

A REPRESENTATION THEOREM FOR SCHAUDER BASES IN HILBERT SPACE

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ABSTRACT. A sequence of vectors $\{f_1, f_2, f_3, \dots\}$ in a separable Hilbert space H is said to be a Schauder basis for H if every element $f \in H$ has a unique norm-convergent expansion

$$f = \sum c_n f_n.$$

If, in addition, there exist positive constants A and B such that

$$A \sum |c_n|^2 \leq \left\| \sum c_n f_n \right\|^2 \leq B \sum |c_n|^2,$$

then we call $\{f_1, f_2, f_3, \dots\}$ a Riesz basis. In the first half of this paper, we show that every Schauder basis for H can be obtained from an orthonormal basis by means of a (possibly unbounded) one-to-one positive self adjoint operator. In the second half, we use this result to extend and clarify a remarkable theorem due to Duffin and Eachus characterizing the class of Riesz bases in Hilbert space.

1. THE MAIN THEOREM

A sequence of vectors $\{f_1, f_2, f_3, \dots\}$ in a separable Hilbert space H is said to be a Schauder basis for H if every element $f \in H$ has a unique norm-convergent expansion

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If, in addition, there exist positive constants A and B such that

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then we call $\{f_1, f_2, f_3, \dots\}$ a Riesz basis. Riesz bases have been studied intensively ever since Paley and Wiener first recognized the possibility of nonharmonic Fourier expansions

$$f = \sum c_n e^{i\lambda_n t}$$

for functions f in $L^2(-\pi, \pi)$ (see, for example, [2] and [5] and the references therein).

In the first half of this paper, we show that every Schauder basis for H can be obtained from some orthonormal basis by means of a (possibly unbounded) one-to-one positive self adjoint operator. In the second half, we use this result to extend

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and clarify a remarkable theorem due to Duffin and Eachus [1] characterizing the class of Riesz bases in Hilbert space.

Theorem 1. *Let H be a separable Hilbert space and $\{f_n\}$ a Schauder basis for H . Then there exists an orthonormal basis $\{e_n\}$ and a one-to-one positive self adjoint transformation T of the Hilbert space such that $T: e_n \rightarrow f_n$ for $n = 1, 2, 3, \dots$.*

Proof. Define a linear transformation S on H by setting

$$S(x) = \sum_{n=1}^{\infty} (x, f_n) f_n$$

for those points x in H for which the series converges *weakly*. If $\{f_n\}$ forms a conditional basis for the Hilbert space, then S may not be bounded and may not even be defined for all elements of the Hilbert space. Let us use the symbols $D(T)$ and $R(T)$ to designate the domain and range, respectively, of a linear transformation T .

We are going to show that S is self adjoint, positive, and one-to-one. In order to define the adjoint transformation S^* , we must first show that $D(S)$ is everywhere dense in H . Let $\{g_n\}$ be the system in H biorthogonal to $\{f_n\}$, so that $(f_n, g_m) = \delta_{nm}$. Then $S: g_n \rightarrow f_n$ ($n = 1, 2, 3, \dots$) and therefore every finite linear combination $c_1 g_1 + \dots + c_n g_n$ also belongs to $D(S)$. Since the system biorthogonal to a Schauder basis is itself a Schauder basis (see [5], p. 29), the set of all such combinations forms a dense subspace of H , and the adjoint of S can be defined.

It follows at once from the definition that S is symmetric,

$$(Sx, y) = (x, Sy)$$

for every x and y in $D(S)$, and hence that $D(S^*) \supseteq D(S)$. Now the domain of S^* consists, by definition, of all y for which (Sx, y) is continuous for x in $D(S)$. Since $D(S)$ is dense, there is a uniquely determined point y^* in H such that $(Sx, y) = (x, y^*)$ for every x in $D(S)$. Accordingly,

$$\sum_{n=1}^{\infty} (x, f_n) (f_n, y) = (x, y^*)$$

whenever $x \in D(S)$ and $y \in D(S^*)$. It is crucial to show that the series on the left converges for *all* x in the Hilbert space. For this purpose, take $x = g_n$ in the expansion above, so that $(f_n, y) = (g_n, y^*)$ for $n = 1, 2, 3, \dots$. Since $\{f_n\}$ is a Schauder basis, the biorthogonal expansion $y^* = \sum (y^*, g_n) f_n$ is valid in norm, and hence weakly. Thus, the series $\sum (y^*, g_n) (f_n, x)$ is convergent for every x in H and y in $D(S^*)$ and hence for these values of x and y ,

$$\sum_{n=1}^{\infty} (y, f_n) (f_n, x)$$

is also convergent. This shows that $\sum (y, f_n) f_n$ converges weakly whenever $y \in D(S^*)$, and hence that $D(S^*) \subseteq D(S)$. Thus, $D(S) = D(S^*)$ and S is self adjoint.

To see that S is one-to-one, suppose that $S(x) = 0$, so that

$$\sum_{n=1}^{\infty} (x, f_n) f_n = 0 \quad (\text{weakly}).$$

Taking the inner product of both sides with g_n ($n = 1, 2, 3, \dots$), we see that $(x, f_n) = 0$ for $n = 1, 2, 3, \dots$, and, since the f_n are complete, x must be zero. Thus, S is one-to-one. That S is positive follows at once since

$$(Sx, x) = \sum_{n=1}^{\infty} |(x, f_n)|^2 \geq 0.$$

Now, the null space of an operator is the orthogonal complement of the range of its adjoint, and since S is one-to-one and self adjoint, its range is dense in H . Therefore, S^{-1} is also one-to-one, positive, and self adjoint. It follows that there exists a unique one-to-one, positive, self adjoint transformation A such that

$$A^2 = S^{-1}$$

(see [3], p. 265). It is to be shown that the system $\{e_n\}$ defined by

$$e_n = A(f_n) \quad (n = 1, 2, 3, \dots)$$

forms an orthonormal basis for H .

That the e_n are well-defined is clear, since

$$f_n \in R(S) = D(S^{-1}) \subset D(A).$$

Orthonormality follows at once from

$$\begin{aligned} (e_n, e_m) &= (Af_n, Af_m) = (f_n, A^2 f_m) \\ &= (f_n, S^{-1} f_m) = (f_n, g_m) = \delta_{nm}. \end{aligned}$$

To establish completeness, let x be a fixed but arbitrary element in $D(S^{-1})$. Then $S^{-1}(x) \in R(S^{-1}) = D(S)$, and hence we can write

$$\begin{aligned} x &= S(S^{-1}x) = \sum_1^{\infty} (S^{-1}x, f_n) f_n \\ &= \sum_1^{\infty} (Ax, Af_n) f_n = \sum_1^{\infty} (Ax, e_n) f_n \end{aligned}$$

where each of the series is weakly convergent to x . If x_n denotes the n th partial sum of the last series, then

$$Ax_n = \sum_1^n (Ax, e_m) e_m \rightarrow P(Ax),$$

where P denotes the orthogonal projection onto the subspace generated by the e_n ($n = 1, 2, 3, \dots$). Since every self adjoint transformation is closed, the relations

$$\begin{aligned} x_n &\rightarrow x \text{ weakly,} \\ Ax_n &\rightarrow P(Ax) \end{aligned}$$

imply that $P(Ax) = Ax$. Thus, P reduces to the identity transformation on a dense subspace of H and hence on all of H . This shows that the system $\{e_n\}$ is complete and so forms an orthonormal basis for H . The required transformation T is obtained by taking $T = A^{-1}$. This completes the proof of the theorem. \square

2. ON A CLASSIC CHARACTERIZATION OF RIESZ BASES

Let us now turn our attention to the class of Riesz bases in a separable Hilbert space H . If $\{f_n\}$ is such a basis, then for *any* orthonormal basis $\{e_n\}$, the mapping

$$T: e_n \rightarrow f_n \quad (n = 1, 2, 3, \dots)$$

can be extended to a bounded invertible operator on all of H . Riesz bases constitute the largest and most tractable class of bases known. It is extremely difficult to exhibit at least one bounded basis for a Hilbert space that is not equivalent to an orthonormal basis. An example in $L^2(-\pi, \pi)$ was discovered by Babenko (see [4], p. 351).

The simplest and perhaps the most obvious way of ensuring that a bounded linear transformation T is invertible is by showing that it is “close” to the identity transformation I in the sense that $\|I - T\| < 1$. This is a stringent requirement to place on a linear operator and one might well suppose that the class of Riesz bases is very small. A remarkable theorem due to Duffin and Eachus [1] shows, surprisingly, that just the opposite is true: *Every Riesz basis for H is obtained in essentially this way.*

Theorem 2. *Let H be a separable Hilbert space and $\{f_n\}$ a Riesz basis for H . Then there exists an orthonormal basis $\{e_n\}$ and a positive number ρ such that the mapping*

$$T: e_n \rightarrow f_n \quad (n = 1, 2, 3, \dots)$$

can be extended to a bounded invertible operator on all of H such that

$$\|I - \rho T\| < 1.$$

Let us briefly review the construction of the orthonormal basis $\{e_n\}$.

Start with any orthonormal basis $\{\varphi_n\}$ for H and let V be the unique bounded invertible operator on H that maps $\varphi_n \rightarrow f_n$. Factorize V by writing $V = PU$, where P is a positive operator and U is a unitary operator. Then $e_n = U(\varphi_n)$, $n = 1, 2, 3, \dots$, is the desired basis.

We are going to show that the basis $\{e_n\}$ so constructed is independent of the choice of $\{\varphi_n\}$ and that it is, in fact, the same basis guaranteed by Theorem 1.

We have

$$V(x) = \sum (x, \varphi_n) f_n$$

and

$$V^*(x) = \sum (x, f_n) \varphi_n,$$

and hence

$$\begin{aligned} VV^*(x) &= \sum_n \sum_m (x, f_m) (\varphi_m, \varphi_n) f_n \\ &= \sum_n (x, f_n) f_n = S(x), \end{aligned}$$

where S is the same operator used in the proof of Theorem 1. We shall continue to use the symbol A to designate the positive square root of S^{-1} . Since $\{f_n\}$ is a Riesz basis, both A and S are bounded invertible operators defined on all of H .

Using the canonical factorization $V = PU$, we find $V^* = U^*P^* = U^{-1}P$, and hence

$$S = VV^* = P^2.$$

Thus,

$$P^{-1} = S^{-1/2} = A$$

and

$$U = P^{-1}V = AV.$$

Since $V: \varphi_n \rightarrow f_n$, it follows that

$$U(\varphi_n) = A(f_n) \quad (n = 1, 2, 3, \dots).$$

Thus the two algorithms of Theorems 1 and 2 generate the same orthonormal basis.

The next theorem shows that this distinguished basis is “closest” to $\{f_n\}$ in a certain precise sense.

Theorem 3. *Let $\{f_n\}$ be a Riesz basis for a separable Hilbert space H and let $\{e_n\}$ be the distinguished orthonormal basis guaranteed by Theorems 1 and 2. If T denotes the operator on H for which*

$$T: e_n \rightarrow f_n \quad (n = 1, 2, 3, \dots),$$

then there exists a positive number ρ with the following property: If $\{h_n\}$ is any other orthonormal basis for H and G the operator for which

$$G: h_n \rightarrow f_n \quad (n = 1, 2, 3, \dots),$$

then

$$\|I - \lambda G\| \geq \|I - \rho T\|$$

for any positive real number λ .

Proof. Since $\{f_n\}$ is a Riesz basis, we know from Theorem 1 that the mapping

$$S(x) = \sum_{n=1}^{\infty} (x, f_n) f_n$$

defines a positive self adjoint invertible operator on all of H and that T is the positive square root of S . If $\alpha = 1/\|S^{-1}\|$ and $\beta = \|S\|$, then

$$\alpha I \leq S \leq \beta I.$$

Here, as usual, the order relation $A \leq B$ between symmetric operators A and B means that $(Ax, x) \leq (Bx, x)$ for all elements x of H . We define

$$\rho = \frac{2}{\sqrt{\alpha} + \sqrt{\beta}}.$$

Select an arbitrary orthonormal basis $\{h_n\}$ for H and let G be the operator for which $G(h_n) = f_n$ ($n = 1, 2, 3, \dots$). Then for any x in H , $x = \sum (x, h_n) h_n$ and hence

$$G(x) = \sum_{n=1}^{\infty} (x, h_n) f_n$$

and

$$G^*(x) = \sum_{n=1}^{\infty} (x, f_n) h_n,$$

so that

$$GG^* = \sum_n \sum_m (x, f_m)(h_m, h_n) f_n = \sum_n (x, f_n) f_n = S(x).$$

We can now calculate $\|G\|$ and $\|G^{-1}\|$. We have

$$\begin{aligned} \|G\|^2 &= \|G^*\|^2 = \sup_{\|x\|=1} (G^*x, G^*x) \\ &= \sup_{\|x\|=1} (GG^*x, x) = \sup_{\|x\|=1} (Sx, x) = \|S\|, \end{aligned}$$

and hence $\|G\| = \sqrt{\beta}$. A similar calculation applied to G^{-1} shows that $\|G^{-1}\| = 1/\sqrt{\alpha}$.

Let ε be an arbitrary positive number and select unit vectors x_1 and x_2 such that

$$\sqrt{\beta} - \varepsilon \leq \|G(x_1)\| \leq \sqrt{\beta}$$

and

$$\sqrt{\alpha} \leq \|G(x_2)\| \leq \sqrt{\alpha} + \varepsilon.$$

For any positive number λ , set $a = \|(I - \lambda G)(x_1)\|$ and $b = \|(I - \lambda G)(x_2)\|$. Then

$$a \geq \|x_1\| - \|\lambda G(x_1)\| \geq |1 - \lambda\sqrt{\beta}| - \lambda\varepsilon,$$

and

$$b \geq \|x_2\| - \|\lambda G(x_2)\| \geq |1 - \lambda\sqrt{\alpha}| - \lambda\varepsilon,$$

and hence

$$\|I - \lambda G\| \geq \max\{a, b\} \geq \max\{|1 - \lambda\sqrt{\alpha}|, |1 - \lambda\sqrt{\beta}|\} - 2\lambda\varepsilon.$$

By inspecting the graphs of $y = |1 - \lambda\sqrt{\alpha}|$ and $|1 - \lambda\sqrt{\beta}|$, we see that the smallest possible value of the second maximum occurs when

$$1 - \lambda\sqrt{\alpha} = \lambda\sqrt{\beta} - 1,$$

in other words, when

$$\lambda = \frac{2}{\sqrt{\alpha} + \sqrt{\beta}} = \rho.$$

Accordingly,

$$\|I - \lambda G\| \geq \|I - \rho G\| \geq \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\beta} + \sqrt{\alpha}} - 2\lambda\varepsilon,$$

and, since ε was arbitrary,

$$\|I - \lambda G\| \geq \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\beta} + \sqrt{\alpha}}$$

for any positive λ .

It remains only to show that this lower bound is in fact attained when $G = T$. Since

$$\alpha I \leq S \leq \beta I$$

and $T = S^{1/2}$, it follows that

$$\sqrt{\alpha}I \leq T \leq \sqrt{\beta}I.$$

Therefore,

$$-\left(\frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\beta}+\sqrt{\alpha}}\right)I \leq I - \rho T \leq \left(\frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\beta}+\sqrt{\alpha}}\right)I,$$

which in turn implies that

$$\|I - \rho T\| \leq \frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\beta}+\sqrt{\alpha}}$$

(see [4], p. 262). This shows that $\|I - \rho T\| = (\sqrt{\beta}-\sqrt{\alpha})/(\sqrt{\beta}+\sqrt{\alpha})$ and the proof is complete. \square

Remark. It is worth pointing out that an orthonormal basis $\{e_n\}$ with this minimum norm property is not unique. To see this, take $H = R^4$ and let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal basis for H . Choose

$$f_1 = e_1/2, \quad f_2 = e_2, \quad f_3 = e_3, \quad f_4 = 3e_4/2.$$

Then the mapping $T: e_n \rightarrow f_n$ extends to a positive self adjoint operator on H . The minimum value of $\|I - \lambda T\|$ occurs when $\lambda = 1$ and

$$\|I - T\| = 1/2.$$

Consider now the orthonormal basis $\{h_1, h_2, h_3, h_4\}$ defined by

$$\begin{aligned} h_1 &= e_1, \\ h_2 &= (\cos \theta)e_2 + (\sin \theta)e_3, \\ h_3 &= (-\sin \theta)e_2 + (\cos \theta)e_3, \\ h_4 &= e_4 \end{aligned}$$

where θ is a small positive number. If $G: h_n \rightarrow f_n$, then the matrix of $I - G$ is

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 - \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & 1 - \cos \theta & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

and

$$\|I - G\| = 1/2$$

provided that θ is sufficiently small. This shows that $\{e_n\}$ is not the only orthonormal basis with the minimum norm property described in Theorem 3.

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