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ELEMENTARY ABELIAN 2-GROUP ACTIONS ON FLAG MANIFOLDS AND APPLICATIONS

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Abstract. Let $\mathcal{N}_*$ denote the unoriented cobordism ring. Let $G = (\mathbb{Z}/2)^n$ and let $Z_*(G)$ denote the equivariant cobordism ring of smooth manifolds with smooth G-actions having finite stationary points. In this paper we show that the unoriented cobordism class of the (real) flag manifold $M = O(m)/(O(m_1) \times \cdots \times O(m_s))$ is in the subalgebra generated by $\bigoplus_{i<2^n} \mathcal{N}_i$, where $m = \sum m_j$, and $2^{n-1} < m \leq 2^n$. We obtain sufficient conditions for indecomposability of an element in $Z_*(G)$. We also obtain a sufficient condition for algebraic independence of any set of elements in $Z_*(G)$. Using our criteria, we construct many indecomposable elements in the kernel of the forgetful map $Z_d(G) \to \mathcal{N}_d$ in dimensions $2 \leq d < n$, for $n > 2$, and show that they generate a polynomial subalgebra of $Z_*(G)$.

1. Introduction

Let $G = (\mathbb{Z}_2)^n$, $n \geq 2$. Denote by $Z_*(G)$ the equivariant cobordism ring of (smooth) closed manifolds with smooth G-actions having finite stationary point sets [2], [1]. The cobordism class of a manifold M, along with an action ϕ of G having finite stationary point set, will be denoted by $[M, \phi]$. Let $R_q(G)$ denote the vector space over $\mathbb{Z}_2$, with basis the set of isomorphism classes of $\mathbb{R}G$-modules of dimension q. If $R_*(G) = \sum_{q\geq 0} R_q(G)$, then $R_*(G)$ is a graded commutative $\mathbb{Z}_2$-algebra with unit. The multiplication in $R_*(G)$ is given by $[V] \cdot [W] = [V \oplus W]$. One can identify $R_*(G)$ with the graded polynomial algebra over $\mathbb{Z}_2$ generated by $\widehat{G} = \mathrm{Hom}_{\mathbb{Z}_2}(G, \mathbb{Z}_2)$ (cf. [1]).

One has an algebra homomorphism $\eta_* : Z_*(G) \to R_*(G)$ where $\eta_*([M, \phi]) = \sum [T_x M]$, the sum being taken over the (finite) set of stationary points of M. By a theorem of Stong [11], we know that η_* is a monomorphism. One has the 'forgetful' homomorphism $\varepsilon_* : Z_*(G) \to \mathcal{N}_*$, the unoriented cobordism ring, $[M, \phi] \mapsto [M]$. Let T^n_* denote the subalgebra of $\mathcal{N}_*$ generated by $\bigoplus_{i<2^n} \mathcal{N}_i$. Then tom Dieck [3] has shown that $\mathrm{Im}\, \varepsilon_* = T^n_*$ (cf. Kosniowski and Stong, section 4 of [5]). In this paper we prove

Theorem 1.1. *Let $G(m_1, \ldots, m_k)$ denote the flag manifold*

$$O(m)/(O(m_1) \times \cdots \times O(m_k)),$$

$m = \sum m_i$. Let $2^{n-1} < m \leq 2^n$. Then $[G(m_1, \ldots, m_k)] \in T^n_*$.

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The problem of determining which flag manifolds are unoriented boundaries was addressed in [10]. The case of Grassmann manifolds was completely settled in [9], which also considers the case of complex and quaternionic Grassmannians. See also Stong [12]. The above theorem gives perhaps the best known result in general. It is easy to see that if two of the numbers $m_1, \ldots, m_k$ are equal, then $G(m_1, \ldots, m_k)$ admits a fixed point free involution and hence bounds. If m is even, then the above theorem does not yield the best possible result (cf. Theorem 2.1 in [10]). But by remark 2.3(iii) in [10], the general problem of determining which flag manifolds bound has been reduced to the consideration of only the case $m = \sum m_i$ is odd, $m_1, \ldots, m_k$ being distinct.

Theorem 1.1 is proved by exhibiting certain $(\mathbb{Z}_2)^n$-actions on $G(m_1, \ldots, m_k)$ with finitely many stationary points.

It is well-known that when $n = 2$, $Z_*(G)$ is isomorphic to the polynomial algebra with one generator $[\mathbb{P}^2, \phi]$, where the action is given as follows: $t_1([x, y, z]) = [-x, y, z]$, $t_2([x, y, z]) = [x, -y, z]$. Here t_1, t_2 denotes a set of generators of G. Hence $\varepsilon_* : Z_*(G) \to T_*^2$ is an isomorphism (cf. [1]). However, for $n \geq 3$, ε_* is not a monomorphism [8] and the structure of $Z_*(G)$ is not known.

A G-manifold (M, ϕ) with finite stationary point set is equivariantly indecomposable if $[M, \phi]$ is an indecomposable element in $Z_*(G)$. Clearly, if $[M, \phi] \in Z_*(G)$ and $[M] \in \mathcal{N}_*$ is indecomposable, then $[M, \phi] \in Z_*(G)$ is indecomposable. An important step towards understanding the structure of $Z_*(G)$ is to know the indecomposable elements, as they generate $Z_*(G)$ as a $\mathbb{Z}_2$-algebra. In section 3 we obtain a sufficient criterion for an element in $Z_*(G)$ to be indecomposable. Since T_*^n is a polynomial algebra, and since by tom Dieck's theorem [3] $\operatorname{Im} \varepsilon_* = T_*^n$, it follows that the exact sequence $0 \to \ker \varepsilon_* \to Z_*(G) \to T_*^n \to 0$ splits. Therefore one can clearly 'lift' indecomposable elements from T_*^n to obtain indecomposable elements in $Z_*(G)$. However, we apply our indecomposability criterion to construct indecomposable elements in $Z_*(G)$ which belong to $\mathcal{K}_* = \ker \varepsilon_*$ in each dimension $2 \leq m \leq n$ except possibly in dimension n when n is even. For the precise statement see Theorem 3.6. This is in striking contrast to the situation in the unoriented cobordism ring $\mathcal{N}_*$, where there is no generator in dimensions $2^j - 1$. Note that indecomposable elements in $\mathcal{K}_*$ cannot arise by 'lifting' indecomposable elements from $\mathcal{N}_*$. We also prove a sufficient criterion for a set of elements in $Z_*(G)$ to be algebraically independent and use it to show that suitable indecomposable elements in $\mathcal{K}_*$ generate a polynomial subalgebra of $Z_*(G)$. In the last section we generalize a result of Conner and Floyd regarding the number of stationary points x such that an irreducible representation occurs at the tangential representation T_xM with a given multiplicity.

2. Action of $(\mathbb{Z}_2)^n$ on flag manifolds

Let $2 \leq m \leq 2^n$, where $n \geq 2$. In this section we shall exhibit certain $G = (\mathbb{Z}_2)^n$ actions on the flag manifold $G(m_1, \ldots, m_k) \cong O(m)/(O(m_1) \times \cdots \times O(m_k))$, $\sum m_i = m$. Indeed this is a routine generalization of the actions of G on projective spaces considered by tom Dieck in [3]. We recall the action of G on the Milnor manifold also considered by tom Dieck [3] as it will be needed in section 3.

Let $\underline{n} = \{1, 2, \ldots, n\}$. We regard $\{e_\alpha \mid \alpha \subset \underline{n}\}$ as the 'standard basis' of $\mathbb{R}^{2^n}$, with its usual innerproduct. For $1 \leq i \leq n$, let $t_i : \mathbb{R}^{2^n} \to \mathbb{R}^{2^n}$ be the $\mathbb{R}$-linear map defined by

$$t_i(e_\alpha) = \begin{cases} -e_\alpha & \text{if } i \in \alpha, \\ e_\alpha & \text{if } i \notin \alpha. \end{cases}$$

Then it is readily checked that $t_i^2 = \mathrm{Id}$, and $t_i t_j = t_j t_i$ for $1 \le i, j \le n$. Therefore we obtain a linear action of G on $\mathbb{R}^{2^n}$.

Lemma 2.1. (i) *The decomposition* $\mathbb{R}^{2^n} = \sum_{\alpha \subset \underline{n}} \mathbb{R}e_\alpha$ *expresses* $\mathbb{R}^{2^n}$ *as a sum of mutually non-isomorphic irreducible* G*-submodules of* $\mathbb{R}^{2^n}$.

(ii) *If* V *is any* G*-submodule of* $\mathbb{R}^{2^n}$, *then* $V = \sum_{\alpha \in S} \mathbb{R}e_\alpha$ *for some* $S \subset \mathcal{P}(\underline{n})$ *with* $\dim V = \#S$.

Proof of (i). It is clear that each $\mathbb{R}e_\alpha$ is a G-submodule. We only have to prove that if $\alpha \ne \beta$, then $\mathbb{R}e_\alpha \not\cong \mathbb{R}e_\beta$. Let $\alpha \ne \beta$, and let $i \in \alpha\Delta\beta$. Say, $i \in \alpha$ and $i \notin \beta$. Then $t_i e_\alpha = -e_\alpha$, and $t_i e_\beta = e_\beta$. This shows that $\mathbb{R}e_\alpha$ cannot be isomorphic to $\mathbb{R}e_\beta$ as G-modules. Part (ii) is an immediate consequence of part (i). □

Let $S \subset \mathcal{P}(\underline{n})$, $\#S = m$. Let $V = \sum_{\alpha \in S} \mathbb{R}e_\alpha \subset \mathbb{R}^{2^n}$. Since the G-action on $\mathbb{R}^{2^n}$ preserves innerproduct and since V is a G-submodule, we obtain a G-action on any flag manifold $G(m_1, \dots, m_k)$, $\sum m_j = m$, modelled on V. Explicitly, if $(A_1, \dots, A_k) \in G(m_1, \dots, m_k)$, and $t \in G$, then $t(A_1, \dots, A_k) = (tA_1, \dots, tA_k)$. Clearly $(A_1, \dots, A_k)$ is a stationary point if and only if each A_j, $1 \le j \le k$, is a G-submodule of V. We conclude from Lemma 2.1 that there are only finitely many stationary points. We shall denote this action on $G(m_1, \dots, m_k)$ by ϕ_S or simply by ϕ when there is no risk of confusion. Thus $[G(m_1, \dots, m_k), \phi_S] \in Z_*(G)$ for every $S \subset \mathcal{P}(\underline{n})$, $\#S = \sum m_j$. In the case of a Grassmann manifold $G_{m,k}$ with G-action ϕ_S, the stationary points are $E_\alpha = \langle e_{\alpha_1}, \dots, e_{\alpha_k} \rangle$, the span of $e_{\alpha_1}, \dots, e_{\alpha_k}$ where $\alpha = \{\alpha_1, \dots, \alpha_k\}$ is any k-element subset of S.

Next, we recall the action of G on Milnor manifolds considered by tom Dieck [3]. Let $S \subset \mathcal{P}(\underline{n})$, $\#S = k+1$, and let $T \subset S$, $\#T = l+1$. Then we can form the product $(P^l \times P^k, \phi_T \times \phi_S)$. Let $H_{l,k}$ be the Milnor manifold

$$\left\{ \left(\left[\sum_{\alpha \in T} x_\alpha e_\alpha \right], \left[\sum_{\beta \in S} y_\beta e_\beta \right] \right) \mid \sum_{\alpha \in T} x_\alpha y_\alpha = 0 \right\}.$$

Then $H_{l,k}$ is a G-stable submanifold of $(P^l \times P^k, \phi_T \times \phi_S)$. It is obvious that there are only finitely many stationary points for the G-action on $H_{l,k}$. We denote this action on $H_{l,k}$ by $\phi_{T,S}$ or simply by ϕ.

Proof of Theorem 1.1. Since $M = G(m_1, \dots, m_k)$, $m = \sum m_i \le 2^n$, was shown to admit a $G = (\mathbb{Z}_2)^n$ action with finite stationary point set, it follows from the result of tom Dieck [3] that $[M] \in T_*^n$. This completes the proof. □

Example 2.2. Take $M = G(40, 42, 45)$. Then the dimension of M is 5370. The above theorem says that $[M] \in T_*^7 = \mathbb{Z}_2[x_2, x_4, \dots, x_{126}]$.

Let $1 \le k < m \le 2^n$. For any $S \subset \mathcal{P}(\underline{n})$, $\#S = m$, we obtain an element $[G_{m,k}, \phi_S]$ in $Z_*(G)$, $G = (\mathbb{Z}_2)^n$. In general distinct choices of S do not necessarily lead to distinct elements $[G_{m,k}, \phi_S]$ in $Z_*(G)$. In fact we have the following rather amusing example.

Example 2.3. Let $m = 2^n - 2$, $1 \leq k < m$, k odd. Then for any $S \subset \mathcal{P}(\underline{n})$, $\#S = m$, one has $[G_{m,k}, \phi_S] = 0$ in $Z_*(G)$.

Proof. First note that $\mathcal{P}(\underline{n})$ has the structure of a Boolean algebra where addition is given by symmetric difference and multiplication, by intersection. For this reason, we write "$\lambda + \mu$" to mean $\lambda \Delta \mu$ in the proof. Let $S \subset \mathcal{P}(\underline{n})$, $\#S = 2^n - 2 = m$. Let α, β be elements of $\mathcal{P}(\underline{n})$ not in S. Let $\gamma = \alpha + \beta$. Then $\gamma \neq \varnothing$. It is easy to see that $f : \delta \mapsto \delta + \gamma$ defines a bijection of S onto itself such that $f^2 = \mathrm{Id}$, and f is fixed point free. It follows that f induces a bijection from the set $\binom{S}{k}$ of k-element subsets of S to itself, denoted by F, where $F(\{\alpha_1, \ldots, \alpha_k\}) = \{\alpha_1 + \gamma, \ldots, \alpha_k + \gamma\}$. Since k is odd, and $f : S \to S$ is fixed point free, it follows that F is fixed point free.

Recall that the stationary points for the G-action ϕ_S on $G_{m,k}$ are

$$E_\alpha = \langle e_{\alpha_1}, \ldots, e_{\alpha_k} \rangle,$$

where $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subset S$, $\#\alpha = k$. From Lam's [6] description of the tangent bundle of Grassmannians, it is easy to see that the tangential representation $T_\alpha G_{m,k}$ of G at E_α is isomorphic to

$$\bigoplus_{\substack{1 \leq i \leq k \\ 1 \leq j \leq m-k}} \mathbb{R}e_{\alpha_i} \otimes \mathbb{R}e_{\beta_j} \cong \bigoplus_{\substack{1 \leq i \leq k \\ 1 \leq j \leq m-k}} \mathbb{R}e_{\alpha_i + \beta_j},$$

where $S \backslash \alpha = \{\beta_1, \ldots, \beta_{m-k}\}$. Since $\alpha_i + \beta_j = \alpha_i + \gamma + \beta_j + \gamma = f(\alpha_i) + f(\beta_j)$ for any $\alpha_i, \beta_j \in S$, it is seen that $T_\alpha G_{m,k} \cong T_{F(\alpha)} G_{m,k}$, as G-modules. Since F has no fixed point, $\eta_*[G_{m,k}, \phi_S] = 0$. By Stong's theorem [11], it follows that $[G_{m,k}, \phi_S] = 0$.

3. Indecomposability

In this section we obtain a sufficient condition for indecomposability of an element in $Z_*(G)$, $G = (\mathbb{Z}_2)^n$. We apply our criterion to show the existence of indecomposable elements in the kernel $\mathcal{K}_*$ of the forgetful homomorphism $\varepsilon_* : Z_*(G) \to \mathcal{N}_*$. We make use of the equivariant characteristic numbers of a G-manifold constructed by tom Dieck [4].

Let $B = \mathbb{Z}_2[b_1, \ldots, b_m, \ldots]$ be the graded $\mathbb{Z}_2$-algebra with $\deg b_m = m$, for $m \geq 1$. Let L_* denote the B-algebra $B[[y_1, \ldots, y_n]]$ of formal power series in $y_1, \ldots, y_n$ with $\deg y_i = 1$, $1 \leq i \leq n$. Let K_* denote the fraction field of L_*. Recall that $R_*(G)$ is the polynomial algebra over $\widehat{G} = \mathrm{Hom}_{\mathbb{Z}_2}(G, \mathbb{Z}_2)$.

As usual we denote the generators of G by $t_1, \ldots, t_n$. For $A \subset \underline{n}$, denote by Y_A the irreducible representation of G given by the character $\chi_A \in \widehat{G}$. Here

$$\chi_A(t_i) = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}$$

Thus $R_*(G) \cong \mathbb{Z}_2[Y_A \mid A \subset \underline{n}]$. One has the subalgebra $\widetilde{R}_*(G)$ of $R_*(G)$ generated by $\{Y_A \mid A \subset \underline{n}, A \neq \varnothing\}$. Note that the algebra map $\eta_* : Z_*(G) \to R_*(G)$ actually has image in $\widetilde{R}_*(G)$, and that η_* is well known to be a monomorphism [11].

Let $\gamma : \widetilde{R}_*(G) \to K_*$ denote the $\mathbb{Z}_2$-algebra homomorphism defined by $\gamma(Y_A) = \frac{1}{y_A} \sum_{r \geq 0} b_r y_A^r \in K_*$, where $b_0 = 1$, $y_A = \sum_{i \in A} y_i$. Then tom Dieck [4] shows that $\gamma \circ \eta$ is a monomorphism and that $\mathrm{Im}(\gamma \circ \eta_*)$ is contained in L_*.

Let $\lambda = \lambda_1, \dots, \lambda_r$, $r \geq 0$, be a non-increasing sequence of natural numbers. We denote by b_λ the product $b_{\lambda_1} b_{\lambda_2} \cdots b_{\lambda_r} \in B$ ($b_\lambda = 1$ if $\lambda = (\ )$, the empty sequence). For any class $[M, \phi] \in Z_d(G)$, one can express $\gamma\eta([M, \phi])$ as $\sum_{|\lambda| \geq 0} \psi_\lambda([M, \phi]) \cdot b_\lambda$, with $\psi_\lambda([M, \phi]) \in \mathbb{Z}_2(y_1, \dots, y_n)$ being a homogeneous element of degree $|\lambda| - d$, where $|\lambda| = \sum_j \lambda_j$. The 'integrality theorem' of tom Dieck [4] implies that $\psi_\lambda([M, \phi]) = 0$ if $|\lambda| < d$, and that $\psi_\lambda([M, \phi]) \in \mathbb{Z}_2[y_1, \dots, y_n]$ if $|\lambda| \geq d$. One can regard ψ_λ as defining a $\mathbb{Z}_2$-linear map $\psi_\lambda : Z_d(G) \to \mathbb{Z}_2[y_1, \dots, y_n]$. When $|\lambda| = d$, $\psi_\lambda([M, \phi])$ coincides with the Stiefel-Whitney s-numbers $s_\lambda[M] \in \mathbb{Z}_2$ (cf. Kosniowski-Stong [5]). Just as the nonvanishing of $s_d(M)$ implies the indecomposability of $[M]$ in $\mathcal{N}_*$, one can expect that $\psi_k([M, \phi])$, k any integer greater than d, may detect the indecomposability of $[M, \phi] \in Z_*(G)$. In fact we have the following

Proposition 3.1. *Let $[M, \phi] \in Z_d(G)$. Suppose that for some $k > d$, either $\psi_k([M, \phi]) \neq 0$ or $\psi_{k-1,1}([M, \phi]) \neq 0$; then $[M, \phi] \in Z_*(G)$ is indecomposable.*

We need the following lemma to prove the above proposition.

Lemma 3.2. *Let $[M, \phi] = [M_1, \phi_1] \cdot [M_2, \phi_2] \in Z_d(G)$, and let $\lambda = \lambda_1, \dots, \lambda_r$. Then*

$$\psi_\lambda([M, \phi]) = \sum_{\lambda = \mu \cdot \nu} \psi_\mu([M_1, \phi_1]) \cdot \psi_\nu([M_2, \phi_2])$$

where $\mu \cdot \nu$ denotes juxtaposition of μ and ν arranged in the non-increasing order.

Proof of Lemma 3.2. This is a straightforward consequence of the fact that γ is an algebra homomorphism. Indeed we have

$$\begin{aligned}
\sum_\lambda \psi_\lambda([M, \phi]) b_\lambda = \gamma([M, \phi]) &= \gamma([M_1, \phi_1]) \cdot \gamma([M_2, \phi_2]) \\
&= \left(\sum_\mu \psi_\mu([M_1, \phi_1]) b_\mu \right) \left(\sum_\nu \psi_\nu([M_2, \phi_2]) b_\nu \right) \\
&= \sum_\mu \sum_\nu \psi_\mu([M_1, \phi_1]) \cdot \psi_\nu([M_2, \phi_2]) b_\mu b_\nu \\
&= \sum_\lambda \left\{ \sum_{\mu \cdot \nu = \lambda} \psi_\mu([M_1, \phi_1]) \cdot \psi_\nu([M_2, \phi_2]) \right\} b_\lambda.
\end{aligned}$$

Hence, comparing the coefficients of b_λ, we obtain

$$\psi_\lambda([M, \phi]) = \sum \psi_\mu([M_1, \phi_1]) \cdot \psi_\nu([M_2, \phi_2]) \in \mathbb{Z}_2[y_1, \dots, y_n].$$

□

Proof of Proposition 3.1. Let $\lambda = k - 1, 1$. If $\lambda = \mu \cdot \nu$, then either $\mu = 1$ of $\nu = 1$. Note that $Z_1(G) = 0$, so that for any $[N, \theta] \in Z_*(G)$, $\psi_1([N, \theta]) = 0$. By the above lemma it follows that $\psi_{k-1,1}$ vanishes on any decomposable element of $Z_*(G)$. Therefore $[M, \phi]$ is indecomposable if $\psi_{k-1,1}([M, \phi]) \neq 0$. The proof for the case $\lambda = k$ is similar. □

The following three examples will be used in the proof of Theorem 3.6.

Example 3.3. Let $3 \leq d = 2^k - 1 \leq n$. Let $S = \{\varnothing, \{1\}, \{2\}, \dots, \{d\}\}$. Then $\sigma = [P^d, \phi_S]$ is indecomposable in $Z_*(G)$. (Note that $[P^d] = 0$ in $\mathcal{N}_*$.)

We prove this by showing that $\psi_{d+2}([P^d, \phi_S]) \neq 0$. Let us write Y_i for $Y_{\{i\}}$, and Y_{ij} for $Y_{\{i,j\}}$, $1 \leq i, j \leq d$. A straightforward calculation shows that

$$\eta_*(\sigma) = \sum_{i=1}^{d} \left(\prod_{j=1, j\neq i}^{d} Y_{ij} \right) Y_i + \prod_{i=1}^{d} Y_i.$$

Let $D = \prod_{k=1}^{d} y_k \prod_{1\leq p<q\leq d}(y_p + y_q)$, and for $1 \leq i \leq d$, let

$$N_i = \sum_{p=1}^{d} y_p^{d+2} + \sum_{p=1, p\neq i}^{d} (y_i^{d+1} y_p + y_p^{d+1} y_i), \qquad C_i = \prod_{p=1, p\neq i}^{d} y_p \prod_{\substack{1\leq p<q\leq d \\ p\neq i, q\neq i}} (y_p + y_q),$$

$$N_0 = \sum_{p=1}^{d} y_p^{d+2}, \qquad C_0 = \prod_{1\leq p<q\leq d} (y_p + y_q).$$

Then $\psi_{d+2}(\sigma) = \frac{1}{D}(\sum_{j=0}^{d} N_j C_j)$ and so

$$D\psi_{d+2}(\sigma) = \sum_{j=0}^{d} N_j C_j. \tag{1}$$

One can show that the monomial $y_1^{d+1} y_2^{d} y_3^{d-2} \cdots y_j^{d-j+1} \cdots y_d$ occurs exactly once on the right hand side of (1). In fact it occurs in $N_1 C_1$ exactly once and in no other $N_j C_j$ $(j \neq 1)$. Hence $\psi_{d+2}(\sigma) \neq 0$. □

Example 3.4. Let $d \leq n$, d even. As before, let $S = \{\varnothing, \{1\}, \ldots, \{d\}\}$. Let $\sigma = [P^d, \phi_S] \in Z_*(G)$. Note that σ is indecomposable, since $\varepsilon_*(\sigma) = [P^d] \in \mathcal{N}_*$ is indecomposable. We claim that $\psi_N([P^d, \phi_S]) \neq 0$ for $N = 2^m$, m sufficiently large.

To see this, note that

$$\eta_*(\sigma) = \sum_{i=1}^{d} \left(\prod_{j=1, j\neq i}^{d} Y_{ij} \right) Y_i + \prod_{j=1}^{d} Y_j.$$

Let $D = \prod_{p=1}^{d} y_p \prod_{1\leq p<q\leq d}(y_p + y_q)$ and for $1 \leq i \leq d$, let

$$N_i = y_i^N + \sum_{p=1, p\neq i}^{d} (y_i + y_p)^N = \sum_{p=1, p\neq i}^{d} y_p^N,$$

as d is even and N is a power of 2, and

$$C_i = \prod_{p=1, p\neq i}^{d} y_p \prod_{\substack{1\leq p<q\leq d \\ p\neq i, q\neq i}} (y_p + y_q).$$

Let $N_0 = \sum_{p=1}^{d} y_p^N$, and $C_0 = \prod_{1\leq p<q\leq d}(y_p + y_q)$. Then $D \cdot \psi_N(\sigma) = \sum_{j=0}^{d} N_j C_j$. One can show that for N sufficiently large—say $N > \binom{d}{2}$—the monomial $y_1^{d-1} y_2^{d-2} \cdots y_{d-1} y_d^N$ occurs in $N_0 C_0$ exactly once and does not occur in any other $N_j C_j$. Hence the aforementioned monomial survives in $\psi_N(\sigma)$. □

Example 3.5. Let $S = \{\varnothing, \{1\}, \dots, \{k\}\}$, $T = \{\varnothing, \{1\}, \dots, \{l\}\}$, $l < k \le n$, k, l both even. Recall that one has an action $\phi_{T,S}$ of G on the Milnor manifold $H_{l,k}$. Let $\sigma = [H_{l,k}, \phi_{T,S}] \in Z_*(G)$. Further assume that $\binom{k+l}{k} \equiv 1 \pmod 2$, so that $[H_{l,k}] \in \mathcal{N}_*$ is indecomposable. We claim that for $N = 2^\nu$ sufficiently large, $\psi_{N+1}(\sigma) \neq 0$.

To establish our claim, first note that the normal bundle to the imbedding $H_{l,k} \overset{i}{\hookrightarrow} P^l \times P^k$ is $i^*(\xi_l \otimes \xi_k)$, where ξ_l is the pull-back of the Hopf line bundle over P^l via the projection $P^l \times P^k \to P^l$. Using this it is easy to see that

$$\eta_*(\sigma) = \sum_{\substack{0 \le i \le l \\ 0 \le j \le k \\ i \ne j}} \left(\prod_{p=0, p\neq i}^{l} Y_{ip} \prod_{q=0, q\neq i,j}^{k} Y_{jq} \right),$$

with the convention that $Y_{0p} = Y_{\{p\}} = Y_{p0}$. Let $D = \prod_{p=1}^k y_p \prod_{1 \le p < q \le k} (y_p + y_q)$. For $1 \le i \le l$, $1 \le j \le k$, and $i \neq j$, write

$$N_{ij} = \sum_{p=l+1}^{k} y_p^{N+1} + \sum_{p=1, p\neq i}^{l} (y_i^N y_p + y_i y_p^N) + \sum_{q=1, q\neq i,j}^{k} (y_j^N y_q + y_j y_q^N),$$

$$C_{ij} = \prod_{p=1, p\neq i,j}^{k} y_p \cdot \prod_{\substack{1 \le p < q \le k \\ q \neq i,j,\ p \neq i,j}} (y_p + y_q) \cdot \prod_{p=l+1}^{k} (y_i + y_p).$$

For $1 \le j \le k$, let

$$N_{0j} = \sum_{p=l+1}^{k} y_p^{N+1} + \sum_{p=1, p\neq j}^{k} (y_j^N y_p + y_j y_p^N),$$

$$C_{0j} = \prod_{\substack{1 \le p < q \le k \\ p \neq j,\ q \neq j}} (y_p + y_q) \cdot \prod_{p=l+1}^{k} y_p.$$

For $1 \le i \le l$, let

$$N_{i,0} = \sum_{p=l+1}^{k} y_p^{N+1} + \sum_{p=1, p\neq i}^{l} (y_i^N y_p + y_i y_p^N),$$

$$C_{i0} = \prod_{\substack{1 \le p < q \le k \\ p \neq i, q \neq i}} (y_p + y_q) \cdot \prod_{q=l+1}^{k} (y_i + y_q).$$

Then a routine verification shows that

$$D\psi_{N+1}(\sigma) = \sum_{\substack{0 \le i \le l \\ 0 \le j \le k \\ i \ne j}} N_{ij} C_{ij}. \tag{2}$$

Consider the monomial $y_1 y_2^{k-2} y_3^{k-3} \cdots y_{l-1}^{k-l+1} y_l^{k-l+1} y_{l+1}^{k-l} \cdots y_{k-2}^3 y_{k-1}^2 y_k^N$. We claim that this monomial survives in the right hand side of (2) for sufficiently large N—say $N > \binom{k}{2}$, $N = 2^\nu$. First note that since N is large, the only monomials which involve y_k^N must come from the terms $N_{ik}C_{ik}$, $1 \le i \le l$. In N_{ik} the only terms which contribute to the above monomial are $y_k^N(\sum_{p=1, p\neq i}^{k-1} y_p)$. It is not difficult to show that the above monomial occurs $(l-1)$-times in $y_k^N(\sum_{p=2}^{k-1} y_p)C_{1k}$, and that it does not occur in $y_k^N(\sum_{p=1, p\neq i}^{k-1} y_p)C_{ik}$ for $i > 1$. Since l is even, it follows that the above monomial survives in $D\psi_{N+1}(\sigma)$. Hence $\psi_{N+1}(\sigma) \neq 0$. □

Let $\mathcal{D}$ denote the ideal of decomposable elements in $Z_*(G)$. Let $\overline{\mathcal{K}}_m$ denote the $\mathbb{Z}/2$-vector space $\mathcal{K}_m/(\mathcal{K}_m \cap \mathcal{D})$. The dimension of $\overline{\mathcal{K}}_m$ is the number of $\mathbb{Z}/2$-linearly independent indecomposable elements in $\mathcal{K}_m$.

Theorem 3.6. *Let $2^{s-1} < n \le 2^s$, $n \ge 3$. Let $d = \dim_{\mathbb{Z}/2} \overline{\mathcal{K}}_m$. Then*
(i) *$d \ge n - 2^r + 2$, when $m = 2^r - 1$, $m \le n$,*
(ii) *$d \ge n - 2^r$, when $2^r < m + 1 < 2^{r+1} \le 2^s$, and m is odd,*
(iii) *$d \ge n - m$, when m is even, $2 \le m < n$.*

Proof. We shall only prove (ii), proofs for other parts being exactly analogous, and make use of Examples 3.3 and 3.4. Let m be odd and $2^r < m+1 < 2^{r+1}$, $r \le s-1$. Then $[H_{l,k}] \in \mathcal{N}_*$ is indecomposable where $k = 2^r$, $l = m+1-k$. Note that $l \ge 2$, and that $k < n$. Now let $\sigma = [H_{l,k}, \phi_{S,T}]$ be as in Example 3.5. By Example 3.5, $\psi_{N+1}(\sigma) \in \mathbb{Z}_2[y_1, \ldots, y_k]$ is not zero. Suppose that $\psi_{N+1}(\sigma) = \sum_{r\ge 0} P_r y_t^r$, with $1 \le t \le k$, $P_r \in \mathbb{Z}_2[y_1, \ldots, y_{t-1}, y_{t+1}, \ldots, y_k]$, is the expression for $\psi_{N+1}(\sigma)$ as a polynomial in y_t with positive degree.

Now, for $1 \le j \le n - 2^r$, let $A_j = \{\varnothing, \{1\}, \ldots, \{t-1\}, \{t+1\}, \ldots, \{k\}, \{k+j\}\}$, and let

$$B_j = \begin{cases} T & \text{if } l < t, \\ T \cup \{k+j\} \backslash \{t\} & \text{if } t \le l. \end{cases}$$

Then, writing $\sigma_j = [H_{l,k}, \phi_{A_j,B_j}]$, we see that $\psi_{N+1}(\sigma_j) = \sum_{r\ge 0} P_r y_{k+j}^r$. Therefore, $\psi_{N+1}(\sigma + \sigma_j) = \sum P_r(y_t^r + y_{k+j}^r) \neq 0$. Hence by Proposition 3.1, it follows that $\sigma + \sigma_j$ is indecomposable. Clearly $\varepsilon_*(\sigma + \sigma_j) = 2[H_{l,k}] = 0$. Therefore $\sigma + \sigma_j \in \mathcal{K}_m$. Writing $u_j = \sigma + \sigma_j$, for $1 \le j \le n - 2^r$, we see that for any sequence of $1 \le j_1 < \cdots < j_p \le n - 2^r$, one has $\psi_{N+1}(u_{j_1} + \cdots + u_{j_p}) = \sum_{1\le q\le p} \psi_{N+1}(u_{j_q}) = \sum P_r(py_t^r + \sum_{1\le q\le p} y_{k+j_q}^r) \neq 0$. This proves that $u_1, \ldots, u_{n-2^r}$ are linearly independent in $\overline{K}_m$, completing the proof. □

Remark 3.7. Note that in the course of the above proof one could as well work with the smallest integer M such that $\psi_M(\sigma) \neq 0$ in the place of the integer $N+1$.

The above theorem suggests the following conjecture. Note that the group $\mathrm{Aut}(\mathbb{Z}_2)^n \cong SL_n(\mathbb{Z}_2)$ acts on $Z_*(G)$ as $\mathbb{Z}_2$-algebra automorphisms. Indeed if $w \in SL_n(\mathbb{Z}_2)$, then $w([M, \phi]) = [M, \phi^w]$, where $\phi^w(t, x) = \phi(w(t), x)$ for all $x \in M$. In particular, let $\sigma \in Z_d(G)$ be indecomposable. Then so is $w(\sigma)$, and $\sigma + w(\sigma) \in \mathcal{K}_d$.

Conjecture. If $w(\sigma) \neq \sigma$, then $\sigma + w(\sigma)$ is an indecomposable element in $\mathcal{K}_d$.

Let $u_1, u_2, \ldots, u_k$ be homogeneous elements in $Z_*(G)$. Assume that there exists an integer N_i such that $f_i := \psi_{N_i}(u_i) \neq 0$. We may suppose that N_i is the smallest such integer. By relabelling the u_i's if necessary, we assume that $N_1 \le N_2 \le \cdots \le$

N_k. Let $i_0 = 0 < i_1 < \cdots < i_p = k$ be such that $N_j = N_{i_r}$ for $i_{r-1}+1 \leq j \leq i_r$, $1 \leq r \leq p$.

Proposition 3.8. *With the above notation, assume that $f_1, \ldots, f_k \in \mathbb{Z}_2[y_1, \ldots, y_n]$ are algebraically independent. Then $u_1, \ldots, u_k$ are algebraically independent.*

Proof. Suppose that $P(u_1, \ldots, u_k) = 0$ where P is a homogeneous polynomial of degree d. Then we can write P as $P = \sum \varepsilon_{\underline{r}} u_1^{r_1} \cdots u_k^{r_k}$ where the sum is over all sequences $\underline{r} = (r_1, \ldots, r_k) \in \mathbb{N}^k$ with $\sum_{i=1}^k r_i |u_i| = d$, $\varepsilon_{\underline{r}} \in \{0,1\}$, $\mathbb{N}$ being the set of non-negative integers. Let $|\underline{r}| = r_1 + \cdots + r_k$. Let $S = \{\underline{r} \mid \varepsilon_{\underline{r}} = 1\}$, and let $s = \min\{|\underline{r}| \mid \underline{r} \in S\}$. Note that $P = \sum_{\underline{r} \in S} u_1^{r_1} \cdots u_k^{r_k}$. For $\underline{r} \in S$, let $\underline{r}^j = r_{i_{j-1}+1} + \cdots + r_{i_j}$, $1 \leq j \leq p$. Let $S_p = \{\underline{r} \in S \mid |\underline{r}| = s\}$, and let $s_p = \min\{\underline{r}^p \mid \underline{r} \in S_p\}$. Similarly, let $S_{p-1} = \{\underline{r} \in S_p \mid \underline{r}^p = s_p\}$, and set $s_{p-1} = \min\{\underline{r}^{p-1} \mid \underline{r} \in S_{p-1}\}$. Having defined $S_p, S_{p-1}, \ldots, S_{p-j+1}$ and $s_p, s_{p-1}, \ldots, s_{p-j+1}$, we define S_{p-j} and s_{p-j} as $S_{p-j} = \{\underline{r} \in S_{p-j+1} \mid \underline{r}^{p-j+1} = s_{p-j+1}\}$ and $s_{p-j} = \min\{\underline{r}^{p-j} \mid \underline{r} \in S_{p-j}\}$. Note that S_1 consists precisely of those $\underline{r} \in S_p$ such that the sequence $(\underline{r}^p, \ldots, \underline{r}^1) = (s_p, \ldots, s_1)$ is the smallest in the lexicographic ordering of $\mathbb{N}^p$, as $\underline{r}$ varies in S_p. Now let

$$\lambda = N_{i_1}^{s_1} \ldots N_{i_p}^{s_p} = \underbrace{N_{i_1}, \ldots, N_{i_1}}_{s_1}, \cdots, \underbrace{N_{i_p}, \ldots, N_{i_p}}_{s_p}.$$

Claim. For $\underline{r} \in S$,

$$\psi_\lambda(u_1^{r_1} \cdots u_k^{r_k}) = \begin{cases} f_1^{r_1} \cdots f_k^{r_k} & \text{if } \underline{r} \in S_1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $\psi_\lambda(u_1^{r_1} \cdots u_k^{r_k}) = 0$ if $|\underline{r}| > s = |\lambda|$. Also it is trivial to see that $\psi_{N_{i_j}^{s_j}}(u_{i_{j-1}+1}^{r_{j-1}+1} \cdots u_{i_j}^{r_j}) = f_{i_{j-1}+1}^{r_{j-1}+1} \cdots f_{i_j}^{r_j}$, $1 \leq j \leq p$. Furthermore $\psi_{M_1 \cdots M_q}(u_{i_{j-1}+1}^{r_1} \cdots u_{i_j}^{r_q}) = 0$, $q = i_j - i_{j-1}$, if some $M_t < N_{i_j}$ by minimality of N_j. By Lemma 3.2 the claim follows. To complete the proof of the lemma, we apply ψ_λ to $P(u_1, \ldots, u_k) = 0$. By the claim above

$$0 = \psi_\lambda(P(u_1, \ldots, u_k)) = \sum_{\underline{r} \in S_1} \psi_\lambda(u_1^{r_1} \cdots u_k^{r_k}) = \sum_{\underline{r} \in S_1} f_1^{r_1} \cdots f_k^{r_k}$$

which contradicts the hypothesis that the f_i's are algebraically independent. ☐

Let $u_1, \ldots, u_r$ be homogeneous elements in $\mathcal{K}_*$ constructed as in the proof of Theorem 3.6. By Remark 3.7 there exist M_i smallest such that $\psi_{M_i}(u_i) \neq 0$. Write $\psi_{M_i}(u_i) = P_i(y_1, \ldots, y_{k_i})$ where P_i involves y_{k_i}.

Theorem 3.9. *With the above notation, suppose $u_1, \ldots, u_r \in \mathcal{K}_*$ are such that $1 \leq k_1, \ldots, k_r \leq n$ are all distinct. Then $u_1, \ldots, u_r$ are algebraically independent.*

Proof. The theorem follows immediately from the above proposition as $\psi_{M_i}(u_i) = P_i(y_1, \ldots, y_{k_i})$, $i = 1, \ldots, r$, are clearly algebraically independent. ☐

In particular, it follows from Theorem 3.6 that $\mathcal{K}_m$ has at least r elements which generate a polynomial algebra on r variables where

(i) $r = n - 2^q + 2$, when $m = 2^q - 1$, $m \leq n$,
(ii) $r = n - 2^q$, when $2^q < m + 1 < 2^{q+1} \leq 2^s$, and m is odd,
(iii) $r = n - m$, when m is even, $2 \leq m < n$.

We conclude this section with the following questions. Let as usual $G = (\mathbb{Z}_2)^n$.

Question 1. Is $Z_*(G)$ finitely generated as a $\mathbb{Z}_2$-algebra?

Question 2. Are there indecomposable elements in $Z_*(G)$ beyond dimension $2^n - 2$?

4. Tangential representations

An important step in determining the structure of $Z_*(\mathbb{Z}_2^2)$ is the following observation of Conner and Floyd (Lemma 32.3, [1]). Let (M, ϕ) be a smooth closed $\mathbb{Z}_2 \times \mathbb{Z}_2$-manifold, and let $\alpha \subset \{1, 2\}$. If Y_α occurs in the representation T_xM with multiplicity p, at some stationary point x. Then it occurs with the same multiplicity at an *even* number of stationary points. We shall show in this section that this phenomenon happens for any $(\mathbb{Z}_2)^n$-manifold with finite stationary point set.

Theorem 4.1. *Let $n \geq 2$ and (M, ϕ) be a closed $(\mathbb{Z}_2)^n$-manifold with finite stationary point set. Suppose that for some stationary point x, and $\alpha \subset \underline{n}$, Y_α occurs with multiplicity p in T_xM. Then it occurs with the same multiplicity at an even number of stationary points.*

Proof. Let $d = \dim M$. We use the notation of section 3. Let $V = Y_{\alpha_1}^{p_1} \cdots Y_{\alpha_k}^{p_k}$, $\sum p_i = d$, $\alpha_1, \ldots, \alpha_k$ distinct. Then

$$\gamma(V) = \frac{1}{y_{\alpha_1}^{p_1} \cdots y_{\alpha_k}^{p_k}} (1 + b_1 y_{\alpha_1} + b_2 y_{\alpha_1}^2 + \cdots)^{p_1} \cdots (1 + b_1 y_{\alpha_k} + b_2 y_{\alpha_k}^2 + \cdots)^{p_k}.$$

Let $r < d$. The coefficient $C_{d-r}(V)$ of b_1^{d-r} in the expression for $\gamma(V)$ can be calculated as follows: If $p_1 =: p \geq r$, then

$$C_{d-r}(V) = \frac{\binom{p}{r}}{y_{\alpha_1}^r} + P_{\alpha_1}(V)$$

where $P_{\alpha_1}(V)$ is a polynomial in $(1/y_{\alpha_1})$ of degree *less* than r with coefficients in $\mathbb{Z}_2[1/y_{\alpha_j} \mid 2 \leq j \leq k]$. If $p < r$, then $C_{d-r}(V) = Q_{\alpha_1}(V)$, a polynomial in $(1/y_{\alpha_1})$ of degree *less* than r with coefficients in $\mathbb{Z}_2[1/y_{\alpha_j} \mid j \neq 1]$. Let $S_{\alpha,r} = \{x \in S \mid Y_\alpha$ occurs with multiplicity $r\}$. Fix $\alpha = \alpha_1$. We need to show that $|S_{\alpha,r}|$ is even for each $r, 1 \leq r \leq d$. Now suppose that $r = s$ is the largest integer such that Y_α occurs with multiplicity s, among T_xM, as x varies over the (finite) set S of stationary points. One knows that $s < d$. Then

$$0 = \underbrace{\psi_{1,\ldots,1}}_{d-s}(M, \phi) = \sum_{x \in S} C_{d-s}(T_xM). \tag{3}$$

Multiplying both sides of (3) by y_α^s we obtain

$$0 = \sum_{x \in S_{\alpha,s}} (1 + y_\alpha^s P_\alpha(T_xM)) + \sum_{x \in S \setminus S_{\alpha,s}} y_\alpha^s Q_\alpha(T_xM). \tag{4}$$

Note that $y_\alpha^s P_\alpha(T_xM)$, $y_\alpha^s Q_\alpha(T_xM)$ are polynomials in y_α without constant terms having coefficients in $\mathbb{Z}_2[1/y_\beta \mid \beta \neq \alpha]$. Clearing the denominators in (4), we obtain an expression

$$0=\left(\sum_{x\in S_{\alpha,s}}1\right)y_{\beta_1}^{k_1}\cdots y_{\beta_t}^{k_t}+\left(\sum_{x\in S_{\alpha,s}}y_\alpha^s P_\alpha(T_xM)\right)y_{\beta_1}^{k_1}\cdots y_{\beta_t}^{k_t} + \left(\sum_{x\in S\setminus S_{\alpha,s}}y_\alpha^s Q_\alpha(T_xM)\right)y_{\beta_1}^{k_1}\cdots y_{\beta_t}^{k_t}$$

in $\mathbb{Z}_2[y_1,\dots,y_n]$, where $\beta_1,\dots,\beta_t\in\mathcal{P}(\underline{n})$, $\alpha\neq\beta_i$, $1\le i\le t$, $k_1,\dots,k_t>0$. Put $y_\alpha=0$ in the above expression. Then we obtain $0=|S_{\alpha,s}|\cdot y_{\beta_1}^{k_1}\cdots y_{\beta_t}^{k_t}$ in $\mathbb{Z}_2[y_1,\dots,y_n]/\langle y_\alpha\rangle$. As $\mathbb{Z}_2[y_1,\dots,y_n]/\langle y_\alpha\rangle$ is again a polynomial algebra in $(n-1)$-variables, we see that $|S_{\alpha,s}|$ must be even.

Inductively assume that Y_α^p occurs in T_xM, if at all, at an even number of stationary points for any $d>p>r$. We must show that Y_α^r occurs in T_xM, if at all, for an even number of stationary points. Proceeding exactly as before we see that

$$0=\sum_{p\ge r}\left(\sum_{x\in S_{\alpha,p}}\binom{p}{r}+y_\alpha^r P_\alpha(T_xM)\right)+\sum_{x\in S\setminus\bigcup_{p\ge r}S_{\alpha,p}}y_\alpha^r Q_\alpha(T_xM).$$

Again note that $y_\alpha^r P_\alpha(T_xM)$ and $y_\alpha^r Q_\alpha(T_xM)$ are polynomials in y_α without constant terms and having coefficients in $\mathbb{Z}_2[1/y_\beta\mid\beta\neq\alpha]$. As before we see that $\sum_{p\ge r}|S_{\alpha,p}|\binom{p}{r}=0 \pmod 2$. Hence $|S_{\alpha,r}|\equiv\sum_{p>r}|S_{\alpha,p}|\binom{p}{r}\equiv 0 \pmod 2$, since by the induction hypothesis each $|S_{\alpha,p}|$, $p>r$, is even.

Remark 4.2. It may be noted that our method can be used to interpret the vanishing of $\psi_\lambda(M,\phi)$ for $|\lambda|<d=\dim M$ in terms of the irreducible representations Y_α, $\alpha\subset\underline{n}$, which occur in T_xM as x varies in the (finite) set of stationary points of (M,ϕ).

References

1. P. E. Conner, *Differentiable periodic maps*, 2nd Ed., L.N.M. (738) Springer-Verlag, 1979. MR **81f:**57018
2. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse Sr-33, Springer-Verlag, 1964. MR **31:**750
3. T. tom Dieck, *Fixpunkte vertauschbarer involutionen*, Archiv der math., 20, 295–298, 1969. MR **42:**3798
4. ______, *Characteristic numbers of G manifolds*. I, Invent. Math. 13, 213–224, 1971. MR **46:**8236
5. C. Kosniowski and R. E. Stong, $(\mathbb{Z}_2)^k$*-Actions and Characteristic numbers*, Indiana Univ. Math. J., 28, 725–743, 1979. MR **81d:**57027
6. K. Y. Lam, *A formula for the tangent bundle of flag manifolds and related manifolds*, Trans. Amer. Math. Soc., 213, 305–314, 1975. MR **55:**4196
7. J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Ann. Math. Stud., 76, Princeton, 1974. MR **55:**13428
8. G. Mukherjee, *Equivariant cobordism of Grassmann and flag manifolds*, Proc. Ind. Acad. Sci., 105, 381–391, 1995. MR **97g:**57045
9. P. Sankaran, *Determination of Grassmann manifolds which are boundaries*, Canad. Math. Bull., 34, 119–122, 1991. MR **92h:**57049
10. P. Sankaran and K. Varadarajan, *Group actions on flag manifolds and cobordism*, Canad. J. Math., 45, 650–661, 1993. MR **94k:**57050

11. R. E. Stong, *Equivariant bordism and* $(\mathbb{Z}_2)^k$*-actions*, Duke Math. J. 37, 779–785, 1970. MR **42:**6847
12. R. E. Stong (reviewer). MR **89d:**57050

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