

UNIQUE CONTINUATION ON THE BOUNDARY FOR DINI DOMAINS

IGOR KUKAVICA AND KAJ NYSTRÖM

(Communicated by Christopher D. Sogge)

ABSTRACT. We show that the normal derivative of a harmonic function which vanishes on an open subset of the boundary of a Dini domain cannot vanish on a subset of positive surface measure.

1. INTRODUCTION

In [L], the following unique continuation question was raised: If u is a harmonic function in a Lipschitz domain Ω which vanishes on an open subset Γ of the boundary $\partial\Omega$, and if u is not identically zero, does it follow that the surface measure of the set $\{x \in \Gamma : \nabla u = 0\}$ is zero? In [L], it was shown that the answer is affirmative if Ω is a $C^{1,1}$ domain. In [AEK], it was proven that the answer is affirmative also for convex domains. Further, in [AE], it was proven that the fact holds for Dini domains, and thus, in particular, for $C^{1,\alpha}$ domains for all $\alpha \in (0, 1]$. The approach in [AE] consists of a local change of variables around points on the boundary, which transforms the harmonic operator to a more general elliptic one, while the boundary becomes (in a certain sense) locally convex with respect to this operator.

The purpose of the present paper is to present a short and elementary proof of the Adolfsson-Escauriaza result. By [AEK], it is sufficient to prove that u satisfies a uniform doubling type condition on Γ . This is done by studying the logarithmic convexity of averages of u^2 over the balls centered at points $x_0 \in \Gamma$, rather than spherical shell averages as in [L], [AEK], and [AE], and take into account the fact that a Dini domain is locally star-shaped with respect to suitably chosen points close to x_0 .

2. THE MAIN RESULTS

In this section, we recall the definition of a Dini domain and state the main results.

Definition 2.1. An open connected domain $\Omega \subseteq \mathbb{R}^d$ is a Dini domain if for each point $x_0 \in \partial\Omega$, there exists a local coordinate system $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, $R_0 > 0$, and a function $\phi: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ such that

- (i) $B_{R_0}(x_0) \cap \Omega = \{(x', x_d) \in B_{R_0}(x_0) : x_d < \phi(x')\}$,
- (ii) $B_{R_0}(x_0) \cap \partial\Omega = \{(x', x_d) \in B_{R_0}(x_0) : x_d = \phi(x')\}$,

Received by the editors May 13, 1996 and, in revised form, July 30, 1996.

1991 *Mathematics Subject Classification.* Primary 31B05.

(iii) $|\nabla\phi(x'_1) - \nabla\phi(x'_2)| \leq \psi(|x'_1 - x'_2|)$ for all $(x'_1, \phi(x'_1)), (x'_2, \phi(x'_2)) \in B_{R_0}(x_0)$ where ψ satisfies

$$(2.1) \quad \int_0^1 \frac{\psi(r)}{r} dr < \infty.$$

Theorem 2.2. *Let Ω be a Dini domain, and let Γ be an open subset of the boundary $\partial\Omega$. If $u \in C(\Gamma \cup \Omega)$ is harmonic in Ω , and if u vanishes on Γ , then, for every $x_0 \in \Gamma$,*

$$\sup_{x_0 \in \Gamma} \sup_{r \in (0,1)} \frac{\int_{B_{2r}(x_0) \cap \Omega} u^2 dx}{\int_{B_r(x_0) \cap \Omega} u^2 dx} < \infty.$$

The following corollary of Theorem 2.2 is our main result.

Theorem 2.3. *Let Ω be a Dini domain, and let Γ be an open subset of the boundary $\partial\Omega$. If $u \in C(\Gamma \cup \Omega)$ is harmonic in Ω , and if u vanishes on Γ , then the surface measure of $\{x \in \Gamma : |\nabla u(x)| = 0\}$ is zero.*

Proof of Theorem 2.3. It is proven in [AEK] (or cf. [L]) that Theorem 2.2 implies Theorem 2.3. □

Theorem 2.2 is proven in the next section.

3. PROOF OF THEOREM 2.2

Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain, and let u be harmonic in Ω .

For any $x_0 \in \overline{\Omega}$, denote

$$H_{x_0}(r) = \int_{B_r(x_0) \cap \Omega} u^2 dx.$$

Lemma 3.1. *Let $B_{R_0}(x_0) \cap \Omega$ be star-shaped with respect to some $x_0 \in \overline{\Omega}$, and assume that u vanishes continuously on $B_{R_0}(x_0) \cap \partial\Omega$. If $0 < r_1 < r_2 < r_3 < R_0$, then*

$$\frac{\log \frac{H_{x_0}(r_2)}{H_{x_0}(r_1)}}{\log \frac{r_2}{r_1}} \leq \frac{\log \frac{H_{x_0}(r_3)}{H_{x_0}(r_2)}}{\log \frac{r_3}{r_2}}.$$

The interior version of the above lemma was obtained in [K].

Proof of Lemma 3.1. Assume $x_0 = 0$; also, denote $H(r) = H_0(r)$ and $B_r = B_r(0)$. Fix an arbitrary $r \in (0, R_0)$. Since $B_{R_0} \cap \Omega$ is star-shaped with respect to 0, and since u vanishes on $B_R \cap \partial\Omega$,

$$(3.1) \quad \begin{aligned} H'(r) &= \int_{(\partial B_r) \cap \Omega} u^2 d\sigma = \int_{\partial(B_r \cap \Omega)} u^2 d\sigma = \frac{1}{r} \int_{B_r \cap \Omega} \operatorname{div}(u^2 x) dx \\ &= \frac{d}{r} H(r) + \frac{2}{r} \int_{B_r \cap \Omega} u x \cdot \nabla u dx. \end{aligned}$$

Denote

$$\begin{aligned} I(r) &= 2 \int_{B_r \cap \Omega} u x \cdot \nabla u dx = - \int_{B_r \cap \Omega} u \nabla u \cdot \nabla(r^2 - |x|^2) dx \\ &= \int_{B_r \cap \Omega} |\nabla u|^2 (r^2 - |x|^2) dx, \end{aligned}$$

where we again used that u vanishes on $B_{R_0} \cap \partial\Omega$ and that u is harmonic. Differentiating the last expression, we get

$$I'(r) = 2r \int_{B_r \cap \Omega} |\nabla u|^2 dx + \int_{(\partial B_r) \cap \Omega} |\nabla u|^2 (r^2 - |x|^2) d\sigma = 2r \int_{B_r \cap \Omega} |\nabla u|^2 dx.$$

Note that, using integration by parts,

$$\begin{aligned} & \frac{1}{r} \int_{B_r \cap \Omega} x \cdot \nabla (|\nabla u|^2) (r^2 - |x|^2) dx \\ &= -\frac{d}{r} I(r) + \frac{2}{r} \int_{B_r \cap \Omega} |x|^2 |\nabla u|^2 dx + \frac{1}{r} \int_{B_r \cap \partial\Omega} |\nabla u|^2 (r^2 - |x|^2) x \cdot n d\sigma \\ &= -\frac{d+2}{r} I(r) + \frac{2}{r} \int_{B_r \cap \Omega} |\nabla u|^2 (r^2 - |x|^2) dx \\ & \quad + \frac{2}{r} \int_{B_r \cap \Omega} |x|^2 |\nabla u|^2 dx + \frac{1}{r} \int_{B_r \cap \partial\Omega} |\nabla u|^2 (r^2 - |x|^2) x \cdot n d\sigma \\ &= -\frac{d+2}{r} I(r) + 2r \int_{B_r \cap \Omega} |\nabla u|^2 dx + \frac{1}{r} \int_{B_r \cap \partial\Omega} |\nabla u|^2 (r^2 - |x|^2) x \cdot n d\sigma, \end{aligned}$$

where n denotes the outward unit normal on $\partial\Omega$. Therefore,

$$\begin{aligned} I'(r) &= \frac{d+2}{r} I(r) + \frac{2}{r} \int_{B_r \cap \Omega} x_k (\partial_k \partial_j u) (\partial_j u) (r^2 - |x|^2) dx \\ & \quad - \frac{1}{r} \int_{B_r \cap \partial\Omega} |\nabla u|^2 (r^2 - |x|^2) x \cdot n d\sigma, \end{aligned}$$

whence, integrating by parts (with respect to j) the second expression on the right-hand side,

$$\begin{aligned} I'(r) &= \frac{d+2}{r} I(r) - \frac{2}{r} \int_{B_r \cap \Omega} |\nabla u|^2 (r^2 - |x|^2) dx + \frac{4}{r} \int_{B_r \cap \Omega} (x \cdot \nabla u)^2 dx \\ & \quad + \frac{2}{r} \int_{B_r \cap \partial\Omega} (x \cdot \nabla u) (n \cdot \nabla u) (r^2 - |x|^2) d\sigma \\ & \quad - \frac{1}{r} \int_{B_r \cap \partial\Omega} |\nabla u|^2 (r^2 - |x|^2) x \cdot n d\sigma. \end{aligned}$$

The sum of the last two terms is nonnegative since on $B_R \cap \partial\Omega$ the tangential derivative of u vanishes and $x \cdot n \geq 0$ almost everywhere on $B_r \cap \partial\Omega$. We get

$$I'(r) \geq \frac{d}{r} I(r) + \frac{4}{r} \int_{B_r \cap \Omega} (x \cdot \nabla u)^2 dx.$$

Denoting by

$$N(r) = \frac{I(r)}{H(r)}$$

the frequency function, we obtain by differentiation

$$\begin{aligned} N'(r) &= \frac{1}{H(r)^2} (I'(r)H(r) - I(r)H'(r)) \\ &\geq \frac{4}{rH(r)^2} \left(\int_{B_r \cap \Omega} (x \cdot \nabla u)^2 dx \int_{B_r \cap \Omega} u^2 dx - \left(\int_{B_r \cap \Omega} u x \cdot \nabla u dx \right)^2 \right) \geq 0, \end{aligned}$$

and we conclude that $N(r)$ is an increasing function of $r \in (0, R_0)$. Now, (3.1) implies

$$(3.2) \quad \log \frac{H(r_2)}{H(r_1)} = d \log \frac{r_2}{r_1} + \int_{r_1}^{r_2} \frac{N(r)}{r} dr \leq (d + N(r_2)) \log \frac{r_2}{r_1}.$$

Similarly,

$$(3.3) \quad \log \frac{H(r_3)}{H(r_2)} = d \log \frac{r_3}{r_2} + \int_{r_2}^{r_3} \frac{N(r)}{r} dr \geq (d + N(r_2)) \log \frac{r_3}{r_2}.$$

The asserted inequality is then obtained by combining (3.2) and (3.3). □

Consider the following setting. Let $\phi: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be continuously differentiable. Denote

$$\Omega = B_2 \cap \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : \phi(x') < x_d\},$$

where $B_2 = B_2(0)$. For every $r \in (0, 2)$, let

$$\Lambda(r) = \sup |n(x_2) - n(x_1)|$$

where the supremum is taken over all $x_1, x_2 \in B_2 \cap \partial\Omega$ such that $|x_2 - x_1| \leq r$, and where $n(x)$ denotes the outward unit normal at $x \in B_2 \cap \partial\Omega$. Note that Λ is non-decreasing. Assume also $\phi(0) = 0$ (i.e., $0 \in \partial\Omega$).

Throughout this section, we assume that $u \in C(\overline{\Omega})$ is harmonic in Ω and that it vanishes on $B_2 \cap \partial\Omega$. Theorem 2.2 will be an immediate consequence of the following lemma.

Lemma 3.2. *Assume*

$$(3.4) \quad \int_0^2 \frac{\Lambda(r)}{r} dr < \infty,$$

$\Lambda(r) \neq 0$ for $r \in (0, 2)$, and $\Lambda(2) \leq 1/32$. Then

$$(3.5) \quad C_1 = \prod_{m=1}^{\infty} \frac{\log \frac{2 + 16\Lambda(2^{-m})}{1 - 16\Lambda(2^{-m})}}{\log \frac{4 - 16\Lambda(2^{-m})}{2 + 16\Lambda(2^{-m})}} < \infty,$$

and for all $x_0 \in B_1 \cap \partial\Omega$, we have

$$H_{x_0}(2r) \leq C_2 H_{x_0}(r), \quad r \in \left(0, \frac{1}{2}\right),$$

where $C_2 = (H_{x_0}(1)/H_{x_0}(1/2))^{2C_1}$.

Proof of Lemma 3.2. Note that the function

$$f(x) = \frac{\log \frac{2+x}{1-x}}{\log \frac{4-x}{2+x}}$$

is increasing and continuously differentiable on $(-2, 1)$. Since also $f(0) = 1$, we have

$$1 \leq f(x) \leq 1 + e^{Cx}, \quad x \in [0, 1/2],$$

for some constant $C > 0$. Hence, the product in (3.5) converges provided

$$\sum_{m=1}^{\infty} \Lambda(2^{-m}) < \infty.$$

However, the last fact follows directly from (3.4) taking into account that Λ is non-decreasing.

Without loss of generality, we may assume $x_0 = 0$ and $n(0) = (0, 1)$.

Fix any $r \in (0, 1)$. Let $(x'_1, \phi(x'_1)), (x'_2, \phi(x'_2)) \in B_1$ be arbitrary points. It is easy to check that $|\nabla \phi(x'_j)| \leq 3\Lambda(r)/2$ for $j = 1, 2$; the mean value theorem implies

$$|\phi(x'_2) - \phi(x'_1)| \leq \frac{3}{2}\Lambda(r)|x'_2 - x'_1|.$$

Denoting $a = 4\Lambda(r)r$ and $y_0 = (0, -a)$, we claim that $B_{r-a}(y_0) \cap \Omega$ is star-shaped with respect to x_0 . Indeed, if $(x'_1, \phi(x'_1)), (x'_2, \phi(x'_2)) \in B_1$ are arbitrary points for which $x'_1 \neq x'_2$ and $x'_2 \neq 0$, then

$$\frac{|\phi(x'_2) + a|}{|x'_2|} \geq \frac{a - |\phi(x'_2)|}{|x'_2|} \geq \frac{4\Lambda(r)r}{|x'_2|} - \frac{3}{2}\Lambda(r) \geq \frac{5}{3}\Lambda(r) > \frac{3}{2}\Lambda(r) \geq \frac{|\phi(x'_2) - \phi(x'_1)|}{|x'_2 - x'_1|},$$

and thus the points $(0, -a)$, $(x'_1, \phi(x'_1))$, and $(x'_2, \phi(x'_2))$ are not collinear.

Next, we claim

$$(3.7) \quad \log \frac{H(r/2)}{H(r/4)} \leq \frac{\log \frac{2 + 16\Lambda(r)}{1 - 16\Lambda(r)}}{\log \frac{4 - 16\Lambda(r)}{2 + 16\Lambda(r)}} \log \frac{H(r)}{H(r/2)}.$$

For this purpose, let $r_1 = r/4 - a$, $r_2 = r/2 + a$, and $r_3 = r - a$. Note that, since $\Lambda(r) < 1/16$, we have $0 < r_1 < r_2 < r_3 < 0$. Lemma 3.1 then implies

$$\log \frac{H_{y_0}(r_2)}{H_{y_0}(r_1)} \leq \frac{\log \frac{r/2 + a}{r/4 - a}}{\log \frac{r - a}{r/2 + a}} \log \frac{H_{y_0}(r_3)}{H_{y_0}(r_2)}.$$

Now, noting that $B_{r_1}(y_0) \subseteq B_{r/4}$, $B_{r_2}(y_0) \supseteq B_{r/2}$, and $B_{r_3}(y_0) \subseteq B_r$, we get (3.7).

By using (3.7) for $r = 2^{-m}$ with $m = 0, 1, \dots$ and then iterating the obtained inequalities, we get

$$H(2^{-m}) \leq \left(\frac{H(1)}{H(1/2)} \right)^{C_1} H(2^{-m-1}), \quad m = 1, 2, \dots,$$

and Lemma 3.2 follows. □

Proof of Theorem 2.2. Note that, in the setting of Lemma 3.2,

$$\sup_{x_0 \in B_1 \cap \partial\Omega} \frac{H_{x_0}(1)}{H_{x_0}(1/2)} < \infty,$$

and Theorem 2.2 follows from Lemma 3.2. \square

ACKNOWLEDGMENTS

The work was partially supported by the NSF Grant DMS-9623161, by the MZT Slovenia Grant J1-7037-0101-95, and by a postdoctoral fellowship from the Swedish Research Council for Natural Sciences.

REFERENCES

- [A] F. J. Almgren, Jr., *Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents*, Minimal Submanifolds and Geodesics, North-Holland, Amsterdam, M. Obata, 1979, pp. 1–6. MR **82g**:49038
- [AE] V. Adolfsson and L. Escauriaza, *$C^{1,\alpha}$ domains and unique continuation at the boundary*, 1996.
- [AEK] V. Adolfsson, L. Escauriaza, and C. Kenig, *Convex domains and unique continuation at the boundary*, Revista Matemática Iberoamericana **11** (1995), 513–525. MR **96j**:31003
- [K] I. Kukavica, *Level sets for the stationary Ginzburg-Landau equation*, 1996, to appear in Calc. Var. PDE.
- [L] F.-H. Lin, *Nodal sets of solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math. **44** (1991), 287–308. MR **92b**:58224

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

E-mail address: kukavica@cs.uchicago.edu

Current address: Department of Mathematics, University of Southern California, Los Angeles, California 90089

E-mail address: kaj@math.uchicago.edu