A WEIGHTED POINCARÉ INEQUALITY WITH A DOUBLING WEIGHT

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Abstract. We show that unbounded John domains (and even a larger class of domains than John domains) satisfy the weighted Poincaré inequality

\[ \inf_{a \in \mathbb{R}} \| u(x) - a \|_{L^q(D, w_1)} \leq C \| \nabla u \|_{L^p(D, w_2)} \]

whenever \( u \) is a Lipschitz function on \( D \), \( w_1 \) is a doubling weight, and weights satisfy certain cube conditions, and \( C = C(D, p, q, w_1, w_2) \).

1. Introduction

In this note we generalize results considering weighted Poincaré inequalities. My work was stimulated by a paper of Chua [C]. If \( D \) is a bounded John domain and if there exists a constant \( C_1 < \infty \) such that the inequality

\[ |Q|^{\frac{1}{n}} \left( \int_Q w_1(x) \, dx \right)^{1/q} \left( \int_Q w_2(x)^{-1/p-1} \, dx \right)^{p-1/p} \leq C_1 \]

holds for all Whitney cubes of \( D \), then

\[ \| u(x) - u_{D, w_1} \|_{L^q(D, w_1)} \leq C_2 \| \nabla u \|_{L^p(D, w_2)} \]

whenever \( w_1 \) is a doubling weight and \( 1 < p < q < \infty \). If \( p = q \), instead of (1.1) we need to require that

\[ |Q|^{\frac{1}{n}} \left( \frac{1}{|Q|} \int_Q w_1^r(x) \, dx \right)^{1/pr} \left( \frac{1}{|Q|} \int_Q w_2^{-r/p-1} \, dx \right)^{p-1/pr} \leq C_1 \]

holds for any \( r > 1 \). These results are implicitly in Chua’s paper.

We study the following generalized inequality:

\[ \inf_{a \in \mathbb{R}} \| u(x) - a \|_{L^q(D, w_1)} \leq C \| \nabla u \|_{L^p(D, w_2)} , \]

where \( u \) is a Lipschitz function and \( 1 < p \leq q < \infty \). If \( D \) satisfies (1.2), we write \( D \in P(q, p) \) with \( w_1 \) and \( w_2 \) and \( C = C_{q, p}(D, w_1, w_2) \). We show that unbounded John domains \( D \) satisfy (1.2) whenever (1.1) holds for \( 1 < p < q < \infty \) (respectively (1.1*) for \( p = q \)) and \( w_1 \) is a doubling weight, Theorem 1.8. We also show that there is a larger class than John domains which satisfy (1.1) for \( 1 < p < q < \infty \) (respectively (1.1*) for \( p = q \)) under above conditions, Theorem 1.3. In this manner
a weighted result for so-called rooms and corridors domains is obtained, Example 4.1.

Our main theorems are Theorems 1.3 and 1.8.

1.3 Theorem. Let a domain $G$ be the union of domains $D_i \in \mathcal{P}(q,p)$ with a doubling weight $w_1$ and a weight $w_2$ such that

$$K_{q,p}(D_i, w_1, w_2) \leq C_0 < \infty, \quad i = 1, 2, \ldots.$$  \hfill (1.4)

Suppose that each domain $D_i$ lies in a cube $Q_i$ with the following three properties.

There are constants $C_i$, $i = 1, 2, 3$, such that

$$\sum_{j=1}^{\infty} \chi_{Q_j}(x) \leq C_1 \chi_{\bigcup_{j=1}^{\infty} Q_j}(x)$$  \hfill (1.5)

for all $x \in \mathbb{R}^n$,

$$Q_i \subset C_2 Q_j,$$  \hfill (1.6)

where $j = 1, 2, \ldots, i$, and

$$K_{q,p}(D_i, w_1, w_2)^q \leq C_3 \min\{ w_1(D_i \cap D_i - 1), w_1(D_i \cap D_i + 1) \}.$$  \hfill (1.7)

Then $G$ is a $(q,p)$-Poincaré domain with $w_1$ and $w_2$.

1.8 Theorem. Let $w_1$ be a doubling weight. Suppose that $D$ is an unbounded John domain. Suppose that there exists a constant $C < \infty$ such that the inequality (1.1) for $1 < p < q < \infty$ (respectively (1.1*) for $p = q$) holds for all cubes $Q \subset D$. Then $D$ is a weighted $(q,p)$-Poincaré domain with $w_1$ and $w_2$.

The proofs for Theorem 1.3 and Theorem 1.8 are given in §3. An example and corollaries in §4 reveal that we have generalized results of Chao et al. [CWZ], Evans and Harris [EH], and Hurri [H].

2. Preliminaries

Notation. Throughout this paper we let $D$ be a domain of euclidean $n$-space $\mathbb{R}^n$, $n \geq 2$. We suppose that $1 \leq p \leq q < \infty$ unless otherwise stated.

The diameter of a set $A$ is written as $\text{dia}(A)$. We write $tQ$ for the cube with the same center as $Q$ and dilated by a factor $t > 1$.

We let $C(\ast, \cdots, \ast)$ denote a constant which depends only on the quantities appearing in the parentheses.

A weight (function) is a nonnegative measurable function on $\mathbb{R}^n$. A weight $w$ is a doubling weight (that is, $w$ satisfies a doubling condition) if there exists a constant $t < \infty$ such that

$$\int_{2Q} w(x) \, dx \leq t \int_Q w(x) \, dx \text{ for all cubes } Q \subset \mathbb{R}^n.$$  \hfill (A_p)

$A_p$-weights are doubling weights.

The average of a function $u$ over a domain $D$ with finite Lebesgue measure $|D|$ is

$$u_D = \frac{1}{|D|} \int_D u(x) \, dx,$$
and with an arbitrary weight $w$

$$u_{D,w} = \frac{1}{\int_D w(x) \, dx} \int_D u(x) w(x) \, dx$$

whenever

$$0 < \int_D w(x) \, dx < \infty.$$  

We write

$$\|u\|_{L^p(D,w)} = \left( \int_D |u(x)|^p w(x) \, dx \right)^{1/p},$$

where $u$ is a Lipschitz function on $D$. The distributional gradient is written as

$$\nabla u = (\partial_1 u, \ldots, \partial_n u).$$

We recall a lemma due to Strömberg and Wheeden.

2.1 Lemma ([StW, Lemma 2.3], [C, Lemma 2.5]). Let $\{Q_\alpha\}_{\alpha \in I}$ be an arbitrary family of cubes in $\mathbb{R}^n$. If $\{a_\alpha\}_{\alpha \in I}$ is a family of nonnegative real numbers and $w$ is a doubling weight, then for $1 \leq p < \infty$ and $N \geq 1$ we have

$$\left\| \sum_\alpha a_\alpha \chi_{NQ_\alpha} \right\|_{L^p(\mathbb{R}^n,w)} \leq C(n, p, N, w) \left\| \sum_\alpha a_\alpha \chi_{Q_\alpha} \right\|_{L^p(\mathbb{R}^n,w)}.$$

The Hölder inequality and the Minkowski inequality yield the following useful lemma.

2.2 Lemma. Let $D$ be a domain and $A \subset D$ be a set such that $\int_A w(x) \, dx < \infty$. Then for each $a \in \mathbb{R}$

$$\|u - u_{A,w}\|_{L^p(D,w)} \leq 2 \left( \frac{\int_D w(x) \, dx}{\int_A w(x) \, dx} \right)^{1/p} \|u - a\|_{L^p(D,w)}$$

where $u$ is a Lipschitz function on $D$.

$(q,p)$-Poincaré domains with weights $w_1$ and $w_2$. Let $D \subset \mathbb{R}^n$ be a domain and let $1 \leq p \leq q < \infty$. Let $w_1$ and $w_2$ be weight functions. If there exists a constant $K = K(D, p, q, w_1, w_2) < \infty$ such that the inequality

$$\inf_{a \in \mathbb{R}} \|u - a\|_{L^q(D,w_1)} \leq K \|\nabla u\|_{L^p(D,w_2)}$$

holds for all Lipschitz functions $u$, then $D$ is a $(q,p)$-Poincaré domain with weights $w_1$ and $w_2$. We write $D \in \mathcal{P}(q,p)$ with $w_1$ and $w_2$.

John domains. Let $E$ be a closed arc with endpoints $a$ and $b$. The subarc between $x$ and $y$ is denoted by $E[x,y]$. For $x$ in $E \setminus \{a,b\}$ write

$$q(x) = \min\{\text{dia}(E[a,x]), \text{dia}(E[b,x])\}.$$  

Let $c \geq 1$. A domain $D$ in $\mathbb{R}^n$ is a $c$-John domain, if each pair of distinct points $a$ and $b$ in $D$ can be joined by an arc $E$ such that

$$c_{\text{ig}} E(a,b) = \bigcup \left\{ B \left( x, \frac{q(x)}{c} \right) \mid x \in E \setminus \{a,b\} \right\} \subset D.$$  

Balls, convex domains, domains with smooth boundaries are John domains as well as a snowflake domain.
Bojarski proved that a bounded $b$-John domain satisfies the $(q,p)$-Poincaré inequality with $w_1 = w_2 = 1$ [B, Chapter 6]. Unbounded John domains are $(\frac{mp}{n-p},p)$-Poincaré domains with $w_1 = w_2 = 1$ [H-S, Corollary 4.6].

We recall the following lemma due to Väisälä.

**2.4 Lemma** ([V, Theorem 4.6]). Let $D$ be an unbounded $b$-John domain. Then there are $b_0$-John domains $D_i$ such that $D_i \subset D_i \subset D_{i+1}$, $i = 1,2,\ldots$, and $D = \bigcup_{i=1}^{\infty} D_i$.

Sawyer and Wheeden have given several sufficient and necessary conditions for weights $w_1$ and $w_2$ so that a cube is a weighted $(q,p)$-Poincaré domain. We write down here one of their theorems which was refined by Chua for a doubling weight $w_1$.

**2.5 Lemma** ([C, Theorem 2.14]). Suppose that $w_1$ is a doubling weight. Then for all Lipschitz functions $u$ and a cube $Q_0$

$$\|u - u_{Q_0,w_1}\|_{L^q(Q_0,w_1)} \leq K_{q,p} \|\nabla u\|_{L^p(Q_0,w_2)}$$

where

$$K_{p,q} = K_{q,p}(w_1) \sup_{Q \subset Q_0} |Q|^{\frac{1}{q}} \left( \int_{Q} w_1(x) \, dx \right)^{1/q} \left( \int_{Q} w_2(x)^{-1/p-1} \, dx \right)^{p-1/p}$$

whenever $p < q$.

If $p = q$, then

$$\|u - u_{Q_0,w_1}\|_{L^p(Q_0,w_1)} \leq K_p \|\nabla u\|_{L^p(Q_0,w_2)}$$

with

$$K_p = K_p(r,w_1) \sup_{Q \subset Q_0} |Q|^{1/n} \left( \int_{Q} w_1(x)^r \, dx \right)^{1/pr} \left( \int_{Q} w_2(x)^{-r/(p-1)} \, dx \right)^{(p-1)/pr}$$

for any $r > 1$.

**Proof.** Let $f = u - u_{Q_0,w_1}$ in [C, Theorem 2.14].

Chua considered Boman’s chain condition domains [C]. An especial case of his theorem [C, Theorem 1.5] is the following lemma.

**2.6 Lemma.** Let $D$ be a bounded $b_0$-John domain and let $w_1$ be a doubling weight. Suppose that there exists a constant $C < \infty$ such that for all cubes $Q$ in $D$ (1.1)

holds whenever $1 < p < q < \infty$ (respectively, (1.1*) holds for $p = q$).

Then

$$\|u - u_{D,w_1}\|_{L^q(D,w_1)} \leq K_{p,q}(b_0,C,w_1) \|\nabla u\|_{L^p(D,w_2)} .$$

**Proof.** Lemma 2.5 and [C, Theorem 1.5].

*Proof for Theorem 1.3.* By the given decomposition of $\mathcal{G}$

$$\int_{\mathcal{G}} |u(y) - u_{D_1,w_1}|^q w_1(y) \, dy \leq \sum_{i=1}^{\infty} \int_{D_i} |u(y) - u_{D_1,w_1}|^q w_1(y) \, dy$$

$$\leq 2^{q-1} \left( \sum_{i=1}^{\infty} \int_{D_i} |u(y) - u_{D_1,w_1}|^q w_1(y) \, dy + \sum_{i=1}^{\infty} \int_{D_i} |u_{D_1,w_1} - u_{D_1,w_1}|^q w_1(y) \, dy \right).$$
Since $D_i \in P(q,p)$ with weights $w_1$ and $w_2$ and $K_{q,p}(D_i, w_1, w_2) \leq C_0$, we obtain

$$\sum_{i=1}^{\infty} \int_{D_i} |u(y) - u_{D_i, w_1}|^q w_1(y) \, dy$$

\[
\leq \sum_{i=1}^{\infty} \left( K_{q,p}(D_i, w_1, w_2) \left( \int_{D_i} |\nabla u(y)|^p w_2(y) \, dy \right)^{1/p} \right)^q
\leq C_0^q \left( \int_{D_i} |\nabla u(y)|^p w_2(y) \, dy \right)^{q/p}.
\]

To estimate the second sum we use the triangle inequality and the weighted $(q,p)$-Poincaré inequality in $D_i$. First,

$$|u_{D_j, w_1} - u_{D_{j+1}, w_1}|^q
= \frac{1}{w_1(D_j \cap D_{j+1})} \int_{D_j \cap D_{j+1}} |u_{D_j, w_1} - u_{D_{j+1}, w_1}|^q w_1(x) \, dx
\leq \frac{2^{q-1}}{w_1(D_j \cap D_{j+1})} \sum_{h=j}^{j+1} K_{q,p}^2(D_h, w_1, w_2) \left( \int_D |\nabla u(x)|^p w_2(x) \, dx \right)^{q/p}.$$  

Hence, the triangle inequality, the condition (1.7), and the engulfing property,

$$D_i \subset Q_i \subset C_2 Q_j, \quad j = 1, 2, \ldots, i,$$

yield that

$$\sum_{i=1}^{\infty} \int_{D_i} |u_{D_i, w_1} - u_{D_{i+1}, w_1}|^q w_1(x) \, dx
\leq \sum_{i=1}^{\infty} \int_{D_i} \left( \sum_{j=1}^{i-1} |u_{D_j, w_1} - u_{D_{j+1}, w_1}| \chi_{D_i}(x) \right)^q w_1(x) \, dx
\leq C_4 \int_{R^n} \left( \sum_{j=1}^{\infty} \frac{1}{w_1(Q_j)^{1/q}} \left( \int_{D_j} |\nabla u(y)|^p w_2(y) \, dy \chi_{C_2 Q_j}(x) \right)^{1/p} \right)^q w_1(x) \, dx.$$  

Lemma 2.1 implies

$$\sum_{i=1}^{\infty} \int_{D_i} |u_{D_i, w_1} - u_{D_{i+1}, w_1}|^q w_1(x) \, dx
\leq C_5 \int_{R^n} \left( \sum_{j=1}^{\infty} \frac{1}{w_1(Q_j)^{1/q}} \left( \int_{D_j} |\nabla u(y)|^p w_2(y) \, dy \chi_{Q_j}(x) \right)^{1/p} \right)^q w_1(x) \, dx
\leq C_6 \sum_{j=1}^{\infty} \frac{1}{w_1(Q_j)} \left( \int_{D_j} |\nabla u(y)|^p w_2(y) \, dy \right)^{q/p} \int_{R^n} \chi_{Q_j}(x) w_1(x) \, dx
\leq C_7 \left( \int_{\mathcal{G}} |\nabla u(y)|^p w_2(y) \, dy \right)^{q/p}.$$  

□
Proof for Theorem 1.8. Since \( D \) is an unbounded John domain, there are \( b_0 \)-John domains \( D_1 \subset D \subset D_{i+1} \) such that \( D = \bigcup_{i=1}^{\infty} D_i \) by Lemma 2.4. We set

\[
    u_i = \frac{1}{\int_{D_i} w_1(x) \, dx} \int_{D_i} u(x) w_1(x) \, dx.
\]

We will use \( D_1 \) to obtain for \( |u_i| \) an upper bound which does not depend on \( i \). The triangle inequality yields

\[
    |u_i| = \left( \int_{D_1} w_1(x) \, dx \right)^{-1} \int_{D_1} |u_i| w_1(x) \, dx
    \leq \left( \int_{D_1} w_1(x) \, dx \right)^{-1} \left( \int_{D_1} |u(x) - u_i| w_1(x) \, dx + \int_{D_1} |u(x)| w_1(x) \, dx \right)
\]

where we may assume that

\[
    0 < \int_{D_1} w(x) \, dx < \infty \quad \text{and} \quad \int_{D_1} |u(x)| w_1(x) \, dx < \infty.
\]

By Lemma 2.6

\[
    \int_{D_1} |u(x) - u_i| w_1(x) \, dx \leq \left( \int_{D_1} w_1(x) \, dx \right)^{1 - \frac{1}{q}} \|u - u_i\|_{L^q(D_i, w_1)}
    \leq \left( \int_{D_1} w_1(x) \, dx \right)^{1 - \frac{1}{q}} \|u - u_i\|_{L^q(D_i, w_1)}
    \leq \left( \int_{D_1} w_1(x) \, dx \right)^{1 - \frac{1}{q}} K_{q,p}(b_0, C, w_1) \|\nabla u\|_{L^p(D,w_2)}.
\]

Thus \( (u_i) \) is a bounded sequence and hence there exists a convergent subsequence \( (u_{i_j}) \) and \( b \in R \) such that \( \lim_{j \to \infty} u_{i_j} = b \).

Since

\[
    \lim_{j \to \infty} \chi_{D_j}(x)|u(x) - u_j|^q = \chi_D(x)|u(x) - b|^q,
\]

Fatou’s lemma and Lemma 2.6 yield that

\[
    \int_D |u(x) - b|^q w_1(x) \, dx = \int_D \lim_{j \to \infty} \chi_{D_j}(x)|u(x) - u_j|^q w_1(x) \, dx
    \leq \lim_{j \to \infty} \int_D \chi_{D_j}(x)|u(x) - u_j|^q w_1(x) \, dx
    \leq \lim_{j \to \infty} \left( K_{q,p}(b_0, C, w_1) \int_{D_j} |\nabla u(x)|^p w_2(x) \, dx \right)^{1/p} q
    \leq \lim_{j \to \infty} \left( K_{q,p}(b_0, C, w_1) \int_D |\nabla u(x)|^p w_2(x) \, dx \right)^{q/p}
    = \left( K_{q,p}(D, w_1) \int_D |\nabla u(x)|^p w_2(x) \, dx \right)^{q/p}.
\]

\[ \square \]
4. Further remarks

Example 4.1 considers the rooms and corridors domain.

4.1 Example. Let $G = \bigcup_{i=1}^{\infty} D_i$ be a domain where the sets $D_i$, $i = 1, 2, \ldots$, are defined as follows: Let $(h_i)$ and $(\delta_{2i})$ be sequences such that $h_i = M^{-i}, M > 1$, and $\delta_{2i} = bM^{-2ai}, b > 0, a > 1$. We set $\eta_{2i} = M^{-2(i+1)}, i = 1, 2, \ldots,$ and $\sum_{i=1}^{k} h_i = d_k$. We define

$$D_{2i-1} = (d_{2i-1} - h_{2i-1}, d_{2i-2}) \cdot (-\frac{1}{2}h_{2i-1}, \frac{1}{2}h_{2i+1})^{n-1},$$

$$D_{2i} = (d_{2i-1} - \frac{1}{2}\eta_{2i} d_{2i-1} + h_{2i} + \frac{1}{2}\eta_{2i}) \cdot (-\frac{1}{2}\delta_{2i}, \frac{1}{2}\delta_{2i})^{n-1}.$$  

By [H] $G$ is an ordinary Poincaré domain, if and only if $p \geq (n-1)(a-1)$. Adjoin the cubes $Q_i$ to the sets $D_i$ as follows: $Q_{2i-1} = D_{2i-1}$ and $Q_{2i} = (d_{2i-1} - \frac{1}{2}\eta_{2i}, d_{2i-1} + h_{2i} + \frac{1}{2}\eta_{2i}) \cdot (-\frac{1}{2}(h_{2i} + \eta_{2i}) , \frac{1}{2}(h_{2i} + \eta_{2i}))^{n-1}.$

We choose $w_1(x) = 1$ and $w_2(x) = d(x, \partial G)^{\alpha p}$. To check (1.7) we need the fact that the weighted Poincaré constant in this case is $|D|^{\frac{1}{q} - \frac{1}{p} - \frac{1}{p} c_n}$ for a c-John domain by [H-S2]. Using this we obtain that weighted Poincaré inequality (1.3) holds whenever

$$n(2 - a - \frac{q}{p} + p(1-a)) + q(1-\alpha) - 1 + a \geq 0.$$ 

This generalizes the results of [EH] and [H] to the weighted case.

The proof for Theorem 1.8 has the following interesting corollaries.

4.2 Corollary. Let $w_1$ be a doubling weight. Suppose that $D$ is an unbounded John domain. Suppose that there exists a constant $C < \infty$ such that for all cubes $Q \subset D$ the inequality (1.1) holds whenever $1 < p < q < \infty$ (respectively, (1.1*) for $p = q$). Then there is a constant $K_{q,p}(w_1) < \infty$ such that the inequality

$$\|u\|_{L^p(D, w_1)} \leq K_{q,p}(D, w_1)\|\nabla u\|_{L^p(D, w_2)}$$

holds for all Lipschitz functions $u$.

Proof. Note that

$$u_j = \frac{1}{\int_{D_j} w_1(x) \, dx} \int_{D_j} u(x)w_1(x) \, dx \to 0$$

when we assume that $\int_D u(x)w_1(x) \, dx < \infty$. This follows from the proof for Theorem 1.8.

An especial case of Corollary 4.2 is the following result.

4.3 Corollary. If $D$ is an unbounded John domain then there exists a constant $C = C(n, p) < \infty$ such that

$$\|u\|_{L^{np/(n-p)}(D)} \leq C\|\nabla u\|_{L^p(D)}$$

holds for all Lipschitz functions $u$. 


Corollary 4.3 generalizes the result of Chen et al. [CWZ] who showed that (4.4) holds for domains with a cone condition.

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