PRIMES OF THE FORM $p = 1 + m^2 + n^2$ IN SHORT INTERVALS

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Abstract. In this note, we prove that for every $\theta \geq 115/121$ and $x \geq x_0(\theta)$, the short interval $(x, x + x^\theta]$ contains at least one prime number of the form $p = 1 + m^2 + n^2$ with $(m, n) = 1$. This improves a similar result due to Huxley and Iwaniec, which requires $\theta \geq 99/100$.

§1. Introduction

The existence of prime numbers in short intervals is an important subject in analytic number theory. Huxley and Iwaniec [3] asked the following question: seek positive numbers $\theta$, as small as possible, such that

(1.1) the short interval $(x, x + x^\theta]$ contains at least one prime number of the form $p = 1 + m^2 + n^2$ with $(m, n) = 1$ for $x \geq x_0(\theta)$.

Naturally this problem can be attacked by applying the half dimensional sieve to the sequence $A := \{p - 1 : x < p \leq x + x^\theta, p \equiv 3 \pmod{8}\}$, where, as in the sequel, the letter $p$, with or without subscript, denotes a prime number. In order to control error terms in formulas of sieves, a Bombieri–Vinogradov type mean-value theorem for short intervals is needed. Huxley and Iwaniec [3] have obtained a satisfactory generalization of Bombieri–Vinogradov’s theorem in the case of short intervals, using a zero-density theorem for Dirichlet $L$-functions. As an application, they have shown that (1.1) is true with $\theta = 99/100$.

The aim of this note is to improve Huxley and Iwaniec’s exponent. More precisely we shall prove the following result.

Theorem. For every $\theta \geq 115/121$ and $x \geq x_0(\theta)$, we have

(1.2) $\sum_{x < p \leq x + x^\theta} b^*(p - 1) \gg x^\theta/(\log x)^{3/2}$,

where

$$b^*(a) := \begin{cases} 1 & \text{if } a = m^2 + n^2 \text{ with } (m, n) = 1, \\ 0 & \text{otherwise.} \end{cases}$$
For comparison, we have $\frac{115}{127} \approx 0.9504$ and $\frac{99}{100} = 0.99$.

We begin in the same way as Huxley and Iwaniec. By (3.2) of [3], we have

\begin{equation}
\sum_{x < p \leq x + x^\theta} b^*(p - 1) = S(A; P_3, x + x^\theta),
\end{equation}

where $S(A; P_3, z) := \# \left\{ a \in A : (a, \prod_{p < z, p \in P_3} p) = 1 \right\}$ and $P_3 := \{ p : p \equiv 3 \pmod{4} \}$. Let $\alpha = \alpha(\theta) \in [2, 3)$ be a parameter to be chosen later. For $z = x^{1/\alpha}$, we write

\begin{equation}
S(A; P_3, x + x^\theta) = S(A; P_3, z) - T,
\end{equation}

where $T := S(A; P_3, z) - S(A; P_3, x + x^\theta)$. A lower bound for $S(A; P_3, z)$ on the right-hand side of (1.4) will be obtained by the half dimensional sieve (as in [3]) in Section 3. We shall give a better upper bound for $T$ than that of Huxley and Iwaniec. This improvement comes from our generalized Bombieri–Vinogradov type mean-value theorem for short intervals that we state in Section 2. One observes that each element $a \in A$ is divisible by an even number of primes from $P_3$ and $2 \parallel a$. Hence, for $z = x^{1/\alpha}$ with $2 \leq \alpha < 3$, we have obviously

\begin{equation}
T \leq \sum_{\substack{x < p \leq x + x^\theta \atop p = 1 + 2np_1p_2}} 1,
\end{equation}

where $p_1 \in P_3, p_2 \in P_3, p_1 \geq p_2 \geq x^{1/\alpha}$ and $n$ is an integer divisible only by primes of the form $p \equiv 1 \pmod{4}$. We define

\[ \mathcal{L} := \{ l = 2np_2 : n \leq x^{1-2/\alpha}, p \mid n \Rightarrow p \equiv 1 \pmod{4}; \quad x^{1/\alpha} < p_2 \leq (x/n)^{1/2}, p_2 \in P_3 \}. \]

For every $l \in \mathcal{L}$, we put

\[ \mathcal{M}(l) := \{ m = lp_1 + 1 : x/l < p_1 \leq (x + x^\theta)/l, (l/2)p_1 \equiv 1 \pmod{4} \}. \]

It is clear that the sum on the right-hand side of (1.5) does not exceed the number of primes in the set $\bigcup_{l \in \mathcal{L}} \mathcal{M}(l)$; thus

\begin{equation}
T \leq \sum_{l \in \mathcal{L}} \left\{ S\left( \mathcal{M}(l) ; P(l), (x/l)^{\theta_1} \right) + O\left( (x/l)^{\theta_1} \right) \right\},
\end{equation}

where $P(l) := \{ p : (p, l) = 1 \}$ and $\theta_1 := (2\theta - 1)/4$. We shall apply an upper bound formula of the linear sieve to treat $S\left( \mathcal{M}(l) ; P(l), (x/l)^{\theta_1} \right)$ in Section 4.

It seems interesting to compare the theorem with the following results: for $x$ sufficiently large, $(x, x + x^{0.973})$ contains at least one prime of the form $p = -2 + P_2$ where $P_2$ denotes an integer having at most two prime factors [8]; for every $\varepsilon > 0$ and $x \geq x_0(\varepsilon)$, there exists at least one prime of the form $p = -2 + a$ in $(x, x + x^{3/4+\varepsilon}]$ where $a$ is a $B$–free integer [7].

\section*{§2. Two Preliminary Lemmas}

We first recall some standard notations (cf. [2]). Let $\mathcal{F}$ be a finite sequence of integers and let $\mathcal{P}$ be a set of prime numbers. For $z \geq 2$, the sifting function is defined as follows:

\[ S(\mathcal{F}; \mathcal{P}, z) := \# \left\{ a \in \mathcal{F} : (a, P(z)) = 1 \right\}, \]
Lemma 1. If there exist positive constants $A_k$ ($k = 1, 2, 3$) such that

\[ 0 \leq \frac{w(p)}{p} < 1 - \frac{1}{A_1}, \]

\[-A_2 \leq \sum_{z_1 \leq p < z_2} \frac{w(p) \log p}{p} - \log \frac{z_2}{z_1} \leq A_3 \quad (z_2 \geq z_1 \geq 2), \]

we then have, for $2 \leq z \leq Q$, that

\[ S(F; \mathcal{P}, z) \leq XV(z) \left\{ F \left( \frac{\log Q}{\log z} \right) + O \left( \frac{A_2}{(\log Q)^{1/14}} \right) \right\} + \sum_{d \leq Q, d \mid P(z)} 3^{\omega(d)} |r(F, d)|, \]

where $F(t) = 2e^t/t$ ($0 < t < 2$).

The following lemma is a direct consequence of Theorem 2 of [6].

Lemma 2. Let $g(l)$ be an arithmetic function satisfying $g(l) \ll 1$ and let

\[ H(x', h, q, a, l) := \sum_{x' < p \leq x'+h} -\frac{1}{\varphi(q)} \int_{x'/l}^{(x'+h)/l} \frac{dt}{\log t}. \]

Then for any $A > 0$, there exists a positive constant $B = B(A)$ such that

\[ \sum_{q \leq Q} \max_{(a, q) = 1} \max_{h \leq x} \max_{x'/l \leq x' \leq x} \left| \sum_{l \leq L} g(l) H(x', h, q, a, l) \right| \ll \frac{x^\theta}{(\log x)^A}, \quad (1.1) \]

\[ \sum_{q \leq Q} \mu(q)^2 \varphi(q)^3 \max_{(a, q) = 1} \max_{h \leq x} \max_{x'/l \leq x' \leq x} \left| \sum_{l \leq L} g(l) H(x', h, q, a, l) \right| \ll \frac{x^\theta}{(\log x)^A}, \quad (2.2) \]

for $x \geq 10$, $\frac{3}{5} + \varepsilon \leq \theta \leq 1$, $Q = x^{\theta-1/2}/(\log x)^2$ and $L = x^{(5\theta-3)/2-\varepsilon}$. 

where $P(z) := \prod_{p < z, p \in \mathcal{P}} p$. If $d$ is a squarefree integer whose prime factors belong to $\mathcal{P}$, we let $\mathcal{F}_d := \{ n \in \mathcal{F} : d \mid n \}$ and use $|\mathcal{F}_d|$ to denote the number of elements of $\mathcal{F}_d$. We write an approximate formula

\[ |\mathcal{F}_d| = \frac{w(d)}{d} X + r(F, d), \]

where $X > 1$ is independent of $d$, and $w(d)$ is a multiplicative function. We define

\[ V(z) := \prod_{p < z, p \in \mathcal{P}} (1 - w(p)/p). \]

As usual, $\mu(n)$ is M"obius’ function, $\varphi(n)$ Euler’s function and $\omega(n)$ the number of distinct prime factors of $n$. Finally we write $\varepsilon$ for an arbitrarily small positive number and $\gamma$ for Euler’s constant.

The first lemma is an upper bound formula of the linear sieve ([2], Theorem 8.3).
\section*{§3. Lower bound for $S(A; P_3, x^{1/\alpha})$}

The following proposition offers the required lower bound for $S(A; P_3, x^{1/\alpha})$.

**Proposition 1.** Assume that $\frac{1}{2} \leq \theta \leq 1$ and $2/(2\theta - 1) \leq \alpha \leq 6/(2\theta - 1)$. We then have

\begin{equation}
S(A; P_3, x^{1/\alpha}) \geq \{W_1(\theta, \alpha) + o(1)\} x^{\theta}/(\log x)^{3/2},
\end{equation}

where

\[ W_1(\theta, \alpha) := \frac{AC_3}{\sqrt{4\theta - 2}} \int_1^{\alpha(\theta - 1/2)} \frac{dt}{\sqrt{t(t-1)}} \]

and \[ A := (1/2\sqrt{2}) \Pi_{p\equiv3(\text{mod}4)} (1 - p^{-2})^{1/2}, \]

\[ C_3 := \Pi_{p\equiv3(\text{mod}4)} (1 - (p - 1)^{-2}) \]

**Proof.** By (3.3) of [3] or Theorem 1 of [4], we have

\begin{equation}
S(A; P_3, x^{1/\alpha}) \geq \{1 + o(1)\} x^{\theta}V(x^{\theta}, Q) \sqrt{\frac{\pi x}{\alpha \log x}} \int_{1}^{\xi} \frac{dt}{\sqrt{t(t-1)}} = E(x, x^\theta, Q),
\end{equation}

where

\[ \xi = \alpha \frac{\log Q}{\log x} \in [1, 3], \quad V(x^{1/\alpha}) = \prod_{p < x^{1/\alpha}} \left(1 - \frac{1}{p - 1}\right), \]

\[ E(x, x^\theta, Q) = \sum_{q \leq Q} \max_{(a, q) = 1} \max_{1 \leq x^{\theta} < x/2 \leq x} \left| \sum_{x^{1/\alpha} < p \leq x^{1/\alpha} + h} 1 - \frac{1}{\varphi(q)} \int_{x^{1/\alpha}}^{x^{1/\alpha} + h} \frac{dt}{\log t} \right|. \]

Let $\chi$ be the non-principal character modulo 4 and let $L(\chi, s)$ be the Dirichlet $L$-function associated with $\chi$. Using the relation \[ L(\chi, 1; y) := \prod_{p < y} (1 - \chi(p)/p)^{-1} \rightarrow L(\chi, 1) = \frac{\pi}{2} (y \rightarrow \infty) \] and Mertens’ formula, we deduce that

\begin{equation}
V(x^{1/\alpha}) = \sqrt{2L(\chi, 1; x^{1/\alpha})} \prod_{p < x^{1/\alpha}} \left(1 - \frac{1}{p - 1}\right)^{1/2}
\end{equation}

\[ \times \prod_{p < x^{1/\alpha}} (1 - p^{-2})^{1/2} \]

\[ = \{1 + o(1)\} 2AC_3(\alpha \pi e^{-\gamma}/\log x)^{1/2} \quad (x \rightarrow \infty). \]

Taking $Q = x^{\theta - 1/2}/(\log x)^B$ and $g(1) = 1, g(l) = 0$ ($l \geq 2$) in (2.1) of Lemma 2, we obtain

\begin{equation}
E(x, x^\theta, Q) \ll x^\theta/(\log x)^2.
\end{equation}

Now the required inequality (3.1) follows from (3.2)–(3.4).

\section*{§4. Upper bound for $T$}

The purpose of this section is to prove Proposition 2 below, which gives a better upper bound for $T$ than that of [3]. To prepare for its proof, we first establish two auxiliary results.
Lemma 3. Let \( u(n) \) be the characteristic function of integers whose prime factors are of the form \( 4m + 1 \) and let \( f(n) := \prod_{p \mid n, p > 2} (p - 1)/(p - 2) \). We then have

\[
(4.1) \quad \sum_{n \leq x} u(n) f(n) = (A/C_1) x/(\log x)^{1/2} + O(x/(\log x)^{3/2}),
\]

where \( C_1 := \prod_{p \equiv 1 (\mod 4)} (1 - (p - 1)^{-2}) \) and \( A \) is defined as in Proposition 1.

Proof. Let \( \chi, L(s, \chi) \) be defined as above. It is clear that \( u(n) f(n) \) is multiplicative and

\[
(4.2) \quad u(p^n) f(p^n) = \begin{cases} (p - 1)/(p - 2) & \text{if } p \equiv 1 (\mod 4), \\ 0 & \text{otherwise} \end{cases}
\]

A simple calculation shows that for \( \Re s > 1 \),

\[
\sum_{n=1}^{\infty} u(n) f(n) n^{-s} = \prod_{p \equiv 1 (\mod 4)} (1 - p^{-s})^{-1} (1 + (p - 2)^{-1} p^{-s}) = \zeta(s)^{1/2} G(s),
\]

where \( \zeta(s) \) is Riemann’s zeta function and

\[
G(s) := (L(s, \chi)(1 - 2^{-s}) \prod_{p \equiv 3 (\mod 4)} (1 - p^{-2s}))^{1/2} \prod_{p \equiv 1 (\mod 4)} (1 + (p - 2)^{-1} p^{-s}).
\]

Using the well known estimation (see [1], Theorem 8.1)

\[
L(s, \chi) \ll (\log (3 + |3m s|) \quad \text{for } \Re s \geq 1 - 1/\log (3 + |3m s|),
\]

we see that \( \sum_{n=1}^{\infty} u(n) f(n) n^{-s} \) is of type \( T(1/2, 1/2; c_0, \delta, M) \), where \( c_0, \delta, M \) are suitable positive constants (see page 185 of [5] for the definition of \( T(1/2, 1/2; c_0, \delta, M) \)). Thus Theorem II.5.3 of [5] is applicable. Now our required result follows immediately from this theorem with \( N = 0 \).

Lemma 4. Let \( \mathcal{L}, f(n), A, C_1 \) be defined as above. Assume that \( 2 \leq \alpha < 3 \). We have

\[
(4.2) \quad \sum_{l \in \mathcal{L}} \frac{f(l)}{l \log^2 (x/l)} = 1 + o(1) \frac{A \alpha}{(\log x)^{3/2}} \int_2^x \frac{t - 2 + (t - 1) \log(t - 1)}{t^2(t - 1)(1 - t/\alpha)^{1/2}} \, dt.
\]

Proof. Let \( Y \) be the sum on the left-hand side of (4.2) and let \( u(n) \) be the function defined as in Lemma 3. We have

\[
Y = \frac{1 + o(1)}{2} \sum_{n \leq x^{1 - 2/\alpha}} \frac{u(n) f(n)}{n} \sum_{x^{1/\alpha} \leq p_2 \leq (x/\Pi)^{1/3}} \frac{1}{p_2 \log^2 (x/np_2)}.
\]

As usual we put \( \pi(t; 4, 3) := \sum_{p \leq t, p \equiv 3 (\mod 4)} 1 \). By the Siegel–Walfisz theorem

\[
\pi(t; 4, 3) = \frac{1}{2} \int_2^t \frac{dv}{\log v} + O(t e^{-\sqrt{\log t}})
\]
and by partial integration, we easily deduce that

\[ Y = \frac{1 + o(1)}{2} \sum_{n \leq x^{1-2/\alpha}} \frac{u(n)f(n)}{n} \int_{x^{1/\alpha}}^{(x/n)^{1/2}} \frac{d\pi(t; 4, 3)}{t\log^2(x/nt)} = \frac{1 + o(1)}{4} \sum_{n \leq x^{1-2/\alpha}} \frac{u(n)f(n)}{n} \int_{x^{1/\alpha}}^{(x/n)^{1/2}} \frac{dt}{\log^2(x/nt) t \log t} = \frac{1 + o(1)}{4 \log^2 x} \sum_{n \leq x^{1-2/\alpha}} \frac{u(n)f(n)}{nh(n)^2} \left( \frac{\alpha h(n) - 2}{\alpha h(n) - 1} + \log \left( \frac{\alpha h(n)}{\alpha h(n) - 1} \right) \right) \]

with \( h(n) := 1 - \log n/\log x \). We define

\[ U(t) := \sum_{n \leq t} u(n)f(n), \quad K(t) := \frac{1}{\log t} \left( \frac{\alpha h(t) - 2}{\alpha h(t) - 1} + \log \left( \frac{\alpha h(t)}{\alpha h(t) - 1} \right) \right). \]

It is not difficult to show that we have, uniformly for \( x \geq 10 \) and \( 1 \leq t \leq x^{1-2/\alpha} \),

\[ K'(t) = -\frac{1}{t^2 \log t} \left( \frac{\alpha h(t) - 2}{\alpha h(t) - 1} + \log \left( \frac{\alpha h(t)}{\alpha h(t) - 1} \right) \right) + O \left( \frac{1}{t^{2 \log x}} \right). \]

Let \( Z \) be the last sum on the right-hand side of (4.3). Noticing that \( U(1-) = K(x^{1-2/\alpha}) = 0 \), using (4.1) and (4.4), we deduce, by partial integration, that

\[ Z = \int_{1-}^{x^{1-2/\alpha}} K(v) dU(v) = -\int_{1-}^{x^{1-2/\alpha}} U(v)K'(v) dv = \frac{A}{C_1} \int_{1}^{2//1} \left( \frac{\alpha h(v) - 2}{\alpha h(v) - 1} + \log \left( \frac{\alpha h(v)}{\alpha h(v) - 1} \right) \right) \frac{dv}{\alpha h(v)^2 (\log v)^{1/2}} + O(1). \]

By the change of variables \( t = \alpha h(v) \), we obtain

\[ Z = \sqrt{\log x} \frac{A\alpha}{C_1} \int_{1}^{\alpha} \frac{t - 2 + (t - 1) \log(t - 1)}{t^2(t - 1)(1 - t/\alpha)^{1/2}} dt + O(1). \]

Inserting this in (4.3) yields (4.2). This completes the proof. \( \square \)

An upper bound for \( T \) is obtained in the following proposition.

**Proposition 2.** Assume that \( \frac{4}{5} < \theta < 1 \) and \( 2 \leq \alpha < \min \{ 3, 2/(5 - 5\theta) \} \). We then have

\[ T \leq \{ W_2(\theta, \alpha) + o(1) \} \frac{x^\theta}{(\log x)^{3/2}}, \]

where

\[ W_2(\theta, \alpha) := \frac{AC_3\alpha}{2\theta - 1} \int_{1/2}^{\alpha} \frac{t - 2 + (t - 1) \log(t - 1)}{t^2(t - 1)(1 - t/\alpha)^{1/2}} dt, \]

and \( A, C_3 \) are defined as in Proposition 1.

**Proof.** For every \( l \in \mathcal{L} \), it is natural to choose, in Lemma 1,

\[ \mathcal{F} = \mathcal{M}(l), \quad \mathcal{P} = \mathcal{P}(l), \quad X = \frac{1}{2} \int_{x/l}^{(x+x^\theta)/l} \frac{dt}{\log t}, \]

\[ w(p) = \begin{cases} p/(p-1) & \text{if } p \in \mathcal{P}(l), \\ 0 & \text{otherwise}. \end{cases} \]
Let $d$ be a squarefree integer whose prime factors belong to $\mathcal{P}(l)$. By the Chinese Remainder Theorem, the system of congruences $(l/2)p_1 \equiv 1 \pmod{4}, lp_1 \equiv -1 \pmod{d}$ has exactly one solution $a^*(\mod 4d)$. Thus we have

$$|\mathcal{M}(l)_d| = \sum_{x/l < p_1 \leq (x+x^*)/l, p_1 \equiv a^* \pmod{4d}} 1,$$

$$r(\mathcal{M}(l), d) = H(x/l, x^\theta/l, 4d, a^*, 1).$$

According to Lemma 1, one has

$$S(\mathcal{M}(l); \mathcal{P}(l), (x/l)^{\theta_1}) \leq XV((x/l)^{\theta_1}) \left\{ F\left(\frac{\log Q}{\theta_1 \log(x/l)}\right) + o(1) \right\} + R(x, x^\theta, Q, l),$$

where $\theta_1 = (2\theta - 1)/4$ and

$$R(x, x^\theta, Q, l) := \sum_{d < Q, d \mid [x/(x/l)^{\theta_1}]^2} 3^{\omega(d)}|r(\mathcal{M}(l), d)|.$$

Using Mertens’ formula, we have

$$V((x/l)^{\theta_1}) = \prod_{p < (x/l)^{\theta_1}, (p, l) = 1} \left(1 - \frac{1}{p - 1}\right) = \{1 + o(1)\} \frac{2C_1C_3e^{-\gamma}f(l)}{x^\theta \log(x/l)},$$

where $f(l), C_1$ are defined as in Lemma 3. Since $\frac{4}{7} < \theta < 1$ and $2 \leq \alpha < 2/(5-5\theta)$, we have $\max_{x \leq l} = x^{1-1/\alpha} \leq x^{(5\theta-3)/2-\varepsilon}$ and $x^\theta/l \geq (x/l)^{3/5+\varepsilon/5}$. Taking $Q = (x/l)^{\theta-1/2}/\log^2(x/l)$ and $g(1) = 1, g(m) = 0 \ (m \geq 2)$, it follows from (2.2) of Lemma 2 that

$$R(x, x^\theta, Q, l) \ll x^\theta/l \log^3(x/l).$$

Combining (4.5)–(4.7), one has the following estimate

$$S(\mathcal{M}(l); \mathcal{P}(l), (x/l)^{\theta_1}) \leq \{1 + o(1)\} \frac{C_1C_3f(l)x^\theta}{\theta_1 \log^2(x/l)}.$$
In order to facilitate the calculation on a computer, we eliminate, by integration by parts, the singularity of two integrands on the right-hand side of (5.2). We have

\[
W(\theta, \alpha) = \sqrt{4\theta - 2} - 4/\alpha + \sqrt{\theta - 1/2} \int_1^{\theta - 1/2} t^{-3/2} dt \\
- 2\alpha^2 \int_2^{\theta - 1/2} \frac{-t^2 + 6t - 4 - 2(t - 1)^2 \log(t - 1)}{(t - 1)^3 t^3} \log(t - 1) \cdot (1 - t/\alpha)^{1/2} dt.
\]

We choose \( \theta = \frac{115}{121} \) and \( \alpha = 2.349 \), which satisfy (5.1). A numerical computation gives us

\[
W\left(\frac{115}{121}, 2.349\right) = 0.314325 \cdots + 0.671128 \cdots \times 0.008853 \cdots - 2 \times 2.349^2 \times 0.028982 \cdots \\
\geq 4 \times 10^{-4}.
\]

This completes the proof.

References


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