

ACTIONS OF COMPACT QUANTUM GROUPS ON C^* -ALGEBRAS

MARCIN MARCINIAK

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper we investigate a structure of the fixed point algebra under an action of compact matrix quantum group on a C^* -algebra \mathcal{B} . We also show that the categories of \mathcal{C} -comodules in \mathcal{B} and inner endomorphisms restricted to the fixed point algebra coincide when the relative commutant of the fixed point algebra is trivial. Next we show a version of the Tannaka duality theorem for twisted unitary groups.

1. INTRODUCTION

In [6] Doplicher and Roberts developed the duality theory for compact groups in the language of abstract symmetric, strict monoidal C^* -categories. This approach has become a useful tool for description of the superselection structure of a quantum field theory in $(3 + 1)$ -dimensional spacetime by the representation theory of a compact gauge group (see [7]). In order to obtain a similar picture in the lower dimensional spaces (*cf.* [9, 10]) one should consider some class of ‘quantum’ groups. In [13, 14] quasi-Hopf $*$ -algebras have been considered as realizations of quantum gauge groups. However, it is evident that we need a more general class of ‘quantum’ groups. This statement can be justified by the remark that recently many attempts concerning generalized symmetries of physical systems have been published (*cf.* Chapter IV in [11], in particular see p. 155). In this paper we shall restrict ourselves to Woronowicz’s compact quantum groups ([23]). We are going to investigate the systems composed of a C^* -algebra \mathcal{B} and a compact quantum group acting on \mathcal{B} . It seems to be interesting whether the theory of Doplicher and Roberts can be generalized to the class of such systems.

Our paper is an attempt to describe the duality theory for compact matrix quantum groups. Let us recall that such groups can be considered as quantum Lie groups. So, we will apply the program of [4, 5] to quantum C^* -systems with compact matrix quantum groups and we will obtain similar results. Particularly, the generalized Cuntz algebra associated with the twisted unitary group $S_qU(2)$ will be considered. In Corollary 5.7 we give some version of the Tannaka duality theorem for $S_qU(2)$. We also show that the fixed point subalgebra has the structure of crossed product of the algebra generated by Jones’ projections by a single endomorphism.

Received by the editors May 7, 1996 and, in revised form, July 16, 1996 and August 8, 1996.
1991 *Mathematics Subject Classification.* Primary 22D25; Secondary 46L60, 81R50.
The author was supported by KBN grant 2 P301 020 07.

The paper is organized as follows: In section 2 we recall some basic definitions concerning compact matrix quantum groups and their representations. In section 3 we prove a kind of duality theorem about equivalences of categories of representations of a quantum group and categories of endomorphisms of a fixed point algebra under an action of the quantum group. In section 4 we establish some basic facts about Cuntz algebras as well as some technical lemmas. In section 5 we study a special example of a quantum C^* -system composed of the Cuntz algebra and the twisted unitary group.

Part of the paper was done at University of Warsaw. The author wishes to thank Professor S.L. Woronowicz for hospitality and financial support. Special thanks are going to Professor W.A. Majewski for the many helpful discussions during the preparation of the paper.

2. COMPACT MATRIX QUANTUM GROUPS

Let us recall some basic definitions about compact matrix quantum groups. For details see [23, 24, 25].

Definition 2.1 ([23]). A pair $G = (\mathcal{C}, u)$ is called a compact matrix quantum group (c.m.q.g.) if \mathcal{C} is a unital C^* -algebra and $u = [u_{ij}]_{i,j=1,\dots,N}$ is a matrix with entries in \mathcal{C} such that the following conditions are fulfilled:

1. The $*$ -subalgebra \mathcal{C}_0 of \mathcal{C} generated by entries of u is dense in \mathcal{C} .
2. There exists a C^* -homomorphism (called a comultiplication) $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ such that $\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj}$ for $i, j = 1, \dots, N$.
3. There exists a linear antimultiplicative mapping (coinverse) $\kappa : \mathcal{C}_0 \rightarrow \mathcal{C}_0$ such that $\kappa(\kappa(C^*)^*) = C$ for $C \in \mathcal{C}_0$, and $\sum_{k=1}^N \kappa(u_{ik})u_{kj} = \delta_{ij}I = \sum_{k=1}^N u_{ik}\kappa(u_{kj})$ for any $i, j = 1, 2, \dots, N$. I denotes the unit of the algebra \mathcal{C} .

Definition 2.2. Let $G = (\mathcal{C}, u)$ be a c.m.q.g. and K be a finite dimensional Hilbert space. A linear mapping $v : K \rightarrow K \otimes \mathcal{C}$ is called a representation of G (or \mathcal{C} -comodule) if $(v \otimes \text{id}_{\mathcal{C}})v = (\text{id}_K \otimes \Delta)v$.

Remark 2.3. One can write $v(e_j) = \sum_{i=1}^d e_i \otimes v_{ij}$, $j = 1, \dots, d$, where $d = \dim K$, e_1, \dots, e_d is an orthonormal basis of K , and $v_{ij} \in \mathcal{C}$ for $i, j = 1, \dots, d$. Definition 2.2 yields

$$(2.1) \quad \Delta(v_{ij}) = \sum_{k=1}^d v_{ik} \otimes v_{kj}, \quad i, j = 1, \dots, d.$$

So, given an orthonormal basis of K one has a correspondence between representations of G and elements $v = [v_{ij}] \in M_d(\mathbf{C}) \otimes \mathcal{C}$ satisfying the equation (2.1), where $M_d(\mathbf{C})$ stands for the C^* -algebra of complex $d \times d$ matrices.

The representation u on \mathbf{C}^N which in the canonical basis is defined by the matrix $[u_{ij}]_{i,j=1,\dots,N}$ is called the fundamental representation of G .

Let v and w be representations of G acting on K and L respectively. A linear mapping $T : K \rightarrow L$ is called an intertwining operator between v and w if $w(Te) = (T \otimes \text{id}_{\mathcal{C}})(v(e))$ for any $e \in K$. The set of all intertwiners between representations v and w will be denoted by $\text{Mor}(v, w)$. The tensor product of representations v and w is the representation $v \otimes w$ acting on $K \otimes L$ defined by $(v \otimes w)(e_k \otimes f_l) = \sum_{i=1}^d \sum_{j=1}^{d'} e_i \otimes f_j \otimes v_{ik}w_{jl}$, $k = 1, \dots, d$, $l = 1, \dots, d'$, where $d = \dim K$, $d' =$

$\dim L$, e_1, \dots, e_d and $f_1, \dots, f_{d'}$ are orthonormal bases of K and L respectively, and $[v_{ik}], [w_{jl}]$ are matrices of v, w in those bases.

A representation v acting on a Hilbert space K is called unitary if the matrix $[v_{ij}]_{i,j=1,\dots,d}$ defined in Remark 2.3 is a unitary element of $M_d(\mathbf{C}) \otimes \mathcal{C}$, i.e. $\sum_{k=1}^d v_{ik}v_{jk}^* = \delta_{ij}I = \sum_{k=1}^d v_{ki}^*v_{kj}$ for any $i, j = 1, \dots, d$.

3. ACTION OF C.M.Q.G. ON C^* -ALGEBRAS

Now, let \mathcal{B} be a unital C^* -algebra. The crucial point in our consideration will be the notion of an action of a c.m.q.g. on the algebra \mathcal{B} (see [8]).

Definition 3.1. Let $G = (\mathcal{C}, u)$ be a c.m.q.g. A morphism $\alpha : \mathcal{B} \rightarrow \mathcal{B} \otimes_{\min} \mathcal{C}$ of C^* -algebras is called an action of G on \mathcal{B} if $(\text{id}_{\mathcal{B}} \otimes \Delta)\alpha = (\alpha \otimes \text{id}_{\mathcal{C}})\alpha$. The triple (\mathcal{B}, α, G) will be called a quantum C^* -system over a c.m.q.g. G .

A Hilbert space in a C^* -algebra \mathcal{B} is a finite dimensional closed linear subspace H of \mathcal{B} such that $\psi^*\psi' \in \mathbf{C}I$ for all $\psi, \psi' \in H$ (cf. [5]). The scalar product (\cdot, \cdot) on the Hilbert space H is defined by $(\psi, \psi')I = \psi^*\psi'$. So, for an orthonormal basis ψ_1, \dots, ψ_d of H one has $\psi_i^*\psi_j = \delta_{ij}I$, $i, j = 1, \dots, d$. Support of H is the projection $1_H = \sum_{i=1}^d \psi_i\psi_i^*$, and it does not depend on the choice of orthonormal basis in H . Moreover, one can define the inner endomorphism of \mathcal{B} associated with H : $\rho_H(B) = \sum_{i=1}^d \psi_i B \psi_i^*$ for $B \in \mathcal{B}$, where ψ_1, \dots, ψ_d is any orthonormal basis of H . Clearly,

$$(3.1) \quad 1_H = \rho_H(I) \quad \text{and} \quad \psi B = \rho_H(B)\psi, \quad \psi \in H, B \in \mathcal{B}.$$

The tensor product of two Hilbert spaces H, H' in \mathcal{B} is the Hilbert space $HH' = \text{span}\{\psi\psi' : \psi \in H, \psi' \in H'\}$. Observe that $\rho_{HH'} = \rho_H\rho_{H'}$.

Definition 3.2. Let (\mathcal{B}, α, G) be a quantum C^* -system. We say that a Hilbert space H in \mathcal{B} is a Hilbert \mathcal{C} -comodule in (\mathcal{B}, α, G) if $\alpha(H) \subset H \otimes \mathcal{C}$ and the representation α_H of G obtained by the restriction of the action α to H is the unitary one.

Lemma 3.3. Let H be a Hilbert \mathcal{C} -comodule in (\mathcal{B}, α, G) . Then $\alpha(1_H) = 1_H \otimes I$.

Proof. Let $[v_{ij}]_{i,j=1,\dots,d}$ be a unitary matrix of the representation α_H in an orthonormal basis ψ_1, \dots, ψ_d of H (cf. Remark 2.3). Then $\alpha(1_H) = \sum_i \alpha(\psi_i)\alpha(\psi_i^*) = \sum_i \sum_{k,l} \psi_k\psi_l^* \otimes v_{ki}v_{li}^* = \sum_{k,l} \psi_k\psi_l^* \otimes \sum_i v_{ki}v_{li}^* = \sum_k \psi_k\psi_k^* \otimes I = 1_H \otimes I. \quad \square$

By $\mathcal{U}(\mathcal{B}, \alpha, G)$ we will denote a category of Hilbert \mathcal{C} -comodules in (\mathcal{B}, α, G) . More precisely, objects of $\mathcal{U}(\mathcal{B}, \alpha, G)$ are Hilbert \mathcal{C} -comodules H in (\mathcal{B}, α, G) and morphisms between two comodules H and H' are intertwiners between representations α_H and $\alpha_{H'}$, i.e.

$$\text{Mor}_{\mathcal{U}}(H, H') = \left\{ T \in \mathcal{B} : \begin{array}{l} T1_H = T = 1_{H'}T, \quad TH \subset H', \\ \alpha_{H'}(T\psi) = (T \otimes I)\alpha_H(\psi), \quad \psi \in H \end{array} \right\}.$$

Let \mathcal{B}^α denote the subalgebra of fixed points under the action α , i.e. $\mathcal{B}^\alpha = \{A \in \mathcal{B} : \alpha(A) = A \otimes I\}$.

Proposition 3.4. Suppose H, H' are Hilbert \mathcal{C} -comodules in (\mathcal{B}, α, G) , and $T \in \text{Mor}_{\mathcal{U}}(H, H')$. Then $T \in \mathcal{B}^\alpha$.

Proof. Lemma 3.3 implies that $\alpha(T) = \alpha(T1_H) = \sum_i \alpha_{H'}(T\psi_i)\alpha(\psi_i^*) = \sum_i (T \otimes I)\alpha_H(\psi_i)\alpha(\psi_i^*) = (T \otimes I)\alpha(1_H) = T \otimes I. \quad \square$

Lemma 3.5. *Let H be an object in $\mathcal{U}(\mathcal{B}, \alpha, G)$. Then $\rho_H(\mathcal{B}^\alpha) \subset \mathcal{B}^\alpha$.*

Proof. Let ψ_1, \dots, ψ_d and $[v_{ij}]_{i,j=1,\dots,d}$ be as in the proof of Lemma 3.3 and $A \in \mathcal{B}^\alpha$. Then $\alpha(\rho_H(A)) = \sum_i \alpha(\psi_i)\alpha(A)\alpha(\psi_i)^* = \sum_{i,j,k} \psi_j A \psi_k^* \otimes v_{ji} v_{ki}^* = \sum_j \psi_j A \psi_j^* \otimes I = \rho_H(A) \otimes I$. \square

Let σ_H be an endomorphism of \mathcal{B}^α obtained by the restriction ρ_H to \mathcal{B}^α . By $\mathcal{I}(\mathcal{B}, \alpha, G)$ we denote the category such that its objects are endomorphisms σ_H with H being an object in $\mathcal{U}(\mathcal{B}, \alpha, G)$ while morphisms of $\mathcal{I}(\mathcal{B}, \alpha, G)$ are defined as follows: for any two endomorphisms σ, σ'

$$\text{Mor}_{\mathcal{I}}(\sigma, \sigma') = \{T \in \mathcal{B}^\alpha : T\sigma(I) = T = \sigma'(I)T, T\sigma(A) = \sigma'(A)T, A \in \mathcal{B}^\alpha\}.$$

Both $\mathcal{U}(\mathcal{B}, \alpha, G)$ and $\mathcal{I}(\mathcal{B}, \alpha, G)$ have the structure of monoidal C^* -categories (cf. [5, 6]).

Theorem 3.6. *Let (\mathcal{B}, α, G) be a quantum C^* -system over a c.m.q.g. (\mathcal{C}, u) . Then there is a faithful functor F from $\mathcal{U}(\mathcal{B}, \alpha, G)$ to $\mathcal{I}(\mathcal{B}, \alpha, G)$. If also $(\mathcal{B}^\alpha)' \cap \mathcal{B} = \mathbf{C}I$, then F is an isomorphism of monoidal C^* -categories.*

Proof. Define F on an object H of $\mathcal{U}(\mathcal{B}, \alpha, G)$ as $F(H) = \sigma_H$ and suppose that $T \in \text{Mor}_{\mathcal{U}}(H, H')$ for some objects H, H' of $\mathcal{U}(\mathcal{B}, \alpha, G)$. By Proposition 3.4 $T \in \mathcal{B}^\alpha$, while (3.2) implies $T\sigma_H(I) = T = \sigma_{H'}(I)T$. (3.2) gives also $T\sigma_H(A)\psi = T\psi A = \sigma_{H'}(A)T\psi$, for $A \in \mathcal{B}^\alpha, \psi \in H$. So, $T\sigma_H(A)1_H = \sigma_{H'}(A)T1_H$, hence $T \in \text{Mor}_{\mathcal{I}}(\sigma_H, \sigma_{H'})$. Then, we can define $F(T) = T$ and F is a faithful functor. Given $\phi \in \mathcal{B}$ such that $\phi A = \sigma_H(A)\phi$ for $A \in \mathcal{B}^\alpha$, one has $\psi^* \phi A = \psi^* \sigma_H(A)\phi = A\psi^* \phi$ for $\psi \in H$. So, $\psi^* \phi \in (\mathcal{B}^\alpha)' \cap \mathcal{B}$. Assuming $(\mathcal{B}^\alpha)' \cap \mathcal{B} = \mathbf{C}I$ one obtain $\psi_i^* \phi = \lambda_i I$ for an orthonormal basis ψ_1, \dots, ψ_d of H . Hence, $\phi = \sigma_H(I)\phi = \sum_i \psi_i \psi_i^* \phi = \sum_i \lambda_i \psi_i \in H$. So, the Hilbert space H is uniquely determined by its inner endomorphism σ_H , i.e. $H = \{\phi \in \mathcal{B} : \phi A = \sigma_H(A)\phi \text{ for } A \in \mathcal{B}^\alpha\}$. Let $T \in \text{Mor}_{\mathcal{I}}(\sigma_H, \sigma_{H'})$ and let $\psi_1, \dots, \psi_d, \psi'_1, \dots, \psi'_{d'}$ be a basis of H (H' respectively). Then, $\psi_j'^* T\psi_i A = \psi_j'^* T\sigma_H(A)\psi_i = \psi_j'^* \sigma_{H'}(A)T\psi_i = A\psi_j'^* T\psi_i, A \in \mathcal{B}^\alpha$. Thus $\psi_j'^* T\psi_i = \lambda_{ji} I$ for $i = 1, \dots, d, j = 1, \dots, d'$. So $T = \sigma_{H'}(I)T\sigma_H(I) = \sum_{i,j} \lambda_{ji} \psi_j' \psi_i^*$, hence $TH \subset H'$. Moreover, $(T \otimes I)\alpha(\psi) = \alpha(T)\alpha(\psi) = \alpha(T\psi)$ for $\psi \in H$. Consequently, we conclude that $T \in \text{Mor}_{\mathcal{U}}(H, H')$. \square

4. PRELIMINARIES ABOUT CUNTZ ALGEBRAS

To describe the action of the quantum group $S_q U(2)$ we will need some basic facts about Cuntz algebras as well as some technical lemmas. Therefore, for the reader's convenience we will provide the necessary material in this section.

Recall that the Cuntz algebra \mathcal{O}_d for $d \in \mathbf{N}, d > 1$, is a C^* -algebra generated by d isometries ψ_1, \dots, ψ_d such that $\psi_i^* \psi_j = \delta_{ij} I, i, j = 1, \dots, d$, and $\sum_{i=1}^d \psi_i \psi_i^* = I$. In [3] Cuntz has showed that \mathcal{O}_d is uniquely defined by these conditions and it is simple, i.e. it contains no nontrivial ideals.

Let $H = \text{span}\{\psi_1, \dots, \psi_d\}$, and let H^k denote the k -th tensor power of H , i.e. $H^k = \text{span}\{\psi_{i_1} \dots \psi_{i_k} : i_1, \dots, i_k = 1, \dots, d\}$. Linear operators from H^k to H^l can be identified with elements of $H^l H^{*k}$, where $H^l H^{*k} = \text{span}\{\phi_1 \phi_2^* : \phi_1 \in H^l, \phi_2 \in H^k\}$. Let us observe that for every $k, l \geq 0, \phi_1 \in H^l, \phi_2 \in H^k$ we can write $\phi_1 \phi_2^* = \sum_{i=1}^d \phi_1 \psi_i \psi_i^* \phi_2^* \in H^{l+1} H^{*(k+1)}$. Hence, there is the natural inclusion $H^l H^{*k} \subset H^{l+1} H^{*(k+1)}$.

By \mathcal{P}_d we will denote the dense $*$ -subalgebra in \mathcal{O}_d generated algebraically by ψ_1, \dots, ψ_d . The structure of \mathcal{P}_d can be described by saying that it is a graded $*$ -algebra, i.e. $\mathcal{P}_d = \bigoplus_{k \in \mathbf{Z}} \mathcal{P}_d^k$, where

$$(4.1) \quad \mathcal{P}_d^k = \bigcup_{n, n+k \geq 0} H^{n+k} H^{*n}.$$

In order to get the similar description of \mathcal{O}_d we will consider the gauge action β of the one dimensional torus T^1 on \mathcal{O}_d which is determined by the condition: $\beta_t(\psi_i) = t\psi_i$ for $t \in T^1, i = 1, \dots, d$. Let $m_k(X) = \int_{T^1} t^{-k} \beta_t(X) dt$ for each $k \in \mathbf{Z}$ and $X \in \mathcal{O}_d$, where dt stands for the normalized Haar measure on the group T^1 . Then m_k is a projection of norm 1 defined on the Banach space \mathcal{O}_d . Let \mathcal{O}_d^k be the range of m_k i.e. $\mathcal{O}_d^k = \{X \in \mathcal{O}_d : m_k(X) = X\}$. Then we have easy

Proposition 4.1. *Let $k \in \mathbf{Z}$. Then*

- i) $\mathcal{O}_d^k = \{\overline{X} \in \mathcal{O}_d : \beta_t(X) = t^k X, t \in T^1\}$,
- ii) $\mathcal{O}_d^k = \mathcal{P}_d^k$.

The projection m_0 is in fact a completely positive conditional expectation onto the fixed point subalgebra \mathcal{O}_d^0 . Moreover, for every $X \in \mathcal{P}_d$ we have $X = \sum_k m_k(x)$.

Let us remark that the fixed point subalgebra \mathcal{O}_d^0 is the inductive limit of matrix algebras (cf. (4.1)), so it has a structure of UHF-algebra of type d^∞ .

Proposition 4.2. *For $X \in \mathcal{O}_d$, if $m_k(X) = 0$ for each $k \in \mathbf{Z}$ then $X = 0$.*

Proof. Use a similar argument to that given in the proof of Proposition 1.10 in [3]. □

Let $G = (\mathcal{C}, u)$ be a c.m.q.g. Given a unitary representation of G on H defined in the basis ψ_1, \dots, ψ_d by a matrix $v = [v_{ij}]_{i,j=1,\dots,d}$ we can determine an action α of G on \mathcal{O}_d by defining it on generators (see [12], [8]):

$$(4.2) \quad \alpha(\psi_j) = \sum_{i=1}^d \psi_i \otimes v_{ij}, \quad j = 1, \dots, d.$$

Definition (4.2) leads to $\alpha(H^k H^{*k}) \subset H^k H^{*k} \otimes \mathcal{C}$ for every $k = 1, 2, \dots$, i.e. $H^k H^{*k}$ is invariant with respect to the action α and so is \mathcal{O}_d^0 .

Further, let us recall that the Haar measure h on G can be considered as a state on \mathcal{C} such that $(h \otimes \text{id}_{\mathcal{C}}) \Delta(a) = h(a)I = (\text{id}_{\mathcal{C}} \otimes h) \Delta(a)$, for any $a \in \mathcal{C}$. It is known ([23]) that each c.m.q.g admits the unique Haar measure. Let us define $E_\alpha = (\text{id}_{\mathcal{O}_d} \otimes h) \alpha$. Then E_α is a projection of norm 1 from \mathcal{O}_d onto the fixed point algebra \mathcal{O}_d^α under the action α ([12]). It is easy to check that the action α of c.m.q.g. G commutes with the action β of T^1 , i.e. $\alpha(\beta_t(X)) = (\beta_t \otimes \text{id}_{\mathcal{C}}) \alpha(X)$, for every $X \in \mathcal{O}_d$ and $t \in T^1$.

Proposition 4.3. *If $X \in \mathcal{O}_d^\alpha$ then $m_k(X) \in \mathcal{O}_d^\alpha$ for each $k \in \mathbf{Z}$.*

Proof. Let $X \in \mathcal{O}_d^\alpha$. Then

$$\begin{aligned} \alpha(m_k(X)) &= \int_{T^1} t^{-k} \alpha(\beta_t(X)) dt = \int_{T^1} t^{-k} (\beta_t \otimes \text{id}_{\mathcal{C}}) (\alpha(X)) dt \\ &= \int_{T^1} t^{-k} (\beta_t \otimes \text{id}_{\mathcal{C}}) (X \otimes I) dt = \int_{T^1} t^{-k} (\beta_t(X) \otimes I) dt \\ &= \int_{T^1} t^{-k} \beta_t(X) dt \otimes I = m_k(X) \otimes I. \quad \square \end{aligned}$$

Proposition 4.4. *For every $k \in \mathbf{Z}$ one has $E_\alpha(\mathcal{O}_d^k) \subset \mathcal{O}_d^k$ and $E_\alpha(\mathcal{P}_d^k) \subset \mathcal{P}_d^k$.*

Proof. Let $X \in \mathcal{O}_d^k$, i.e. $\beta_t(X) = t^k X$. Then

$$\begin{aligned} \beta_t(E_\alpha(X)) &= (\beta_t \otimes \text{id}_{\mathbf{C}})(\text{id}_{\mathcal{O}_d} \otimes h)\alpha(X) = (\text{id}_{\mathcal{O}_d} \otimes h)(\beta_t \otimes \text{id}_{\mathbf{C}})\alpha(X) \\ &= (\text{id}_{\mathcal{O}_d} \otimes h)\alpha(\beta_t(X)) = t^k E_\alpha(X). \end{aligned}$$

Hence $X \in \mathcal{O}_d^k$. The second inclusion is implied by the obvious one $E_\alpha(\mathcal{P}_d) \subset \mathcal{P}_d$. \square

Let α^n denote the restriction of α to the space H^n . Our notation follows from the observation that such a restriction can be considered as the n -th tensor power of the representation v (cf. the definition of α in (4.2)). We shall need the category \mathcal{U} of tensor powers of the representation α , i.e. $\text{Obj } \mathcal{U} = \{\alpha^n : n = 0, 1, 2, \dots\}$ (where $\alpha^0 : \mathbf{C}I \rightarrow \mathbf{C}I \otimes \mathbf{C} : I \rightarrow I \otimes I$) and $\text{Mor}(\alpha^n, \alpha^m)$ are the intertwining elements. Let us remark that \mathcal{U} is a monoidal C^* -category. By Proposition 3.4 we know that morphisms of the category \mathcal{U} are in the fixed point algebra \mathcal{O}_d^α .

Let $\rho(X) = \sum_i \psi_i X \psi_i^*$, $X \in \mathcal{O}_d$, be the canonical inner endomorphism of \mathcal{O}_d and σ be its restriction to the subalgebra \mathcal{O}_d^α . We define a category \mathcal{I} with $\text{Obj } \mathcal{I} = \{\sigma^n : n = 0, 1, 2, \dots\}$ and $\text{Mor}(\sigma^n, \sigma^m) = \{T \in \mathcal{O}_d^\alpha : T\sigma^n(A) = \sigma^m(A)T, A \in \mathcal{O}_d^\alpha\}$. As in the proof of Theorem 3.6 one can see that $\text{Mor}(\alpha^n, \alpha^m) \subset \text{Mor}(\sigma^n, \sigma^m)$. The converse inclusion will be showed in the special case when G is exactly the $S_q U(2)$.

We conclude this section with the following two lemmas.

Lemma 4.5. *Let $p \in \mathbf{N}$ and $S \in H^p$ be an isometry, i.e. $S^*S = I$. If $X \in \mathcal{O}_d^0$ commutes with S or with S^* then $X \in \mathbf{C}I$.*

Proof. The assumption of the lemma implies $X = S^{*n} X S^n$ for every $n \in \mathbf{N}$. On the other hand, let us observe that for every $r \in \mathbf{N}$ we can choose $n_r \in \mathbf{N}$ and $X_r \in H^{pn_r} H^{*pn_r}$ such that $\|X - X_r\| < r^{-1}$. The conditions $S^{*n_r} \in H^{*pn_r}$ and $S^{n_r} \in H^{pn_r}$, for each r , imply: there exist $\lambda_r \in \mathbf{C}$ such that $S^{*n_r} X_r S^{n_r} = \lambda_r I$. Consequently, we have

$$\begin{aligned} \|X - \lambda_r I\| &= \|S^{*n_r} X S^{n_r} - S^{*n_r} X_r S^{n_r}\| = \|S^{*n_r} (X - X_r) S^{n_r}\| \\ &\leq \|X - X_r\| < r^{-1}. \end{aligned}$$

Hence, $\lim_{r \rightarrow \infty} \lambda_r I = X$, and $X \in \mathbf{C}I$. \square

Lemma 4.6. *Let $p \in \mathbf{N}$ and $S \in H^p$ be an isometry. If $X \in \mathcal{O}_d$ commutes with S and S^* then $X \in \mathbf{C}I$.*

Proof. Let $X \in \mathcal{O}_d$. Then, $m_l(XS) = \int t^{-l} \beta_t(XS) dt = \int t^{-l} \beta_t(X) t^p S dt = m_{l-p}(X)S$ for $l \in \mathbf{Z}$. In the same way we can check that $m_l(SX) = S m_{l-p}(X)$. Hence, if X commutes with S then $m_k(X)$ commutes with S for every $k \in \mathbf{Z}$. Similarly, if X commutes with S^* then $m_k(X)$ commutes with S^* . So (cf. Proposition 4.1), for each $k \in \mathbf{Z}$ elements $m_k(X)^* m_k(X)$, $m_k(X) m_k(X)^*$ of \mathcal{O}_d^0 commute with S and by preceding lemma they are multiples of unity. Hence, $m_k(X)$ is a multiple of unitary element $U \in \mathcal{O}_d$. Using the argument given in [2] (see the proof of Theorem 3.2, see also [1]) one can show that if there is a unitary U in \mathcal{O}_d^k then $k = 0$. Namely, taking U^* instead U if necessary, we can assume that $k \geq 0$. Then, $V = U \psi_1^{*k} \in \mathcal{O}_d^0$, $VV^* = I$ and $V^*V = \psi_1^k \psi_1^{*k}$, where ψ_1 is one of the generators of \mathcal{O}_d . If ω is the trace on the UHF-algebra \mathcal{O}_d^0 then

$d^{-k} = \omega(\psi_1^k \psi_1^{*k}) = \omega(V^*V) = \omega(VV^*) = \omega(I) = 1$. Hence $k = 0$. It follows that $m_k(X) = 0$ unless $k = 0$, and $m_0(X) \in \mathbf{CI}$ by Lemma 4.5. Hence, by Proposition 4.2 $X \in \mathbf{CI}$. \square

5. ACTION OF $S_qU(2)$

Recall ([22]) that $S_qU(2)$ is the pair (\mathcal{A}, u) where \mathcal{A} is the C^* -algebra generated by two elements a, c fulfilling the following relations:

$$(5.1) \quad \begin{aligned} a^*a + c^*c &= I, & aa^* + q^2cc^* &= I, \\ cc^* &= c^*c, & qca &= ac, & qc^*a &= ac^*, \end{aligned}$$

where $q \in [-1, 1]$, $q \neq 0$, and u is the matrix

$$(5.2) \quad u = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}.$$

Now, we want to describe the structure of the fixed point algebra \mathcal{O}_2^α . For this we need some lemma about the Haar measure on $S_qU(2)$. For $k \in \mathbf{Z}$, $n, m \in \mathbf{N}_0$ ($\mathbf{N}_0 = \mathbf{N} \cup \{0\}$) we define $a_{knm} = a^k c^n c^{*m}$ where for negative k we set $a^k = a^{*|k|}$. Then it is known ([22], Theorem 1.2) that the set $\{a_{knm} : k \in \mathbf{Z}, n, m \in \mathbf{N}_0\}$ is the linear basis of the $*$ -algebra \mathcal{A}_0 generated by elements a, c . Hence, for every $b \in \mathcal{A}_0$ one has $b = \sum_{k,n,m} \lambda_{knm} a_{knm}$, where coefficients λ_{knm} are uniquely defined and $\lambda_{knm} = 0$ for almost every system (k, n, m) .

Lemma 5.1. *Let $r \in \mathbf{N}$ and let $b_1, b_2, \dots, b_r \in \{a, a^*, c, c^*\}$. If $b_1 b_2 \dots b_r = \sum \lambda_{knm} a_{knm}$ then for every k, n, m we have that $\lambda_{knm} \neq 0$ implies $k + n + m \equiv r \pmod{2}$.*

Proof. Induction with respect to r . If $r = 1$ then the statement is obvious. Suppose that it is true for some r and let b_1, \dots, b_r, b_{r+1} be as in the lemma. Then $b_1 \dots b_r b_{r+1} = \sum \lambda_{knm} a_{knm} b_{r+1}$ and by the inductive assumption, if $\lambda_{knm} \neq 0$ then $k + n + m - r$ is even. Relations (5.1) and simple calculations lead to: for every k, n, m one has $a_{knm} b_{r+1} = \sum_{k', n', m'} \lambda_{k' n' m'}^{knm} a_{k' n' m'}$, where $\lambda_{k' n' m'}^{knm} \neq 0$ for at most two systems (k', n', m') such that $k' + n' + m' - k - n - m$ is odd. So, $b_1 \dots b_r b_{r+1} = \sum_{k', n', m'} \lambda'_{k' n' m'} a_{k' n' m'}$, where

$$(5.3) \quad \lambda'_{k' n' m'} = \sum_{k, n, m} \lambda_{knm} \lambda_{k' n' m'}^{knm}$$

and the summation is over such systems (k, n, m) that $k' + n' + m' - k - n - m$ is odd. Suppose now that $\lambda'_{k' n' m'} \neq 0$ for some k', n', m' . It means that in (5.3) there is a nonzero ingredient, i.e. there is a system (k, n, m) such that $k' + n' + m' - k - n - m$ is odd and $\lambda_{knm} \neq 0$. $k + n + m - r$ is even by the inductive assumption. Then, $k' + n' + m' - (r + 1)$ is even too. \square

Let us fix some important elements in \mathcal{O}_2^α (see [24]):

$$(5.4) \quad \theta = \psi_1 \psi_1 \psi_1^* \psi_1^* + \psi_2 \psi_2 \psi_2^* \psi_2^* + q \psi_2 \psi_1 \psi_2^* \psi_1^* + q \psi_1 \psi_2 \psi_1^* \psi_2^* + (1 - q^2) \psi_2 \psi_1 \psi_1^* \psi_2^*,$$

$$(5.5) \quad S = (1 + q^2)^{-\frac{1}{2}} (\psi_1 \psi_2 - q \psi_2 \psi_1).$$

Remark 5.2. The selfadjoint element $\theta \in H^2H^{*2}$ fulfills the following relations:

- H1) $\sigma^n(\theta)\sigma^{n+1}(\theta)\sigma^n(\theta) = \sigma^{n+1}(\theta)\sigma^n(\theta)\sigma^{n+1}(\theta)$ for $n \in \mathbf{N}_0$,
- H2) $\sigma^n(\theta)\sigma^m(\theta) = \sigma^m(\theta)\sigma^n(\theta)$ for $n, m \in \mathbf{N}_0$ such that $|n - m| > 1$,
- H3) $\theta^2 = (1 - q^2)\theta + q^2I$.

Moreover, it is known ([24]) that

- H4) for every $n \geq 2$ the C^* -algebra $\mathcal{O}_2^\alpha \cap H^n H^{*n}$ is generated by elements $\sigma^m(\theta)$, $m = 0, \dots, n - 2$.

For $q = 1$ the element θ is just the flip operator on $H^2H^{*2} \cong M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$. Moreover, if $1 \leq m \leq n - 1$ then $\sigma^{m-1}(\theta)$ interchanges m -th and $(m + 1)$ -th factors in $H^n H^{*n} \cong M_2(\mathbf{C})^{\otimes n}$. Let $s_1, \dots, s_{n-1} \in S_n$ be reflections generating the symmetric group S_n . Then the mapping $s_m \mapsto \sigma^{m-1}(\theta)$ can be extended to a representation of the group S_n in the algebra $H^n H^{*n}$. By taking inductive limits we obtain a representation of S_∞ on the UHF-algebra $\mathcal{O}_2^0 \cong M_2(\mathbf{C})^{\otimes \infty}$ which is nothing but the one constructed in [17] and [16] (see also [18]). Of course this representation can be extended to a representation π of the algebra $\mathbf{C}(S_\infty)$ (the inductive limit of the group algebras $\mathbf{C}(S_n)$) in the algebra \mathcal{O}_2^0 . Thus we arrive at the permutation symmetry (see [4]) and we have $\pi(\mathbf{C}(S_\infty)) = \mathcal{O}_2^{SU(2)} \cap \mathcal{O}_2^0$.

For an arbitrary $q \in [-1, 1] \setminus \{0\}$ let g_1, \dots, g_{n-1} be the generators of the Hecke algebra $H_n(q^2)$ (see [21]). Then due to the relations H1)–H3) from Remark 5.2 the mapping $g_m \mapsto \sigma^{m-1}(\theta)$, $m = 1, \dots, n - 1$, can be extended to a representation of $H_n(q^2)$ in $H^n H^{*n}$. Moreover, the similar inductive limits argument leads to a representation π_q of $H(q^2)$ in \mathcal{O}_2^0 .

Proposition 5.3. 1. *The element $S \in H^2$ defined in (5.5) is an isometry and $SS^* \in H^2H^{*2}$ is a spectral projection of θ .*

Moreover, S fulfills the following relations:

- 2. $S^*\sigma(S) = -(q + q^{-1})^{-1}I$,
- 3. $\sigma^n(SS^*)\sigma^m(SS^*) = \sigma^m(SS^*)\sigma^n(SS^*)$ for $n, m \in \mathbf{N}_0$ such that $|n - m| > 1$,
- 4. $\sigma^n(SS^*)\sigma^m(SS^*)\sigma^n(SS^*) = (q + q^{-1})^{-2}\sigma^n(SS^*)$ for $n, m \in \mathbf{N}_0$ such that $|n - m| = 1$.

Proof. 1. $\theta = I - (1 + q^2)SS^* = (I - SS^*) - q^2SS^*$. The other points are consequences of simple calculations. \square

Remark 5.4. 5.3.3 and 5.3.4 show that the family $\{\sigma^l(SS^*) : l \in \mathbf{N}_0\}$ can be considered as the system of Jones' projections (see [12]).

Theorem 5.5. \mathcal{O}_2^α is the smallest C^* -subalgebra of \mathcal{O}_2 which contains S and is stable under the canonical endomorphism ρ .

Proof. Let $C^*(S, \rho)$ denote the smallest C^* -algebra containing S and invariant under ρ . It is obvious that $C^*(S, \rho) \subset \mathcal{O}_2^\alpha$ (cf. Lemma 3.5). In order to prove the converse inclusion, first we will show that $\mathcal{O}_2^\alpha \cap \mathcal{P}_2 \subset C^*(S, \rho)$. By Proposition 4.3 it suffices to prove that $\mathcal{O}_2^\alpha \cap \mathcal{P}_2^k \subset C^*(S, \rho)$ for every $k \in \mathbf{Z}$. Consider the case $k = 0$. $\mathcal{O}_2^\alpha \cap \mathcal{P}_2^0 = \bigcup_n \mathcal{O}_2^\alpha \cap H^n H^{*n}$ and from Remark 5.2.H4) we see that $\mathcal{O}_2^\alpha \cap \mathcal{P}_2^0$ is generated by $\sigma^m(\theta)$, $m \in \mathbf{N}_0$. Proposition 5.3 implies that $\theta \in C^*(S, \rho)$, so $\mathcal{O}_2^\alpha \cap \mathcal{P}_2^0 \subset C^*(S, \rho)$. Next, let $X \in \mathcal{O}_2^\alpha \cap \mathcal{P}_2^{2l}$ for some $l \in \mathbf{Z}$. By taking X^* if necessary, we can assume that $l \geq 0$. Then $XS^{*l} \in \mathcal{O}_2^\alpha \cap \mathcal{P}_2^0 \subset C^*(S, \rho)$ and $X = XS^{*l}S^l \in C^*(S, \rho)$. It remains to show that $\mathcal{O}_2^\alpha \cap \mathcal{P}_2^{2l+1} = \{0\}$ for every $l \in \mathbf{Z}$. Proposition 4.4 yields

$$(5.6) \quad \mathcal{O}_2^\alpha \cap \mathcal{P}_2^{2l+1} = E_\alpha(\mathcal{P}_2^{2l+1}).$$

Let X be an element of \mathcal{P}_2^{2l+1} of the form

$$(5.7) \quad X = \psi_{i_1} \dots \psi_{i_n} \psi_{i_{n+1}} \dots \psi_{i_{n+2l+1}} \psi_{j_n}^* \dots \psi_{j_1}^*$$

for some $n \in \mathbf{N}$ and $i_1, \dots, i_{n+2l+1}, j_1, \dots, j_n = 1, 2$. Then

$$E_\alpha(X) = \sum_{\substack{p_1, \dots, p_{n+2l+1} \\ r_1, \dots, r_n}} h(u_{p_1 i_1} \dots u_{p_{n+2l+1} i_{n+2l+1}} u_{r_n j_n}^* \dots u_{r_1 j_1}^*) \\ \times \psi_{p_1} \dots \psi_{p_{n+2l+1}} \psi_{r_n}^* \dots \psi_{r_1}^*$$

where matrix elements u_{ij} are given by (5.2). Thus, Lemma 5.1 implies that the element $u_{p_1 i_1} \dots u_{p_{n+2l+1} i_{n+2l+1}} u_{r_n j_n}^* \dots u_{r_1 j_1}^*$ is a linear combination of elements a_{knm} such that $k + n + m$ is odd. But $h(a_{knm}) \neq 0$ if and only if $k = 0$ and $n = m$ ([23]), so $h(u_{p_1 i_1} \dots u_{p_{n+2l+1} i_{n+2l+1}} u_{r_n j_n}^* \dots u_{r_1 j_1}^*) = 0$ for every $i_1, \dots, i_{n+2l+1}, j_1, \dots, j_n, p_1, \dots, p_{n+2l+1}, r_1, \dots, r_n$. Therefore, $E_\alpha(X) = 0$. \mathcal{P}_2^{2l+1} is spanned by elements of the form (5.7) (see (4.1)), so $E_\alpha(X) = 0$ for every $X \in \mathcal{P}_2^{2l+1}$ and by (5.6) $\mathcal{O}_2^\alpha \cap \mathcal{P}_2^{2l+1} = \{0\}$. Secondly, we know that $\mathcal{O}_2^\alpha \cap \mathcal{P}_2$ is dense in \mathcal{O}_2^α (see Lemma 6 of [12]). Therefore it is clear that $\mathcal{O}_2^\alpha \subset C^*(S, \rho)$. \square

Lemma 4.6 and the above theorem lead to

Lemma 5.6. *The subalgebra \mathcal{O}_2^α has the trivial relative commutant in \mathcal{O}_2 , i.e. $(\mathcal{O}_2^\alpha)' \cap \mathcal{O}_2 = \mathbf{C}I$.*

Consequently, Theorem 3.6 and Lemma 5.6 give

Corollary 5.7. *For every $X \in \mathcal{O}_2^\alpha$ the following conditions are equivalent:*

- i) $X \in \text{Mor}(\alpha^n, \alpha^m)$,
- ii) $X\sigma^n(Y) = \sigma^m(Y)X$, for $Y \in \mathcal{O}_2^\alpha$.

We conclude this section with a discussion of the structure of the fixed point algebra \mathcal{O}_2^α . To this end let us recall that the authors of [12] have considered the twisted unitary group ${}_q U(2)$ and its action α' on \mathcal{O}_2 is. They have proved that the fixed point algebra $\mathcal{O}_2^{\alpha'}$ of this action is exactly $\mathcal{O}_2^\alpha \cap \mathcal{O}_2^0$.

Let $\tau(A) = SAS^*$, $A \in \mathcal{O}_2^{\alpha'}$, be the endomorphism of $\mathcal{O}_2^{\alpha'}$ induced by the isometry S . From the proof of Theorem 5.3 we can observe that every element $A \in \mathcal{O}_2^\alpha \cap \mathcal{O}_2^{2l}$ is of the form

$$(5.8) \quad A = \begin{cases} BS^l, & \text{if } l \geq 0, \\ S^{*l}B, & \text{if } l < 0, \end{cases}$$

where $B \in \mathcal{O}_2^{\alpha'}$. Then, \mathcal{O}_2^α has the structure of crossed product of $\mathcal{O}_2^{\alpha'}$ by the endomorphism τ (see [15]). Indeed, we have the following universal property (cf. Lemma 3.8 of [5])

Proposition 5.8. *Let $\varphi_0 : \mathcal{O}_2^{\alpha'} \rightarrow \mathcal{B}$ be a morphism of unital C^* -algebras and $R \in \mathcal{B}$ be an isometry of \mathcal{B} such that $R\varphi_0(A)R^* = \varphi_0(\tau(A))$, for $A \in \mathcal{O}_2^{\alpha'}$. Then φ_0 extends uniquely to a morphism $\varphi : \mathcal{O}_2^\alpha \rightarrow \mathcal{B}$ with $\varphi(S) = R$.*

Proof. It is the simple consequence of (5.8). \square

REFERENCES

- [1] Araki H., Carey A.L., Evans D.E., *On O_{n+1}* , J. Operator Theory **12** (1984), 247-264 MR **86g**:46082
- [2] Bratteli O., Evans D.E., *Derivations tangential to compact groups: the non-abelian case*, Proc. London Math. Soc. **52** (1986), 369-384 MR **87f**:46123
- [3] Cuntz J., *Simple C^* -algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173-185 MR **57**:7189
- [4] Doplicher S., Roberts J.E., *Duals of Compact Lie Groups Realized in the Cuntz Algebras and Their Actions on C^* -Algebras*, J. Funct. Anal. **74** (1987), 96-120 MR **89a**:22011
- [5] Doplicher S., Roberts J.E., *Compact group actions on C^* -algebras*, J. Operator Theory **19** (1988), 283-305 MR **90f**:46104
- [6] Doplicher S., Roberts J.E., *A new duality theory for compact groups*, Invent. Math. **98** (1989), 157-218 MR **90k**:22005
- [7] Doplicher S., Roberts J.E., *Why There is a Field Algebra with a Compact Gauge Group Describing the Superselection Structure in Particle Physics*, Comm. Math. Phys. **131** (1990), 51-107 MR **91k**:81082
- [8] Fannes M., Nachtergaele B., Werner R.F., *Quantum Spin Chains with Quantum Group Symmetry*, Comm. Math. Phys. **174** (1996), 477-507 CMP 96:07
- [9] Fredenhagen K., Rehren K.-H., Schroer B., *Superselection Sectors with Braid Group Statistics and Exchange Algebras I*, Comm. Math. Phys. **125** (1989), 201-226 MR **91c**:81047
- [10] Fredenhagen K., Rehren K.-H., Schroer B., *Superselection Sectors with Braid Group Statistics and Exchange Algebras II*, Rev. Math. Phys. - Special Issue (1992), 113-157 MR **94g**:81119
- [11] Haag R., *Local Quantum Physics*, Springer-Verlag, Berlin Heidelberg 1992 MR **94d**:81001
- [12] Konishi Y., Nagisa M., Watatani Y., *Some remarks on actions of compact matrix quantum groups on C^* -algebras*, Pacific J. Math. **153** (1992), 119-127 MR **93c**:46121
- [13] Mack G., Schomerus V., *Conformal Field Algebras with Quantum Symmetry from the Theory of Superselection Sectors*, Comm. Math. Phys. **134** (1990), 139-196 MR **92i**:81298
- [14] Mack G., Schomerus V., *Quasi Hopf quantum symmetry in quantum theory*, Nucl. Phys. B **370** (1992), 185-230 MR **93c**:81081
- [15] Paschke W.L., *The crossed product of a C^* -algebra by an endomorphism*, Proc. Amer. Math. Soc. **80** (1980), 113-118 MR **81m**:46081
- [16] Powers R.T., Price G., *Derivations Vanishing on $S(\infty)$* , Comm. Math. Phys. **84** (1982), 439-447 MR **83k**:46056
- [17] Price G., *Extremal Traces on Some Group-Invariant C^* -Algebras*, J. Funct. Anal. **49** (1982), 145-151 MR **84e**:46075
- [18] Størmer E., *Symmetric States of Infinite Tensor Products of C^* -algebras*, J. Funct. Anal. **3** (1969), 48-68 MR **39**:3327
- [19] Wang S., *Free Products of Compact Quantum Groups*, Comm. Math. Phys. **167** (1995), 671-692 MR **95k**:46104
- [20] Wang S., *Tensor products and crossed products of compact quantum groups*, Proc. London Math. Soc. **71** (1995), 695-720 MR **96i**:46074
- [21] Wenzl H., *Hecke algebras of type A_n and subfactors*, Invent. Math. **92** (1988), 349-383 MR **90b**:46118
- [22] Woronowicz S.L., *Twisted $SU(2)$ group. An example of a non-commutative differential calculus*, R.I.M.S. Publ. Kyoto University **23** (1987), 117-181 MR **88h**:46130
- [23] Woronowicz S.L., *Compact Matrix Pseudogroups*, Comm. Math. Phys. **111** (1987), 613-665 MR **88m**:46079
- [24] Woronowicz S.L., *Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups*, Invent. Math. **93** (1988), 35-76 MR **90e**:22033
- [25] Woronowicz S.L., *Compact quantum groups*, preprint, University of Warsaw.

INSTITUTE OF MATHEMATICS, GDAŃSK UNIVERSITY, WITA STWOSZA 57, 80-952 GDAŃSK, POLAND

E-mail address: matmm@univ.gda.pl