PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 1, January 1998, Pages 273–278
S 0002-9939(98)04488-8

EIGENVALUES OF SOME DISTAL FUNCTIONS

JIRŌ EGAWA

(Communicated by James West)

Dedicated to Professor Junji Kato on his sixtieth birthday

Abstract. In this paper we construct distal functions of another type discussed by Salehi (1991). Let $a(t)$ be an almost periodic function with the mean value 0, which has unbounded integral, and Φ a continuous periodic function with the prime period 1. If Φ satisfies some additional condition, then $f(t) = \Phi(\int_0^t a(s)\,ds)$ is a distal function, which is not almost periodic, and the set of eigenvalues of f is the module of a.

1. Preliminaries

Let X be a metric space with the metric d_X. Z, R and C denote the set of integers, the set of real numbers and the set of complex numbers, respectively. A continuous mapping $\pi : X \times R \to X$ is said to be *a flow on* (*a phase space*) X if π satisfies the following two conditions:

(1) $\pi(x, 0) = x$ for $x \in X$.

(2) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for $x \in X$ and $t, s \in R$.

For $A \subset X$ and $B \subset R$ we denote the set $\{\pi(x, t); x \in A, t \in B\}$ by $\pi(A, B)$. The closure of $A \subset X$ is denoted by $\overline{A}$. The orbit of π through $x \in X$ is denoted by $O_\pi(x)$, that is, $O_\pi(x) = \pi(x, R)$. $M \subset X$ is called *an invariant set of* π if $O_\pi(x) \subset M$ for each $x \in M$. The restriction of π to an invariant set M of π is denoted by $\pi|M$. A non-empty compact invariant set of π is called *a minimal set* if $\overline{O_\pi(x)} = M$ for each $x \in M$. If X is itself a minimal set of π, we say that π is *a minimal flow on* X. We say that π is *equicontinuous* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_X(\pi(x, t), \pi(y, t)) < \varepsilon$ for $d_X(x, y) < \delta$ and $t \in R$. π is said to be *distal* if $\inf_{t\in R}\{d_X(\pi(x, t), \pi(y, t))\} > 0$ for $x, y \in X$ $(x \neq y)$.

Let π be a minimal flow on a compact metric space X. $\lambda \in R$ is called an *eigenvalue of* π if there exists a continuous mapping χ_λ from X to $K = \{\xi \in C; |\xi| = 1\}$ such that $\chi_\lambda(\pi(x, t)) = \exp(2\pi i\lambda t)\chi_\lambda(x)$ for $(x, t) \in X \times R$. In this case χ_λ is said to be *an eigenfunction belonging to* λ. The set of eigenvalues of π is denoted by $\Lambda(\pi)$. We can easily see that $\Lambda(\pi)$ is a countable subgroup of the additive group R.

Received by the editors November 28, 1995.

1991 *Mathematics Subject Classification.* Primary 54H20.

Key words and phrases. Equicontinuous, distal, minimal flow, almost periodic function, eigenvalues.

$C(R,C)$ denotes the set of continuous functions from R to C with the compact-open topology. Then $C(R,C)$ is a complete metric space. Define a flow η on $C(R,C)$ by $\eta(f,t)=f_t$ for $(f,t)\in C(R,C)\times R$, where $f_t(s)=f(t+s)$ for $s\in R$. For $f\in C(R,C)$ we denote the hull of f by $H(f)$, that is, $H(f)=\overline{O_\eta(f)}=\overline{\{f_t\}}_{t\in R}$. The restriction of η to $H(f)$ $(f\in C(R,C))$ is denoted by η_f. $L\subset R$ is said to be *relatively dense* if there exists an $l>0$ such that $[l-t,l+t]\cap L\neq\varnothing$ for each $t\in R$. $f\in C(R,C)$ is said to be *an almost periodic function* if for each $\varepsilon>0$ there exists a relatively dense set $A_\varepsilon\subset R$ such that $|f(t+\tau)-f(t)|<\varepsilon$ for $t\in R$ and $\tau\in A_\varepsilon$. It is well known that, if f is an almost periodic function, then $H(f)$ is compact and η_f is an equicontinuous minimal flow on $H(f)$. $f\in C(R,C)$ is said to be *a distal function* if $H(f)$ is compact, and if η_f is a distal flow on $H(f)$. It is known ([4], p. 37) that, if f is a distal function, then η_f is a minimal flow on $H(f)$.

Let π and ρ be flows on X and Y, respectively. A continuous mapping h from X to Y is said to be *a homomorphism from* π *to* ρ if $h(\pi(x,t))=\rho(h(x),t)$ for $(x,t)\in X\times R$. Furthermore, if h is a homeomorphism from X onto Y, then h is called *an isomorphism from* π *to* ρ. In this case, we say that π and ρ are isomorphic.

Proposition 1.1. *Let* π *and* ρ *be minimal flows on compact metric spaces* X *and* Y*, respectively. If* π *and* ρ *are isomorphic, then* $\Lambda(\pi)=\Lambda(\rho)$.

Proof. Easy.

2. SOME DISTAL FLOWS

In this section π is an equicontinuous minimal flow on a compact metric space X, ϕ is a continuous mapping from $X\times R$ to R satisfying the condition

$$(*)\qquad \phi(x,t+s)=\phi(x,t)+\phi(\pi(x,t),s)\quad\text{for } x\in X \text{ and } t,s\in R,$$

and ρ is a flow on $X\times K$ defined by

$$\rho((x,\xi),t)=(\pi(x,t),\xi\exp(2\pi i\phi(x,t)))\quad\text{for } (x,\xi)\in X\times K \text{ and } t\in R.$$

Proposition 2.1. *If* ϕ *is not uniformly continuous, then* ρ *is a distal minimal flow on* $X\times K$*, which is not equicontinuous, such that* $\Lambda(\rho)=\Lambda(\pi)$.

Proof. See [3], Theorems 4.3 and 4.5.

Remark 2.2. (1) $\lim_{t\to\infty}\frac{1}{t}\phi(x,t)$ exists for $x\in X$ and this limit is constant.

(2) Assume that $\lim_{t\to\infty}\frac{1}{t}\phi(x,t)=0$ for some $x\in X$. Then ϕ is not uniformly continuous if and only if $\phi(x,t)$ is unbounded on R for each $x\in X$.

Proof. (1) See [1], Lemma 2.

(2) We assume that ϕ is not uniformly continuous on $X\times R$. Assume there exists $x_0\in X$ such that $|\phi(x_0,t)|\leqq M$ for $t\in R$. Put $\overline{\phi}(x,t)=\frac{1}{4M}\phi(x,t)$ for $(x,t)\in X\times R$. Then we have $|\overline{\phi}(x_0,t)|\leqq\frac{1}{4}$, $\overline{\phi}$ satisfies the condition $(*)$, and it is not uniformly continuous. Define a flow $\overline{\rho}$ on $X\times K$ by $\overline{\rho}((x,\xi),t)=(\pi(x,t),\xi\exp(2\pi i\overline{\phi}(x,t)))$ for $(x,\xi)\in X\times K$ and $t\in R$. By Proposition 2.1 $\overline{\rho}$ must be a minimal flow on $X\times K$. But, since $|\overline{\phi}(x_0,t)|\leqq\frac{1}{4}$, $\overline{O_{\overline{\rho}}(x_0,1)}\neq X\times K$. This is a contradiction. Conversely, if ϕ is uniformly continuous on $X\times R$, there exists a continuous function $\Psi: X\to R$ such that

$$\phi(x,t)=-\Psi(\pi(x,t))+\Psi(x)\quad\text{for } (x,t)\in X\times R$$

([1]). Hence $\phi(x,t)$ is bounded on R for each $x\in X$.

Remark 2.3. Let $C(X)$ be a set of real-valued continuous functions on X with the topology of uniform convergence, and let

$$C_0(X) = \{b \in C(X); \lim_{t\to\infty} \frac{1}{t}\int_0^t b(\pi(x,s))\,ds = 0\}.$$

For $b \in C_0(X)$, put $\phi_b(x,t) = \int_0^t b(\pi(x,s))\,ds$ for $(x,t) \in X \times R$. Then ϕ_b satisfies $(*)$ and there exists $b \in C_0(X)$ such that ϕ_b is not bounded on R ([5], p. 362).

Let Φ be a periodic continuous function on R with the prime period 1. For $\tau \in R$, put $\Phi_\tau(t) = \Phi(\tau + t)$ for $t \in R$, and $S = \{\Phi_\tau; \tau \in R\}$ with the topology of uniform convergence. Then we can easily see that S is homeomorphic to K. Define a flow $\tilde\rho$ on $X \times S$ by $\tilde\rho((x,\Phi_\tau),t) = (\pi(x,t), \Phi_{\tau+\phi(x,t)})$ for $(x,\Phi_\tau) \in X \times S$ and $t \in R$, where ϕ is as in Proposition 2.1.

Proposition 2.4. *If ϕ is not uniformly continuous on $X \times R$, then $\tilde\rho$ is a distal minimal flow on $X \times S$ which is not equicontinuous, and $\Lambda(\tilde\rho) = \Lambda(\pi)$.*

Proof. Let h be the mapping from $X \times S$ to $X \times K$ defined by

$$h(x,\Phi_\tau) = (x, \exp(2\pi i\tau)).$$

Then h is a homeomorphism and we have

$$\begin{aligned} h(\tilde\rho(x,\Phi_\tau),t) &= h(\pi(x,t), \Phi_{\tau+\phi(x,t)}) \\ &= (\pi(x,t), \exp(2\pi i\tau)\exp(2\pi i\phi(x,t))) \\ &= \rho(h(x,\Phi_\tau),t) \end{aligned}$$

for $(x,\Phi_\tau) \in X \times S$ and $t \in R$. Hence ρ and $\tilde\rho$ are isomorphic. Since notions of minimality, distality and equicontinuity are invariant under isomorphisms, it is a distal minimal flow on $X \times S$ which is not equicontinuous. Furthermore, we have $\Lambda(\tilde\rho) = \Lambda(\pi)$ by Proposition 1.1.

3. Some distal functions

Proposition 3.1. *Let π be an equicontinuous minimal flow on a compact metric space X. Then*

(1) *If $\pi(x_0,t_n) \to x_0$ as $n \to \infty$ for some sequence $\{t_n\} \subset R$ and $x_0 \in X$, then we have $\pi(x,t_n) \to x$ as $n \to \infty$ for each $x \in X$.*

(2) *If $\pi(x_0,t_n) \to x_0$ and $\pi(x_0,s_n) \to x_0$ as $n \to \infty$ for some sequences $\{t_n\}, \{s_n\} \subset R$ and $x_0 \in X$, then $\pi(x_0,t_n+s_n) \to x_0$ as $n \to \infty$.*

Proof. Easy.

Proposition 3.2. *Let π be a distal minimal flow on a compact metric space X.*

(1) *For each $(x,y) \in X \times X$, there exists a sequence $\{t_n\} \subset R$ ($t_n \to \infty$ as $n \to \infty$) such that $(\pi(x,t_n),\pi(y,t_n)) \to (x,y)$ as $n \to \infty$.*

(2) *If F is a continuous function on X, then, for each $x \in X$, $f^x(t) = F(\pi(x,t))$ ($t \in R$) is a distal function, and $H(f^x) = \{F(\pi(y,\cdot)); y \in X\}$.*

Proof. (1) Since the product flow $\pi \times \pi$ on $X \times X$ defined by $\pi \times \pi((x,y),t) = (\pi(x,t),\pi(y,t))$ for $(x,y) \in X \times X$ and $t \in R$ is distal ([4], p. 37), it follows that $\overline{O_{\pi\times\pi}((x,y))}$ is a minimal set of $\pi \times \pi$ ([4], p. 37). Hence (1) holds.

(2) Let h be a mapping from X into $C(R,C)$ defined by $h(x) = F(\pi(x,\cdot))$. Then h is a homomorphism from π to η. Hence, for each $x \in X$, $f^x(t) = F(\pi(x,t))$ for $t \in R$ is a distal function ([4], p. 37). The second assertion is obvious.

Proposition 3.3. *Let f be an almost periodic function on R. Then $\Lambda(\eta_f) = \widetilde{\Lambda}_f$, where*

$$\Lambda_f = \left\{ \lambda \in R; \lim_{t\to\infty} \int_0^t f(s)\exp(-2\pi i\lambda s)\,ds \neq 0 \right\}$$

and $\widetilde{\Lambda}_f$ is the least additive subgroup of R containing Λ_f.

Proof. See [2].

Lemma 3.4. *Let π be an equicontinuous minimal flow on a compact metric space X, let $\phi : X \times R \to R$ satisfy $(*)$ which is not uniformly continuous on $X \times R$, and let Φ be a continuous periodic function on R with the prime period 1. If $\Phi_\tau(\phi(x,t)) = \Phi_{\tau'}(\phi(x,t))$ for $x, y \in X$, $\tau, \tau' \in [0,1)$ and $t \in R$, then we have $\Phi_\tau(n\phi(x,t)) = \Phi_{\tau'}(n\phi(y,t))$ for each natural number n and $t \in R$.*

Proof. Let $t \in R$. Since a flow ρ on $X \times K$ defined by

$$\rho((x,\xi),t) = (\pi(x,t), \xi\exp(2\pi i\phi(x,t)))$$

for $(x,\xi) \in X \times K$ and $t \in R$ is a distal minimal flow on $X \times K$, there exists a sequence $\{t_n\} \subset R$ $(t_n \to \infty$ as $n \to \infty)$ such that $\pi(\pi(x,t),t_n) \to \pi(x,t), \pi(\pi(y,t),t_n) \to \pi(y,t), \phi(\pi(x,t),t_n) \to \phi(x,t) \pmod 1$ and $\phi(\pi(y,t),t_n) \to \phi(y,t) \pmod 1$ as $n \to \infty$ by Proposition 3.2. We shall show the lemma by mathematical induction. Since $\Phi_\tau(\phi(x,t) + \phi(\pi(x,t),t_n)) = \Phi_\tau(\phi(x,t+t_n)) = \Phi_{\tau'}(\phi(y,t+t_n)) = \Phi_{\tau'}(\phi(y,t) + \phi(\pi(y,t),t_n))$ for each n, we have $\Phi_\tau(2\phi(x,t)) = \Phi_{\tau'}(2\phi(y,t))$. Assume that

$$\phi(x, t + t_k^1 + t_k^2 + \cdots + t_k^{m-1}) \to m\phi(x,t) \pmod 1$$

and

$$\phi(y, t + t_k^1 + t_k^2 + \cdots + t_k^{m-1}) \to m\phi(y,t) \pmod 1$$

as $k \to \infty$ for $m \geqq 2$, where $\{t_k^i\}$ are subsequences of $\{t_n\}$ for $1 \leqq i \leqq m-1$. Then $\Phi_\tau(m\phi(x,t)) = \Phi_{\tau'}(m\phi(y,t))$. Since $\pi(\pi(x,t), t_k^1 + t_k^2 + \cdots + t_k^{m-1}) \to \pi(x,t)$ and $\pi(\pi(y,t), t_k^1 + t_k^2 + \cdots + t_k^{m-1}) \to \pi(y,t)$ as $n \to \infty$ by Proposition 3.1, we can choose a subsequence $\{k_n\}$ of natural numbers so that

$$\phi(\pi(\pi(x,t), t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}), t_n)) \to \phi(x,t) \pmod 1$$

and

$$\phi(\pi(\pi(y,t), t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}), t_n)) \to \phi(y,t) \pmod 1$$

as $n \to \infty$. For this sequence $\{k_n\}$ we have

$$\begin{aligned}
&\Phi_\tau(\phi(x, t + t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}) + \phi(\pi(\pi(x,t), t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}), t_n)) \\
&\quad = \Phi_\tau(\phi(x, t + t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1} + t_n)) \\
&\quad = \Phi_{\tau'}(\phi(y, t + t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1} + t_n)) \\
&\quad = \Phi_{\tau'}(\phi(y, t + t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}) + \phi(\pi(\pi(y,t), t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}), t_n)).
\end{aligned}$$

Hence we have $\Phi_\tau((m+1)\phi(x,t)) = \Phi_{\tau'}((m+1)\phi(y,t))$. Since $t \in R$ is arbitrary, we obtain the relation $\Phi_\tau(n\phi(x,t)) = \Phi_{\tau'}(n\phi(y,t))$ for each natural number n and $t \in R$.

Lemma 3.5. *Let* $\alpha, \beta \in R$. *If* α *is an irrational number, and if* $\beta \neq \frac{k}{m} - \frac{l}{m}\alpha$ $(k, l, m \in Z, m \neq 0)$, *then, for each* $(x, y) \in [0, 1) \times [0, 1)$, *there exists a sequence* $\{n_i\} \subset Z$ $(n_i > 0)$ *such that* $n_i\alpha \to x \,(\mathrm{mod}\, 1)$ *and* $n_i\beta \to y \,(\mathrm{mod}\, 1)$ *as* $i \to \infty$.

Proof. Consider a flow π on $T^2 = K \times K$ defined by

$$\pi((\xi_1, \xi_2), t) = (\xi_1 e^{2\pi i\alpha t}, \xi_2 e^{2\pi i\beta t})$$

for $(\xi_1, \xi_2) \in T^2$ and $t \in R$. Then, since α and β are rationally independent by the assumption, π is a minimal flow on T^2. We can easily verify that $\Lambda(\pi) = \{l\alpha + m\beta; l, m \in Z\}$ and that $1 \notin \{\frac{r}{k}; k \in Z - \{0\}, r \in \Lambda(\pi)\}$. Hence the discrete flow π_1 defined by $\pi((\xi_1, \xi_2), 1)$ for $(\xi_1, \xi_2) \in T^2$ is minimal ([2]). Hence the orbit of π_1 is dense in T^2. The lemma follows.

Theorem. *Let* π *be an equicontinuous minimal flow on a compact metric space* X, ϕ *a continuous function from* $X \times R$ *to* R, *and* Φ *a continuous periodic function with the prime period* 1. *If* ϕ *and* Φ *satisfy the following conditions*:

(1) $\phi(x, t + s) = \phi(x, t) + \phi(\pi(x, t), s)$ *for* $x \in X$ *and* $t, s \in R$,
(2) $\lim_{t\to\infty} \frac{\phi(x,t)}{t} = 0$,
(3) ϕ *is not uniformly continuous on* $X \times R$ *and* $\phi(x, \cdot) \neq \phi(y, \cdot)$ *for each pair of distinct points* $x, y \in X$,
(4) *for each* $\tau, \tau' \in [0, 1)$, *there exists* $t_{\tau\tau'} \in [0, 1)$ *such that* $\Phi_\tau(t_{\tau\tau'}) \neq \Phi_{\tau'}(-t_{\tau\tau'})$,

then $f(t) = \Phi(\phi(x_0, t))$ *for* $x_0 \in X$ *and* $t \in R$ *is a distal function, which is not almost periodic, and we have* $\Lambda(\eta_f) = \Lambda(\pi)$.

Proof. It is enough to show η_f and $\tilde{\rho}$ in Proposition 2.4 are isomorphic. Define $h : X \times S \to R$ by $h((x, \Phi_\tau)) = \Phi_\tau(0) = \Phi(\tau)$ for $(x, \Phi_\tau) \in X \times S$. Then h is continuous, $h(\tilde{\rho}((x, \Phi_\tau), t)) = \Phi(\tau + \phi(x, t))$ for $(x, \Phi_\tau) \in X \times S$ and $t \in R$, and h is a homomorphism from $\tilde{\rho}$ to η_f. Assume that $h((x, \Phi_\tau)) = h((y, \Phi_{\tau'}))$, that is, $\Phi_\tau(\phi(x, t)) = \Phi_{\tau'}(\phi(y, t))$ for $(x, \Phi_\tau), (y, \Phi_{\tau'}) \in X \times S$ and $t \in R$. If $x \neq y$, then $\phi(x, \cdot) \not\equiv \phi(y, \cdot)$ by the assumption. Put $\delta(t) = \phi(x, t) - \phi(y, t)$ for $t \in R$. Then, since $\delta(0) = 0$ and $\delta(t) \not\equiv 0$, there exists t_1 such that $\delta(t_1)$ is irrational. Put $\alpha = \phi(x, t_1)$ and $\beta = \phi(y, t_1)$. We can assume without loss of generality that α is irrational. If $\beta = -\alpha + \frac{l}{k}$ $(k, l \in Z, k \neq 0)$, then $\Phi_\tau(nk\alpha) = \Phi_{\tau'}(-nk\alpha + nl) = \Phi_{\tau'}(-nk\alpha)$ for $n \in Z$ $(n \geqq 1)$ by Lemma 3.4. Since $k\alpha$ is irrational, there exists a sequence $\{n_p\} \subset Z$ $(n_p \geqq 1)$ such that $n_pk\alpha \to t \,(\mathrm{mod}\, 1)$ as $p \to \infty$ for each $t \in [0, 1)$. Hence we have $\Phi_\tau(t) \equiv \Phi_{\tau'}(-t)$. This is a contradiction to the assumption on Φ. Hence β is not of the form $-\alpha + \frac{l}{k}$. If $\beta = \frac{k}{m} - \frac{l}{m}\alpha$ $(k, m, l \in Z, m \neq 0, |m| \neq |l|)$, then we have $\Phi_\tau(nm\alpha) = \Phi_{\tau'}(kn - nl\alpha) = \Phi_{\tau'}(-nl\alpha)$ for $n \in Z$ $(n \geqq 1)$ by Lemma 3.4. Hence we have $\Phi_\tau(mt) = \Phi_{\tau'}(-lt)$. If $|m| < |l|$, we obtain $\Phi_\tau(t) = \Phi_{\tau'}(-\frac{l}{m}t)$ for $t \in R$, and $\Phi_\tau(t + \frac{m}{l}) = \Phi_{\tau'}(-\frac{l}{m}(t + \frac{m}{l})) = \Phi_{\tau'}(-\frac{l}{m}t - 1) = \Phi_{\tau'}(-\frac{l}{m}t) = \Phi_\tau(t)$. This implies Φ_τ has a period $|\frac{m}{l}| < 1$. This is a contradiction. If $\beta \neq \frac{k}{m} - \frac{l}{m}\alpha$ $(k, l, m \in Z, m \neq 0)$, then, by Lemma 3.5, for each $t \in [0, 1)$, there exists a sequence $\{n_p\} \subset Z$ $(n_p \geqq 1)$ such that $n_p\alpha \to t \,(\mathrm{mod}\, 1)$ and $n_p\beta \to 0 \,(\mathrm{mod}\, 1)$ as $p \to \infty$. Hence we have $\Phi_\tau(t) \equiv \Phi_{\tau'}(0)$. This is a contradiction. The above implies that $\phi(x, t) = \phi(y, t)$ for $t \in R$. Since $\phi(x, t)$ is unbounded, we have $\Phi_\tau = \Phi_{\tau'}$. These imply $x = y$ and $\tau = \tau' \,(\mathrm{mod}\, 1)$. Hence h is an injection, that is, h is an isomorphism from $\tilde{\rho}$ to η_f. It follows that f is a distal function, which is not almost periodic, and $\Lambda(\eta_f) = \Lambda(\pi)$.

Remark. A periodic function $\Phi(t) = \sin(2\pi t) + \sin(4\pi t)$ (for $t \in R$) satisfies the conditions in the Theorem.

Corollary. *Let $a(t)$ be an almost periodic function such that*

(1) $\lim_{t\to\infty} \frac{1}{t} \int_0^t a(s)\,ds = 0$,

(2) $\int_0^t a(s)\,ds$ *is unbounded with respect to* t,

and let Φ be a continuous periodic function satisfying the conditions in the Theorem. Then $f(t) = \Phi(\int_0^t a(s)\,ds)$ is a distal function, which is not almost periodic, and $\Lambda(\eta_f) = \widetilde{\Lambda}_a$.

Proof. In the Theorem, put $\pi = \eta_a$ and $\phi(a^*, t) = \int_0^t a^*(s)\,ds$ for $a^* \in H(f)$ and $t \in R$. For the second assertion, use Proposition 3.3.

References

1. J. Egawa, *Reparametrization and equicontinuous flows*, Proc. Japan Acad. **54** (1978), 202–205. MR **80a:**54075
2. ———, *Eigenvalues of compact minimal flows*, Math. Seminar Notes (Kobe University) **10** (1982), 281–291. MR **84f:**54054
3. ———, *Eigenvalues of harmonizable minimal flows*, Mathematica Japonica **31** (1986), 351–367. MR **87i:**54086
4. R. Ellis, *Lectures on topological dynamics*, W. A. Benjamin, 1969. MR **42:**2463
5. R. A. Johnson, *Almost periodic functions with unbounded integral*, Pacific J. of Math. **87** (1980), 347–362. MR **82e:**42013
6. E. Salehi, *Distal functions and unique ergodicity*, Trans. Amer. Math. Soc. **323** (1991), 703–713. MR **91e:**43009

DIVISION OF MATHEMATICS AND INFORMATICS, FACULTY OF HUMAN DEVELOPMENT, KOBE UNIVERSITY, TURUKABUTO 3-11, NADA, KOBE 657, JAPAN

E-mail address: `egawa@main.h.kobe-u.ac.jp`