EXTENDING THE FORMULA TO CALCULATE
THE SPECTRAL RADIUS OF AN OPERATOR

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Abstract. In a Banach space, Gelfand’s formula is used to find the spectral radius of a continuous linear operator. In this paper, we show another way to find the spectral radius of a bounded linear operator in a complete topological linear space. We also show that Gelfand’s formula holds in a more general setting if we generalize the definition of the norm for a bounded linear operator.

1. Introduction and basic definitions

In all that follows $E$ stands for a linear vector space over the field $\mathbb{C}$ of complex numbers. $E[t]$ will denote a complete locally convex topological vector space, with a Hausdorff topology $t$, and $T : E \to E$ will be a linear map. Finally, $\vartheta(t)$ will be the filter of all balanced, convex and closed $t$-neighborhoods of zero (in $E$).

Definition 1. The linear operator $T : E[t] \to E[t]$ is said to be a bounded operator, if there is a neighborhood $U \in \vartheta(t)$ such that $T(U)$ is a bounded set. If in the definition above $T(U)$ is a relatively compact set, then $T$ is said to be a compact operator. Any compact operator is a bounded operator, and any bounded operator is continuous (with the $t$-topology) (see [5]).

We recall that, given any topological linear space $E[\omega]$ and $S : E[\omega] \to E[\omega]$ a linear operator, the resolvent of $S$ is the set

$$\rho_{\omega}(S) = \left\{ \xi \in \mathbb{C} \mid \xi I - S : E[\omega] \to E[\omega] \text{ is bijective and has a continuous inverse} \right\}. $$

The spectrum of $S$ is defined by $\sigma_{\omega}(S) = \mathbb{C} \setminus \rho_{\omega}(S)$ (the set-theoretic complement in $\mathbb{C}$ of the resolvent set), and the spectral radius by

$$sr_{\omega}(S) = \sup\left\{ |\lambda| \mid \lambda \in \sigma_{\omega}(S) \right\}. $$

Definition 2. A net $\{x_{\alpha}\}_{\alpha \in J} \subset E$ is said to be $t$-ultimately bounded ($t$-ub) if, given any $V \in \vartheta(t)$, there is a positive real number $r$ and an index $\alpha_0 \in J$, both depending on $V$, such that $x_{\alpha} \in rV \forall \alpha \geq \alpha_0$.

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Let us denote by $\Gamma$ the set of all $t$-ub nets in $E$.

**Remark 1.** Any bounded, convergent or Cauchy net is a $t$-ub net. For more details about $t$-ub nets we refer the reader to [1].

**Definition 3.** Let $\xi \in C$. We will say that $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_{\xi,t}} 0$ ($T^n = T \circ T \circ \ldots \circ T$, $n$ times) if, given both $V \in \vartheta(t)$ and $\{x_\alpha\}_{\alpha \in \Gamma}$, there exist $\alpha_0 \in J$ and $n_0 \in N$ such that $\frac{1}{\xi^n}T^n(x_\alpha) \in V$ $\forall \alpha \geq \alpha_0$ and $\forall n \geq n_0$.

**Definition 4.** $\gamma_\xi(T) = \inf \left\{ |\xi| : \frac{1}{\xi^n}T^n \xrightarrow{\Gamma_{\xi,t}} 0 \right\}$.

**Remark 2.** It is shown by Vera [4] that for a bounded operator $T$, we have:

(i) $\gamma_\xi(T) < \infty$, and for any $\xi \in C$ such that $\gamma_\xi(T) < |\xi|$, $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_{\xi,t}} 0$.

(ii) $sr_\xi(T) \leq \gamma_\xi(T)$, where $sr_\xi(T)$ is the spectral radius of $T$.

(iii) When $E[t]$ is a Banach space, $\gamma_\xi(T) = r_\xi(T)$.

In [2] it was proved, based in the above result, that $\gamma_\xi(T) = sr_\xi(T)$ when $T$ is a compact operator. In this paper we extend that result to any bounded operator.

2. Main Results

From now on let $T : E[t] \rightarrow E[t]$ be a bounded operator and let $U \in \vartheta(t)$ be such that $T(U)$ is a bounded set.

Let $P_U$ be the functional of Minkowski (see [3]) generated by $U$, which is a seminorm on $E$. Let $E[P_U]$ denote the vector space $E$ with the topology given by the seminorm $P_U$.

**Remark 3.** The topology on $E$ given by the seminorm $P_U$ is coarser than the topology $t$ ($P_U \leq t$).

**Proposition 1.** $T : E[P_U] \rightarrow E[P_U]$ is a bounded operator (hence a continuous one).

**Proof.** Since $T(U)$ is a bounded set and $P_U \leq t$, $T(U)$ is also a $P_U$-bounded set in $E[P_U]$. \hfill $\square$

**Definition 5.** $\gamma_{P_U}(T) = \inf \left\{ |\xi| : \frac{1}{\xi^n}T^n \xrightarrow{\Gamma_{P_U}} 0 \right\}$.

Here $\Gamma_{P_U}$ convergence means that, given any net $\{x_\alpha\}_{\alpha \in \Gamma} \subset E$ such that for all $\alpha$, $P_U(x_\alpha) \leq r$ for some positive real number $r$ ($P_U$-bounded net), then $P_U(\frac{1}{\xi^n}T^n x_\alpha) \rightarrow 0$ as a net in $R$ whose index set is $N \times J$.

**Proposition 2.** $\gamma_{P_U}(T) = \gamma_\xi(T)$.

**Proof.** Let $\xi \in C$ be such that $\gamma_{P_U}(T) < |\xi|$, and let $V \in \vartheta(t)$ and $\{x_\alpha\}_{\alpha \in \Gamma}$ be given. Since $\frac{1}{\xi^n}T(U)$ is a bounded set, there is a positive real number $r_1$ such that $\frac{1}{r_1^\xi}T(U) \subset V$. In [1] is shown that $\{x_\alpha\}_{\alpha \in \Gamma} \Rightarrow \{r_1 x_\alpha\}_{\alpha \in \Gamma}$. This implies that there exist both $\alpha_0 \in J$ and $r_2 > 0$ such that $r_1 x_\alpha \in r_2 U \forall \alpha \geq \alpha_0$, i.e., $P_U(r_1 x_\alpha) \leq r_2$, that is, the net $\{x_\alpha\}_{\alpha \geq \alpha_0}$ is a $P_U$-bounded net; therefore, $\exists \alpha_1 \in J (\alpha_1 \geq \alpha_0)$ and $n_1 \in N$ such that $P_U(\frac{1}{\xi^n}T^n(x_\alpha)) < 1 \forall \alpha \geq \alpha_1$, $n \geq n_1$, that is, $\frac{1}{\xi^{n+1}}T^{n+1} x_\alpha = \frac{1}{r_1^\xi} T (\frac{1}{\xi^n}T^n r_1 x_\alpha) \in \frac{1}{r_1^\xi} T(U) \subset V$ $\forall \alpha \geq \alpha_1$, $n \geq n_1$. 


that is, \( \frac{1}{\xi} T^n \xrightarrow{n \to \infty} 0 \), and therefore, \( \gamma(T) \leq ||\xi||. \) This implies that \( \gamma_{P_U}(T) \leq \gamma(T). \)

On the other hand, let \( \gamma(T) < ||\xi|| \) and \( \{x_\alpha\}_J \), a \( P_U \)-bounded net; that is, \( x_\alpha \in rU \) for all \( \alpha \) and some \( r > 0 \). Then \( \{\frac{1}{\xi} T x_\alpha\}_J \subset \frac{1}{\xi} T(U) \), where \( \frac{1}{\xi} T(U) \) is a \( t \)-bounded set; therefore, \( \{\frac{1}{\xi} T x_\alpha\}_J \subset \Gamma \). Since \( \frac{1}{\xi} T^n \xrightarrow{n \to \infty} 0 \), given \( \epsilon > 0 \), \( \exists \alpha_0 \in J \) and \( n_0 \in N \) such that \( \frac{1}{\xi} T^n \xi = \frac{1}{\xi} T^n(\frac{1}{\xi} T x_\alpha) \in \epsilon U \) \( \forall \alpha \geq \alpha_0 \), \( n \geq n_0 \); that is, \( P_U(\frac{1}{\xi} T^n+1 x_\alpha) \leq \epsilon \) for those indices. This says that \( \frac{1}{\xi} T^n x_\alpha \) is \( P_U \)-convergent to 0; therefore, \( \gamma_{P_U}(T) \leq ||\xi||. \) This implies that \( \gamma(T) \leq \gamma_{P_U}(T) \).

\[ \text{Definition 6.} \]
\[ L(E) = \left\{ S : E[t] \to E[t] \mid S \text{ is a linear and continuous operator} \right\}, \]
\[ L_U(E) = \left\{ S \in L(E) \mid S(U) \text{ is a bounded set} \right\}, \]
\[ L_U(E) \text{ is a vector subspace of the complex vector space } L(E). \]

\[ \text{Remark 4.} \text{ For the bounded operator } T \text{ that we have been working on we have } T, T^n, \lambda T^n \in L_U(E) \text{ for all } n \in N \text{ and all } \lambda \in C. \]

Moreover, for any \( S \in L(E) \), \( S \circ T, T \circ S \in L_U(E). \)

\[ \text{Definition 7.} \text{ For any operator } S \in L_U(E), \text{ we define, taking into account that } S(U) \text{ is a bounded set, the following real number:} \]
\[ ||S||_U = \sup \{ P_U(S x) \mid x \in U \}. \]

It easy to check that \( ||S^n||_U \leq ||S||_U^n \forall S \in L_U(E) \text{ and } \forall n \in N. \)

\[ \text{Theorem 1.} \text{ If } S_n \xrightarrow{n \to \infty} S \text{ in } L(E), \text{ then } ||S_n \circ T - S \circ T||_U \to 0. \]

\[ \text{Proof.} \text{ Let us prove it by way of contradiction.} \]

Let \( \epsilon > 0 \) be such that there exist natural numbers \( n_1 < n_2 < n_3 < \ldots \) such that \( \epsilon < ||S_{n_1} \circ T - S \circ T||_U \); hence, for each of those \( n_k \) there is \( x_{n_k} \in U \) such that \( P_U(S_{n_k} \circ T - S \circ T x_{n_k}) > \epsilon \). Since \( \{T x_{n_k}\} \subset T(U) \), it is a bounded sequence; hence, for \( V = \epsilon U \in \partial_t(t) \) there is an index \( m_0 \in N \) such that \( (S_{n_k} - S)(T x_{n_k}) \in V \) for all \( n, n_k \geq m_0 \); this implies that \( P_U((S_{n_k} \circ T - S \circ T x_{n_k})) \leq \epsilon \), which yields a contradiction. \( \square \)

\[ \text{Proposition 3.} \rho(t) \subset \rho_{P_U}(T). \]

\[ \text{Proof.} \text{ Let us suppose first that } \gamma(t) < 1. \text{ Let } \xi \in \rho(t) \text{ be such that } ||\xi|| > \gamma(t). \text{ Then } S = \sum_{k=0}^{\infty} \frac{1}{\xi} T^k \text{ is a continuous operator and } S = (\xi I - T)^{-1}. \]

Set \( S_n = \sum_{k=0}^{n} \frac{1}{\xi} T^k \). Then \( S_n \xrightarrow{n \to \infty} S \), and from Theorem 1 it follows that \( ||S_n \circ \frac{1}{\xi} T - S \circ \frac{1}{\xi} T||_U \to 0 \). On the other hand, \( S_n \circ \frac{1}{\xi} T = S_{n+1} - \frac{1}{\xi} I \) and \( S \circ \frac{1}{\xi} T = S - \frac{1}{\xi} I \); hence \( ||S_{n+1} - S||_U \to 0 \). Thereby, given \( \{x_m\}_N \subset E \) such that \( P_U(x_m) \to 0 \), then \( P_U(S x_m) \leq P_U((S - S_n) x_m) + P_U(S_n x_m) \to 0 \). This proves that \( S : E[P_U] \to E[P_U] \) is a continuous operator; hence \( \xi \in \rho_{P_U}(T) \).

Now let \( \xi \in \rho(t) \) be such that \( ||\xi|| \leq \gamma(t) \). Then \( \frac{1}{\xi} > 1 > \gamma(t) \), which means that \( \frac{1}{\xi} I - T : E[P_U] \to E[P_U] \) is a continuous operator. Since \( \xi I - T = \ldots \)
Remark 6. \( N = \left\{ x \in E \mid P_U(x) = 0 \right\} \).

Remark 5. Since \( \left\{ x \in E \mid P_U(x) \leq 1 \right\} \subset U \), \( N \subset U \).

Theorem 2. \( N \) is a closed linear subspace of \( E \), and \( T(x) = 0 \) for all \( x \in N \).

Proof. The first claim follows from the fact that
\[
P_U(\xi x + y) \leq |\xi| P_U(x) + P_U(y).
\]
For the second claim let’s take \( x \in N \); then \( mx \in N \) for \( m = 1, 2, \ldots \). Let \( V \) be any balanced, convex and closed \( t \)-neighborhood of \( 0 \). Since \( \{ mT(x) \}_{m=1,2,3,\ldots} \subset T(N) \subset T(U) \) and the latter set is bounded, there exists \( r \in \mathbb{R}^+ \) such that \( \{ mT(x) \} \subset rV \Rightarrow T(x) \in rV \subset V \) when \( m > r \). Since \( V \) was an arbitrary neighborhood of zero and \( E[\| t \|] \) is Hausdorff, then \( T(x) = 0 \).

Definition 9. Let \( E/N \) be the quotient linear space and let \( \hat{P}_U \) be the norm on it defined by \( \hat{P}_U(x + N) = P_U(x) \) (see [3]).

Remark 6. \( (E/N)[\hat{P}_U] \) will denote the vector space \( E/N \) with the topology given by the norm \( \hat{P}_U \).

Definition 10. Let \( \hat{T} : (E/N) \rightarrow (E/N) \) be defined by \( \hat{T}(x + N) = T(x) + N \).

Remark 7. It is easy to show that \( \hat{T} \) is a well defined linear map.

Proposition 4. \( \hat{T} : (E/N)[\hat{P}_U] \rightarrow (E/N)[\hat{P}_U] \) is a linear and bounded operator (hence \( \hat{T} \) is continuous).

Proof. \( U/N \) is the unit ball in \( (E/N)[\hat{P}_U] \) and \( \hat{T}(U/N) = (T(U) + N)/N \). The latter set is \( \hat{P}_U \)-bounded because the canonical projection \( E[P_U] \rightarrow (E/N)[\hat{P}_U] \) is a continuous map.

Remark 8. Since \( (E/N)[\hat{P}_U] \) is a norm space we can define, as usual, the norm of \( \hat{T} \), and this will be denoted by \( ||\hat{T}||_{\hat{P}_U} \).

Proposition 5. \( \gamma_{\hat{P}_U}(\hat{T}) = \gamma_{P_U}(T) \).

Proof. Set \( |\xi| > \gamma_{P_U}(T) \). Let \( \{ x_n + N \}_J \) be a \( \hat{P}_U \)-bounded net in \( E/N \); then \( \{ x_n \}_J \) is a \( P_U \)-bounded net in \( E \); hence, given \( \epsilon > 0 \), there are indices \( \alpha \in J \) and \( n_0 \in \mathbb{N} \) such that \( \frac{1}{\xi} T^n x = \epsilon U \ \forall \alpha \geq \alpha_0 \) and \( n \geq n_0 \). Thus
\[
\frac{1}{\xi^n} T^n(x_n + N) = \frac{1}{\xi^n} T^n x_n + N \in \epsilon (U/N), \ \alpha \geq \alpha_0, \ n \geq n_0
\]
This implies that \( \gamma_{\hat{P}_U}(T) \leq |\xi| \). Hence \( \gamma_{\hat{P}_U}(T) \leq \gamma_{P_U}(T) \).
Set $|\xi| > \gamma_{P_U}(T)$. Let $\{x_\alpha\}_J$ be a $P_U$-bounded net in $E$. Then $\{x_\alpha + N\}_J$ is a $P_U$-bounded net in $E/N$; hence, given $\epsilon > 0$, there are indices $\alpha_0 \in J$ and $n_0 \in \mathbb{N}$ such that \[ \frac{1}{n_0}T^n(x_\alpha + N) \in \epsilon (U/N) \forall \alpha \geq \alpha_0, n \geq n_0. \] This implies that for those indices $\frac{1}{n_0}T^n x_\alpha = cu_\alpha + z_\alpha$, $u_\alpha \in U$, $z_\alpha \in N$; hence $P_U(\frac{1}{n_0}T^n x_\alpha) \leq P_U(cu_\alpha) + P_U(z_\alpha) \leq \epsilon + 0 = \epsilon$, and thus $|\xi| > \gamma_{P_U}(T)$. This implies that $\gamma_{\tilde{P}_U}(T) \geq \gamma_{P_U}(T)$.

**Proposition 6.** $\rho_{P_U}(T) = \rho_{\tilde{P}_U}(\tilde{T})$.

**Proof.** $\xi \in \rho_{P_U}(T) \Rightarrow \xi I - T: E[P_U] \to E[P_U]$ is bijective and has a continuous inverse.

Let us show that $A: (E/N)[\tilde{P}_U] \to (E/N)[\tilde{P}_U]$ defined by $A(x + N) = (\xi I - T)^{-1}(x) + N$, which is a linear and continuous map, is the inverse function of $\xi I - \tilde{T}$. For this, $A(\xi I - T)(x + N) = A(\xi I - T)(x + N) = (\xi I - T)^{-1}(\xi I - T)(x) + N = x + N$. In a similar way it can be proved that $(\xi I - \tilde{T}) \circ A = I$. This implies that $\xi \in \rho_{\tilde{P}_U}(\tilde{T})$.

It is just routine to prove the set contention in the other way around.

**Definition 11.** $(E/N)[\tilde{P}_U]$ will denote the completion (as a normed space) of $(E/N)[P_U]$, and $\tilde{T}$ will denote the natural extension of $T$.

**Remark 9.** $(E/N)[\tilde{P}_U]$ is a Banach space. Besides, since $\tilde{T}$ is a bounded operator, $\tilde{T}: (E/N)[\tilde{P}_U] \to (E/N)[\tilde{P}_U]$ is a bounded operator (see [3]).

**Remark 10.** Since $(E/N)[\tilde{P}_U]$ is a Banach space we can define, as usual, the norm of $\tilde{T}$; this will be denoted by $||\tilde{T}||_{\tilde{P}_U}$.

**Proposition 7.** $\gamma_{\tilde{P}_U}(\tilde{T}) = \gamma_{P_U}(T)$.

**Proof.** Since $\tilde{T}$ is an extension of $T$, the proof follows immediately from the definitions of $\gamma_{\tilde{P}_U}(\tilde{T})$ and $\gamma_{P_U}(T)$.

**Proposition 8.** $\rho_{\tilde{P}_U}(\tilde{T}) = \rho_{P_U}(T)$.

**Proof.** If $\xi \in \rho(\tilde{T})$, then $\xi I - \tilde{T}: (E/N)[\tilde{P}_U] \to (E/N)[\tilde{P}_U]$ is bijective and has a continuous inverse, so that both $\xi I - \tilde{T}$ and $(\xi I - T)^{-1}$ have a continuous extension to $(E/N)$, which are precisely $\xi I - \tilde{T}$ and $(\xi I - T)^{-1}$ respectively. This implies that $\xi \in \rho(T)$.

On the other hand, if $\xi \in \rho(T)$, then $\xi I - T: (E/N)[P_U] \to (E/N)[P_U]$ is bijective and has a continuous inverse; hence the restrictions of those functions to $(E/N)[P_U]$ are precisely $\xi I - T$ and its inverse function, which are continuous functions for being the restrictions of continuous ones. Then $\xi \in \rho(T)$.

**Theorem 3.** $\gamma_{\tilde{t}}(T) = sr_{\tilde{t}}(T)$.

**Proof.** By Remark 2 (ii) it suffices to show that $sr_{\tilde{t}}(T) \geq \gamma_{\tilde{t}}(T)$. Also, from Remark 2 (iii) we get

$$sr_{\tilde{P}_U}(\tilde{T}) = \gamma_{\tilde{P}_U}(\tilde{T})$$
because \( \widetilde{E/N}[\widetilde{P}_U] \) is a Banach space. From Propositions 2, 5 and 7 we obtain

\[ \gamma_t(T) = \gamma_{\widetilde{P}_U}(\widetilde{T}) \]

(2)

From Propositions 3, 6 and 8 we obtain

\[ \rho_t(T) \subset \rho_{\widetilde{P}_U}(\widetilde{T}) ; \]

this implies that

\[ sr_{\widetilde{P}_U}(\widetilde{T}) \leq sr_t(T) . \]

(3)

From (1), (2) and (3) we finally get

\[ \gamma_t(T) \leq sr_t(T) . \]

\[ \square \]

3. A generalization of Gelfand’s formula

In this part we prove that Gelfand’s formula (see [3]) applies for a bounded operator defined on a topological vector space. Following the notation from the sections above, we will show that we can use \( ||T||_U \) in Gelfand’s formula to calculate the spectral radius of \( T \).

**Proposition 9.** For any \( T \in L_U(E) \), \( ||T||_U = ||\widetilde{T}||_{\widetilde{P}_U} \).

**Proof.** Set \( r > ||T||_U \). Then \( T(U) \subset rU \); hence \( P_U(Tx) \leq r \) for all \( x \in U \). This implies that \( ||T||_{\widetilde{P}_U} \leq r \), and therefore \( ||T||_{\widetilde{P}_U} \leq ||T||_U \).

Set \( r < ||T||_U \). Then there exists \( x \in U \) such that

\[ r < P_U(Tx) = \widetilde{P}_U(\widetilde{T}(x + N)) \leq ||\widetilde{T}||_{\widetilde{P}_U} . \]

This implies that \( ||\widetilde{T}||_{\widetilde{P}_U} = ||T||_U . \)

\[ \square \]

**Corollary 1.** \( ||\widetilde{T}||_{\widetilde{P}_U} = ||\widetilde{T}||_{\widetilde{P}_U} = ||T||_U \).

**Theorem 4.** \( sr_t(T) = \lim_{n \to \infty} \|T^n\|_U^{\frac{1}{n}} \) for any \( T \in L_U(E) \).

**Proof.** We recall first that \( T \in L_U(E) \Rightarrow T^n \in L_U(E) \). From (1), (2), and Theorem 3 we obtain

\[ sr_t(T) = sr_{\widetilde{P}_U}(\widetilde{T}) . \]

(4)

Because \( \widetilde{E/N}[\widetilde{P}_U] \) is a Banach space, Gelfand’s formula holds:

\[ sr_{\widetilde{P}_U}(\widetilde{T}) = \lim_{n \to \infty} \|\widetilde{T^n}\|_{\widetilde{P}_U}^{\frac{1}{n}} . \]

(5)

Finally, using (4) and (5) and the corollary above, we obtain

\[ sr_t(T) = \lim_{n \to \infty} \|T^n\|_U^{\frac{1}{n}} . \]

\[ \square \]
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