabla_X h^3)(Y,Z), \\ C^4(X,Y,Z) &= (\nabla_X h^4)(Y,Z) + \tau(X)h^3(Y,Z). \end{aligned} \tag{2.12}$$

3. Planar ∇-geodesics

While in equiaffine geometry the complex paraboloid is the only (positive) definite surface with planar ∇-geodesics [Vr], we get in the centroaffine case the following interesting class of surfaces:

Theorem 1. *Let $x(U)$ be a (positive) definite oriented surface with induced connection ∇. If all ∇-geodesics are planar curves, then $x(U)$ is centroaffinely equivalent to a surface parametrized by*

$$\tilde{x}(s,t) = \frac{1}{\epsilon - \frac{h}{2}(s^2 - t^2) - st}(s, t, s^2 - t^2, 1), \quad h \in \mathbb{R}, \epsilon \in \{0, \pm 1\}.$$

Proof. A first step is to prove that the assumption of planar ∇-geodesics implies that

$$h^3(X,X)C^4(X,X,X) - h^4(X,X)C^3(X,X,X) = 0 \quad \text{for all } X \in \mathfrak{X}(U), \tag{3.1}$$

and that there exists a function λ on U such that

$$S = -\lambda \operatorname{Id}. \tag{3.2}$$

Let (v_1, v_2, n, x) be a centroaffine frame field along x. Let $p \in U, X \in T_pM$, and let γ be a ∇-geodesic with $\gamma(0) = p$ and $\gamma'(0) = X$. Since γ is a planar curve, $\gamma', D_{\gamma'}\gamma'$

and $D_{\gamma'}D_{\gamma'}\gamma'$ must be linearly dependent. Using $\nabla_{\gamma'}\gamma' = 0$, (2.1), (2.2), (2.5) and (2.12), we get at p:

$$\begin{aligned} D_X\gamma' &= h^3(X,X)n + h^4(X,X)x, \\ D_XD_X\gamma' &= X(h^3(X,X))n + h^3(X,X)(-S(X) + \tau(X)x) \\ &\quad + X(h^4(X,X))x + h^4(X,X)X \\ &= -h^3(X,X)S(X) + h^4(X,X)X + C^3(X,X,X)n + C^4(X,X,X)x. \end{aligned}$$

Therefore, (3.1) must be satisfied and $S(X) = -\lambda X$ if $h^3(X,X) \neq 0$. By continuity this implies (3.2). An evaluation of (3.1) in terms of the centroaffine tangential frame $\{v_1, v_2\}$, using (2.3) and (2.4) and the notation $C^\alpha(v_i, v_j, v_k) = C^\alpha_{ijk}$ $(\alpha \in \{3,4\}, i,j,k \in \{1,2\})$, gives

$$\begin{aligned} &C^4_{111} = 0, && C^4_{222} = 0, \\ &C^4_{112} = \tfrac{2}{3}C^3_{111}, && C^4_{221} = -\tfrac{2}{3}C^3_{222}, \\ &C^3_{112} = -\tfrac{1}{3}C^3_{222}, && C^3_{221} = -\tfrac{1}{3}C^3_{111}. \end{aligned} \tag{3.3}$$

Our next step is a study of the integrability conditions (2.6) – (2.12). It was shown in [Sch1] that they impose (among others) algebraic relations such that we can express any of the other invariants in terms of C^3, h^3 and h^4. Using (3.3) and the notation

$$a := C^3_{111}, \quad b := C^3_{222},$$

we can show that

$$\begin{aligned} &\nabla_{v_1}v_1 = -\tfrac{1}{2}av_1, \quad \nabla_{v_1}v_2 = \tfrac{1}{3}bv_1 - \tfrac{1}{6}av_2, \\ &\nabla_{v_2}v_1 = \tfrac{1}{6}bv_1 - \tfrac{1}{3}av_2, \quad \nabla_{v_2}v_2 = \tfrac{1}{2}bv_2, \\ &\tau = 0, \quad S = -h\,\mathrm{Id}, \quad h = \text{const}. \end{aligned} \tag{3.4}$$

The proof is a straightforward computation, where we obtain the induced connection and the normal connection from (2.12) (cf. [Sch3, (11), (12)]), and then $h = \text{const}$ from (2.9). Finally, (2.6) gives the following non-linear system of first order partial differential equations for a and b:

$$\begin{aligned} &v_1(a) = -\tfrac{1}{6}a^2 - 3h, && v_1(b) = -\tfrac{1}{6}ab + 3, \\ &v_2(a) = \tfrac{1}{6}ab - 3, && v_2(b) = \tfrac{1}{6}b^2 - 3h. \end{aligned} \tag{3.5}$$

Note that the system has no constant solution, in particular $a \not\equiv 0$ and $b \not\equiv 0$. Thus $[v_1, v_2] = \frac{1}{6}(bv_1 + av_2) \not\equiv 0$. However, we will show that there exist a function ρ and coordinate functions s and t such that

$$\rho v_1 = \tfrac{\partial}{\partial s}, \quad \rho v_2 = \tfrac{\partial}{\partial t}, \tag{3.6}$$

namely ρ is a solution of

$$v_1(\rho) = -\tfrac{1}{6}a\rho, \quad v_2(\rho) = \tfrac{1}{6}b\rho. \tag{3.7}$$

Let f be a function and put $\rho = e^f$. Then (3.7) is equivalent to $v_1(f) = -\frac{1}{6}a$ and $v_2(f) = \frac{1}{6}b$. To show that there exists a solution f we need to check the integrability condition: $[v_1, v_2](f) = \frac{1}{6}(bv_1 + av_2)(f) = 0$. Next we show that the distribution $\{\rho v_1, \rho v_2\}$ is integrable: $[\rho v_1, \rho v_2] = \rho^2\frac{1}{6}(bv_1 + av_2) + \rho(-\frac{1}{6}a\rho)v_2 - \rho(\frac{1}{6}b\rho)v_1 = 0$, which proves (3.6).

The problem left is to find explicit solutions of the system (3.5) and (3.7), where by now we can write the system in the form:

$$\begin{aligned} &\tfrac{\partial\rho}{\partial s} = -\tfrac{1}{6}a\rho^2, && \tfrac{\partial a}{\partial s} = (-\tfrac{1}{6}a^2 - 3h)\rho, && \tfrac{\partial b}{\partial s} = (-\tfrac{1}{6}ab + 3)\rho, \\ &\tfrac{\partial\rho}{\partial t} = \tfrac{1}{6}b\rho^2, && \tfrac{\partial a}{\partial t} = (\tfrac{1}{6}ab - 3)\rho, && \tfrac{\partial b}{\partial t} = (\tfrac{1}{6}b^2 - 3h)\rho. \end{aligned} \tag{3.8}$$

Note that there are relations like $\frac{\partial a}{\partial s}\rho - a\frac{\partial\rho}{\partial s} = -3h\rho^2$, which we can integrate. Thus it is easy to prove that, possibly after a translation of s and t, we get :

$$\tfrac{a}{\rho}(s,t) = -3(hs+t), \qquad \tfrac{b}{\rho}(s,t) = 3(s-ht). \tag{3.9}$$

Now the equations for ρ in (3.8) become $\frac{\partial\rho}{\partial s} = \frac{1}{2}(hs+t)\rho^3$ and $\frac{\partial\rho}{\partial t} = \frac{1}{2}(s-ht)\rho^3$, i.e.,

$$\tfrac{\partial}{\partial s}(-\rho^{-2}) = hs+t \quad \text{and} \quad \tfrac{\partial}{\partial t}(-\rho^{-2}) = s-ht. \tag{3.10}$$

Integration leads to

$$-(\rho(s,t))^{-2} = \begin{cases} \frac{h}{2}(s^2-t^2)+st, & C=0, \\ C(\epsilon + \frac{h}{2}(s^2-t^2)+st), & C\neq 0, \end{cases} \tag{3.11}$$

where $C \in \mathbb{R}$ depends on a given initial value, $\epsilon := \operatorname{sign} C$, and s and t possibly are rescaled. We still do not have explicit solutions for a, b and ρ. Fortunately we only need the products $a\rho, b\rho$ and ρ^2 to write down the structure equations (2.1) in coordinates, since for example $\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t} = \rho\nabla_{v_1}\rho v_2 = (-\frac{1}{6}a\rho)\frac{\partial}{\partial t} + \rho\frac{1}{3}b\frac{\partial}{\partial s} - \rho\frac{1}{6}a\frac{\partial}{\partial t}$ (cf. (3.6), (3.7),(3.4)) and $h^3(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) = \rho^2 h^3(v_1, v_2)$, etc. Using (2.3) and (2.4), we obtain:

$$\begin{aligned} \tfrac{\partial^2}{\partial s^2}x &= 2(hs+t)\rho^2\tfrac{\partial}{\partial s}x + \rho^2 n, \\ \tfrac{\partial^2}{\partial s\partial t}x &= (s-ht)\rho^2\tfrac{\partial}{\partial s}x + (hs+t)\rho^2\tfrac{\partial}{\partial t}x + \rho^2 x, \\ \tfrac{\partial^2}{\partial t^2}x &= 2(s-ht)\rho^2\tfrac{\partial}{\partial t}x - \rho^2 n. \end{aligned} \tag{3.12}$$

It turns out that we obtain simpler equations if we compute the second derivatives of $x\rho^{-2}$, using (3.10) and (3.12):

$$\tfrac{\partial^2}{\partial s^2}(x\rho^{-2}) = -hx+n, \quad \tfrac{\partial^2}{\partial s\partial t}(x\rho^{-2}) = 0, \quad \tfrac{\partial^2}{\partial t^2}(x\rho^{-2}) = hx-n. \tag{3.13}$$

Noticing that $\Delta(x\rho^{-2}) = 0$, we can easily integrate and obtain the general solution:

$$x = \rho^2(A(s^2-t^2) + Bs + Et + F),$$

where A, B, E and F are linearly independent vectors in $\mathbb{R}^4$ (since x is definite). By a centroaffine transformation we can map A, B, E and F to the standard basis vectors e_3, e_1, e_2 and e_4 and, if $C \neq 0$, rescale C to 1, which gives $\tilde{x}$. $\square$

Theorem 2. *Let $h \in \mathbb{R}, \epsilon \in \{0, \pm 1\}$ and $V = \{(s,t) \in \mathbb{R}^2 | \epsilon > \frac{h}{2}(s^2-t^2)+st\}$. Then the centroaffine surface $x_{h,\epsilon}\colon V \to \mathbb{R}^4$, defined by*

$$x_{h,\epsilon}(s,t) = \frac{1}{\epsilon - \frac{h}{2}(s^2-t^2) - st}(s, t, s^2-t^2, 1),$$

has the following properties:

(i) *the induced connection ∇ is projectively flat,*
(ii) *the ∇-geodesics are planar curves,*
(iii) *$S = -hId$ and $n = hx + 2e_3$,*
(iv) *$\tau = 0$, and so the normal bundle is flat,*
(v) *$x_{h,\epsilon}(V)$ is an open part of the intersection of the quadratic hypersurfaces given by $x_1^2 - x_2^2 = x_3x_4$ and $\epsilon x_4^2 - \frac{h}{2}x_3x_4 - x_1x_2 - x_4 = 0$.*

Proof. Since most of the proof follows from the foregoing, we will just briefly recall the main steps. First we define on V a function $\rho^{-2}(s,t) = \epsilon - \frac{h}{2}(s^2 - t^2) - st$, and a vector field $n = \frac{\partial^2}{\partial s^2}(x\rho^{-2}) + hx$ in $\mathbb{R}^4$ (cf. (3.11) and (3.13)). It is easily checked that $\tilde{F} = \{\frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}, n, x\}$ is a basis of $\mathbb{R}^4$ for every $(s,t) \in V$ (actually a first order frame). We compute the structure equations (2.1) and (2.2) with respect to $\tilde{F}$ and obtain that (2.1) is equivalent to (3.12) and that (2.2) gives $S = -h\,\mathrm{Id}$, $\tau^3 = 0$ and $\tau^4 = 0$. Thus $F = \{\frac{1}{\sqrt{\rho^2}}\frac{\partial x}{\partial s}, \frac{1}{\sqrt{\rho^2}}\frac{\partial x}{\partial t}, n, x\}$ is a centroaffine frame field along x, and we have proved (iii) and (iv). Furthermore, (3.12) implies that

$$\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} = 2(hs+t)\rho^2\frac{\partial}{\partial s}, \qquad \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t} = (s-ht)\rho^2\frac{\partial}{\partial s} + (hs+t)\rho^2\frac{\partial}{\partial t}$$

and

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = 2(s-ht)\rho^2\frac{\partial}{\partial t}.$$

If we define a 1-form η on V by $\eta = \rho^2(hs+t)ds + \rho^2(s-ht)dt$, then $\nabla_X Y - D^0_X Y = \eta(X)Y + n(Y)X$ for the standard flat connection D^0 on $\mathbb{R}^2$ and arbitrary tangential vector fields $X, Y \in \mathfrak{X}(V)$, which proves (i). To show (ii), we can either prove (3.3) (resp. (3.1)) or give a direct proof, since we know by (i) that the ∇-pregeodesics are just the images of the straight lines in $V \subset \mathbb{R}^2$. We will do the latter. Let $\gamma(u) = x(au+b, cu+d)$ be an arbitrary ∇-(pre)geodesic. Then the coefficient functions of γ in $\mathbb{R}^4$ are $\gamma^1(u) = \rho^2(au+b), \gamma^2(u) = \rho^2(cu+d), \gamma^3(u) = \rho^2(pu^2+qu+r)$ and $\gamma^4(u) = \rho^2$, where we omitted the argument of ρ^2 and $p = a^2 - c^2, q = 2(ab-cd), r = b^2 - d^2$. Furthermore $\rho^{-2}(au+b, cu+d) = mu^2 + nu + o$ with $m = -\frac{h}{2}p - ac, n = -\frac{h}{2}q - (ad+bc)$ and $o = \epsilon - \frac{h}{2}r - bd$. If we define $\beta = \frac{m}{p}, \alpha = \frac{1}{a}(n - \beta q)$ and $\delta = o - \alpha b - \beta r$, then we get the following two linear equations:

$$-\frac{c}{a}\gamma^1 + \gamma^2 + (\frac{c}{a} - d)\gamma^4 = 0,$$
$$\alpha\gamma^1 + \beta\gamma^3 + \delta\gamma^4 = 1,$$

and thus γ is a planar curve. □

Remark. Finally we would like to remark that by Theorem 2, (iii), the surface $x_{h,\epsilon}$ is umbilic related to n in the terminology of [N-S]. Using a similar argument as in Theorem 1, it is easy to show that a surface is umbilic if and only if every ∇-geodesic lies in a 3-dimensional linear subspace.

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References

[N-S] K. Nomizu and T. Sasaki, *Centroaffine immersions of codimension two and projective hypersurface theory*, Nagoya Math. J. **132** (1993), 63–90. MR **94j:**53013

[N-Vr] K. Nomizu and L. Vrancken, *A new equiaffine theory for surfaces in* $\mathbb{R}^4$, Internat. J. Math. **4** (1993), 127–165. MR **94f:**53014

[Sch1] Ch. Scharlach, *Centroaffine differential geometry of surfaces in* $\mathbb{R}^4$, Dissertation, Technische Universität Berlin, 1994.

[Sch2] Ch. Scharlach, *Centroaffine first order invariants of surfaces in* $\mathbb{R}^4$, Results Math. **27** (1995), 141–159. MR **96a:**53016

[Sch3] Ch. Scharlach, *Centroaffine differential geometry of (positive) definite oriented surfaces in* $\mathbb{R}^4$, Preprint No. 451/1995, FB Mathematik, TU Berlin, submitted.

[Vr] L. Vrancken, *Affine surfaces whose geodesics are planar curves*, Proc. Amer. Math. Soc. **123** (1995), 3851–3854. MR **96b:**53021

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