A BOUND FOR THE NILPOTENCY OF A GROUP OF SELF HOMOTOPY EQUIVALENCES

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Abstract. Let $E_{\Omega}(X)$ be the group of homotopy classes of self-homotopy equivalences of $X$ such that $\Omega f \simeq 1_{\Omega X}$. We prove that $E_{\Omega}(X)$ is a nilpotent group and that $\text{nil} E_{\Omega}(X) \leq \text{cat}(X) - 1$.

Given a pointed space $X$ of the homotopy type of a CW-complex, let $E(X)$ denote the group of based homotopy classes of self homotopy equivalences of $X$ ([1] is an excellent survey on this object). A considerable amount of work has been dedicated to obtaining finiteness properties, not only of $E(X)$, but also of certain interesting subgroups which preserve additional geometrical structure (see for example [2],[5],[6],[8]). This note goes in this direction: Let $E_{\Omega}(X)$ be the kernel of the obvious map $E(X) \rightarrow E(\Omega X)$ (i.e. homotopy classes of equivalences $f: X \rightarrow X$ such that $\Omega f \simeq 1_{\Omega X}$) and, as usual, denote by $\text{cat}(X)$ the Lusternik-Schnirelmann category of $X$. Then we prove:

Theorem. If $\text{cat}(X)$ is finite then $E_{\Omega}(X)$ is a nilpotent group and $\text{nil} E_{\Omega}(X) \leq \text{cat}(X) - 1$.

Remarks. (a) Observe that $E_{\Omega}(X)$ is a subgroup of the group $E_{\#}(X)$ consisting of homotopy classes of equivalences inducing the identity on the homotopy groups of $X$. Therefore it is known to be nilpotent for finite complexes in view of [4, Thm. B]. Note also that, in general, this inclusion is proper as is shown in the following example communicated to us by F. Cohen: It is known [4, Cor. 1.3] that, given a prime $p \geq 3$ and $n \geq 1$, $p^n$ is an exponent for $S^{2n+1}$ at $p$. Therefore, if we consider $\rho$ the $p^n$-th power map on the space $X = (\Omega^{2n-3} \mathbb{S}^{2n+1}(2n+1))_{(p)}$ and call $\sigma = 1 + \rho$, it follows that $\pi_\ast(\sigma) = 1_{\pi_\ast(X)}$. On the other hand $\Omega(\rho)$ is essential [9, Thm. 1] and thus $\Omega(\sigma)$ cannot be homotopic to the identity.

(b) However, for rational spaces it is well known that $E_{\Omega}(X) = E_{\#}(X)$ since in this case $\Omega X$ has the homotopy type of a product of Eilenberg-Mac Lane spaces of type $(n_i, \mathbb{Q})$ in which the integers $\{n_i\}$ describe the degrees of a basis of $\pi_\ast(X)$. Hence, the theorem above could be seen as a generalization of [6, Thm. 1]

The rest of the paper is devoted to the proof of the theorem above. To simplify the notation we shall not distinguish between a homotopy class and a map which represents it. Also, equality of homotopy classes (or maps) will often mean homotopy between its representatives.

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To start, let us recall the characterization of the LS category of a space $X$ given in [7]. The $n$-th Ganea fibration of $X$, $F_n(X) \rightarrow E_n(X) \xrightarrow{i_n} X$, is defined by an inductive procedure in the following way: $p_0$ is just the path fibration $\Omega X \rightarrow PX \xrightarrow{p_0} X$. Next consider $C$ the homotopy cofibre of the inclusion $F_{n-1}(X) \rightarrow E_{n-1}(X)$ and extend $p_{n-1}$ to a map $C \rightarrow X$. The associated fibration to this map $F_n(X) \rightarrow E_n(X) \xrightarrow{i_n} X$ is by definition the $n$-th Ganea fibration of $X$. $E_n(X)$ is called the $n$-th Ganea space for $X$. As a general picture we have:

\[
\begin{array}{cccc}
\Omega X & F_1(X) & F_{n-1}(X) & F_n(X) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
PX & \xrightarrow{i_1} E_1(X) & \cdots & E_{n-1}(X) \xrightarrow{i_n} E_n(X) \\
p_0 & p_1 & \cdots & p_{n-1} \\
X & p_n \\
& f & & Y
\end{array}
\]

Then, we shall make use of the following facts:

(1) cat $X \leq n$ if and only if $p_n$ admits a homotopy section.

(2) $F_n(X)$ has the homotopy type of the join of $n + 1$ copies of $\Omega X$.

(3) For each space $X$ and each integer $n$, the fibration $E_n(X) \xrightarrow{p_n} X$ defines an augmented functor, that is to say, given $f: X \rightarrow Y$, there exists a (functorial) map $E_n(f): E_n(X) \rightarrow E_n(Y)$ such that the following diagram commutes:

\[
\begin{array}{ccc}
E_n(X) & \xrightarrow{E_n(f)} & E_n(Y) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

Next, given a map $f: X \rightarrow X$ representing an element on $\mathcal{E}_\Omega(X)$ define the length of $f$, $l(f)$, as the biggest integer $n$ for which $fp_n = p_n$. Since $E_1(X)$ has the homotopy type of $\Sigma \Omega X$ and (up to homotopy) $p_1: \Sigma \Omega X \rightarrow X$ is the adjoint to the identity, $l(f)$ is at least 1. Also, observe that if $l(f) = n$, then $fp_m = f$ for any $m \leq n$. Next, define $G_n$ as the subgroup of $\mathcal{E}_\Omega(X)$ consisting of equivalences of length at least $n$. Clearly $G_1 = \mathcal{E}_\Omega(X)$ and $G_{n+1} \subset G_n$.

Lemma. $[G_1, G_n] \subset G_{n+1}$.

Proof. First, recall [10] that given a cofibration sequence $Y \rightarrow Z \rightarrow C$, the coaction $\nu: C \rightarrow \Sigma Y \vee C$ induces a natural action of the group $[\Sigma Y, X]$ on $[C, X]$. Explicitly, given $\beta \in [\Sigma Y, X]$ and $\alpha \in [C, X]$, define $\alpha\beta = (\beta, \alpha) \circ \nu$. The orbits of this action are precisely $i_s^{-1}(h)$, $h \in [Z, X]$, with $i_s: [C, X] \rightarrow [Z, X]$ induced by $i$. That is to say, given maps $\alpha_1, \alpha_2: C \rightarrow X$, $\alpha_1 i \sim \alpha_2 i$ if and only if there exists $\beta: \Sigma Y \rightarrow X$ such that $\alpha_1\beta = \alpha_2$.

Note also that given $\gamma \in [X, W]$ and $\varphi \in [C, C]$, $\gamma\alpha\beta = (\gamma\alpha)\gamma\beta$ and $\alpha\beta\varphi = (\alpha\varphi)\beta\Sigma\varphi$, with $\Sigma\varphi \in [\Sigma Y, \Sigma Y]$ induced by $\varphi$ by collapsing $Z$. We return to the proof of the lemma. Let $f, g: X \rightarrow X$ be maps satisfying $fp_1 = p_1$ and $gp_n = p_n$. We will prove that $fp_{n+1} = gp_{n+1}$. For that we shall apply the considerations above to the cofibration sequence $F_n(X) \rightarrow E_n(X) \xrightarrow{i_{n+1}} E_{n+1}(X)$. Since $gp_n = p_n$
and $p_n = p_{n+1}^4 n_{n+1}$, there exists $h: \Sigma F_n(X) \to X$ such that $g p_{n+1} = p^h n_{n+1}$. Observe that:

(i) Since $h$ factors as the composite $\Sigma F_n(X) \xrightarrow{k} \Sigma \Omega X \xrightarrow{p} X$, we have $f h = f p_1 k = p_1 k = h$.
(ii) On the other hand, since $\Omega f = 1$, via (2), it follows that $F_n(f) = \ast^{n+1} \Omega f = 1$.

Finally we can write:

\[ fg p_{n+1} = f p^h n_{n+1} = (f p_{n+1})^{f h} = (f p_{n+1})^h = (f p_{n+1})^{h \Sigma F_n(f)} = (p_{n+1} E_{n+1}(f))^{h \Sigma F_n(f)} = p^h_{n+1} E_{n+1}(f) = g p_{n+1} E_{n+1}(f) = g f p_{n+1}. \]

Proof of the theorem. Observe that if $\text{cat} X = m$ then $G_m = \{1\}$. Indeed, given $f \in G_n$ and in view of (1), $f = f p_m \sigma = p_m \sigma = 1$ with $\sigma$ section of $p_n$. Hence, by lemma above we have a finite decreasing sequence of normal subgroups

\[ E_{\Omega}(X) = G_1 \supset G_2 \supset \ldots \supset G_m = \{1\} \]

in which $[G_1, G_n] \subset G_{n+1}$ and thus the theorem follows.

References

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