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A NOTE ON KAMENEV TYPE THEOREMS FOR SECOND ORDER MATRIX DIFFERENTIAL SYSTEMS

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ABSTRACT. Some oscillation criteria are given for the second order matrix differential system $Y'' + Q(t)Y = 0$, where Y and Q are $n \times n$ real continuous matrix functions with $Q(t)$ symmetric, $t \in [t_0, \infty)$. These results improve oscillation criteria recently discovered by Erbe, Kong and Ruan by using a generalized Riccati transformation $V(t) = a(t)\{Y'(t)Y^{-1}(t) + f(t)I\}$, where I is the $n \times n$ identity matrix, $f \in C^1$ is a given function on $[t_0, \infty)$ and $a(t) = \exp\{-2\int^t f(s)\,ds\}$.

Consider the second order linear differential system

$$Y'' + Q(t)Y = 0, \qquad t \in [t_0, \infty), \tag{1}$$

where Y and Q are $n \times n$ real continuous matrix functions with $Q(t)$ symmetric. A solution $Y(t)$ of (1) is said to be a nontrivial solution if $\det Y(t) \neq 0$ for at least one $t \in [t_0, \infty)$, and a nontrivial solution $Y(t)$ of (1) is said to be prepared if

$$Y^*(t)Y'(t) - (Y^*(t))'Y(t) \equiv 0, \qquad t \in [t_0, \infty), \tag{2}$$

where for any matrix A, the transpose of A is denoted by A^*. System (1) is said to be oscillatory on $[t_0, \infty)$ in case the determinant of every nontrivial prepared solution vanishes on $[T, \infty)$ for each $T > t_0$.

For matrix system (1), many authors have given some important simple oscillation criteria (see [1], [2], [3], [6]). We particularly mention the results of Erbe, Kong and Ruan [3] who proved the following theorem.

Erbe, Kong and Ruan's Theorem. Let $H(t,s)$ and $h(t,s)$ be continuous on $D = \{(t,s) : t \geq s \geq t_0\}$ such that $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $t > s \geq t_0$. We assume further that the partial derivative $\frac{\partial}{\partial s}H(t,s) = H_s(t,s)$ is nonpositive and continuous for $t \geq s \geq t_0$ and $h(t,s)$ is defined by

$$H_s(t,s) = -h(t,s)[H(t,s)]^{1/2}, \qquad (t,s) \in D.$$

Finally, we assume that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \lambda_1 \left[\int_{t_0}^{t} \left(H(t,s)Q(s) - \frac{1}{4}h^2(t,s)I \right) ds \right] = \infty, \tag{3}$$

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where $\lambda_1[A] \geq \lambda_2[A] \geq \cdots \geq \lambda_n[A]$ denotes the usual ordering of the eigenvalues of the symmetric matrix A, I is the $n \times n$ identity matrix. Then system (1) is oscillatory.

However, if $Q(t) = \mathrm{diag}(\frac{\gamma}{t^2}, \frac{\alpha}{t^2})$ in (1), where $\gamma \geq \alpha > 0$ are constants, then

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{H(t,1)}\lambda_1 \left[\int_1^t \left(H(t,s)Q(s) - \frac{1}{4}h^2(t,s)I\right) ds\right] \\ \leq \limsup_{t\to\infty} \int_1^t \frac{\gamma}{s^2} ds = \gamma < \infty,\end{aligned}$$

where $H(t,s), h(t,s)$ are defined as in Erbe, Kong and Ruan's theorem. Thus the above mentioned criteria of Erbe, Kong and Ruan cannot be applied to the Euler differential system

$$Y'' + \mathrm{diag}\left(\frac{\gamma}{t^2}, \frac{\alpha}{t^2}\right) Y = 0, \tag{4}$$

where Y is a 2×2 matrix, $\gamma \geq \alpha > 0$ are constants. In fact, the Euler differential system (4) is oscillatory if $\gamma > \frac{1}{4}$ and nonoscillatory if $\gamma \leq \frac{1}{4}$.

The purpose of this note is to improve Erbe, Kong and Ruan's oscillation criteria by using a generalized Riccati transformation. Our main results are the following theorems.

Theorem 1. *Let $H(t,s)$ and $h(t,s)$ be continuous on $D = \{(t,s) : t \geq s \geq t_0\}$ such that $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $t > s \geq t_0$. We assume further that the partial derivative $\frac{\partial}{\partial s}H(t,s) = H_s(t,s)$ is nonpositive and is continuous for $t \geq s \geq t_0$ and $h(t,s)$ defined by*

$$H_s(t,s) = -h(t,s)[H(t,s)]^{1/2}, \qquad (t,s) \in D.$$

If there exists a function $f \in C^1[t_0, \infty)$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\lambda_1 \left[\int_{t_0}^t \left(H(t,s)R(s) - \frac{1}{4}a(s)h^2(t,s)I\right) ds\right] = \infty, \tag{5}$$

where $a(t) = \exp\{-2\int^t f(s)ds\}, R(t) = a(t)\{Q(t) + f^2(t)I - f'(t)I\}$, then equation (1) is oscillatory.

Proof. Suppose to the contrary that there exists a prepared solution $Y(t)$ of (1) which is not oscillatory. Without loss of generality, we may suppose that $\det Y(t) \neq 0$ for $t \geq t_0$. Define

$$V(t) = a(t)(Y'(t)Y^{-1}(t) + f(t)I), \qquad t \geq t_0.$$

This and (1) imply

$$\begin{aligned}V'(t) &= -2f(t)V(t) + a(t)(Y''(t)Y^{-1}(t) - [Y'(t)Y^{-1}(t)]^2 + f'(t)I) \\ &= -\frac{1}{a(t)}V^2(t) - R(t), \qquad t \geq t_0.\end{aligned} \tag{6}$$

Multiplying (6) with t replaced by s, by $H(t,s)$ and integrating from t_0 to t, we obtain

$$\begin{aligned}
&\int_{t_0}^{t} H(t,s)R(s)\,ds \\
&\quad = -\int_{t_0}^{t} H(t,s)V'(s)ds - \int_{t_0}^{t} \frac{H(t,s)}{a(s)}V^2(s)ds \\
&\quad = -H(t,s)V(s)|_{t_0}^{t} - \int_{t_0}^{t} \left(-H_s(t,s)V(s) + \frac{H(t,s)}{a(s)}V^2(s)\right) ds \\
&\quad = H(t,t_0)V(t_0) - \int_{t_0}^{t} \left(-H_s(t,s)V(s) + \frac{H(t,s)}{a(s)}V^2(s)\right) ds \\
&\quad = H(t,t_0)V(t_0) - \int_{t_0}^{t} \left(h(t,s)\sqrt{H(t,s)}V(s) + \frac{H(t,s)}{a(s)}V^2(s)\right) ds \\
&\quad = H(t,t_0)V(t_0) + \frac{1}{4}\int_{t_0}^{t} a(s)h^2(t,s)I\,ds \\
&\qquad - \int_{t_0}^{t} \left[\sqrt{\frac{H(t,s)}{a(s)}}V(s) + \frac{1}{2}\sqrt{a(s)}h(t,s)I\right]^2 ds.
\end{aligned}$$

Hence we have

$$\int_{t_0}^{t} \left(H(t,s)R(s) - \frac{1}{4}a(s)h^2(t,s)I\right) ds \le H(t,t_0)V(t_0), \qquad t \ge t_0.$$

It follows that

$$\lambda_1 \left[\int_{t_0}^{t} \left(H(t,s)R(s) - \frac{1}{4}a(s)h^2(t,s)I\right) ds\right] \le \lambda_1[H(t,t_0)V(t_0)]. \tag{7}$$

Since $H(t,t_0) > 0$ for $t > t_0$, dividing (7) by $H(t,t_0)$, we get

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\lambda_1 \left[\int_{t_0}^{t} \left(H(t,s)R(s) - \frac{1}{4}a(s)h^2(t,s)I\right) ds\right] \\
&\qquad \le \frac{1}{H(t,t_0)}\lambda_1[H(t,t_0)V(t_0)] = \lambda_1[V(t_0)].
\end{aligned} \tag{8}$$

Taking the upper limit in both sides of (8) as $t \to \infty$, the right-hand side is always bounded, which contradicts condition (5). This completes the proof.

Under a modification of the hypotheses of Theorem 1, we can obtain the following result.

Theorem 2. *In Theorem 1, if condition* (5) *is replaced by the conditions*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s)h^2(t,s)\,ds < \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\lambda_1 \left[\int_{t_0}^{t} H(t,s)R(s)\,ds\right] = \infty,$$

then system (1) *is oscillatory.*

If $f(t) = 0$, then Theorems 1 and 2 reduce to the Erbe, Kong and Ruan criterion [3].

Let $H(t,s) = (t-s)^\alpha$, $t \geq s \geq t_0$, where $\alpha > 1$ is an integer. By Theorem 1, we have the following result.

Corollary 1. *Let $\alpha > 1$ be an integer, suppose that there exists a function $f \in C^1[t_0, \infty)$ satisfying*

$$\lim_{t\to\infty} \sup \frac{1}{t^\alpha} \lambda_1 \left[\int_{t_0}^t \left((t-s)^\alpha R(s) - \frac{\alpha^2}{4}(t-s)^{\alpha-2} a(s) I \right) ds \right] = \infty,$$

where $a(t) = \exp\{-2\int^t f(s)ds\}$, $R(t) = a(t)\{Q(t) + f^2(t)I - f'(t)I\}$. Then Eq. (1) *is oscillatory.*

If $Q(t) = q(t)$, a scalar function, take $f(t) = 0$; then Corollary 1 reduces to the Kamenev criterion [5].

Example. Consider the Euler differential system (4) for $\gamma > \frac{1}{4}$ and let $f(t) = -\frac{1}{2t}$, then $a(t) = t$ and $f'(t) = \frac{1}{2t^2}$. This implies that

$$\int_1^t (t-s)^{\alpha-2} s \, ds = \left(\frac{t}{\alpha(\alpha-1)} + \frac{1}{\alpha} \right) (t-1)^{\alpha-1}.$$

Since $\alpha > 1$ is an integer, it follows from the book of Hardy, Littlewood and Pólya [4, Theorem 41], that

$$(t-s)^\alpha \geq t^\alpha - \alpha s t^{\alpha-1}, \quad \text{for } t \geq s \geq 1.$$

Then

$$\begin{aligned}
&\lim_{t\to\infty} \sup \frac{1}{t^\alpha} \lambda_1 \left[\int_1^t \left((t-s)^\alpha R(s) - \frac{\alpha^2}{4}(t-s)^{\alpha-2} a(s) I \right) ds \right] \\
&\quad = \lim_{t\to\infty} \sup \frac{1}{t^\alpha} \int_1^t \left((t-s)^\alpha s \left[\frac{\gamma}{s^2} + \frac{1}{4s^2} - \frac{1}{2s^2} \right] - \frac{\alpha^2}{4}(t-s)^{\alpha-2} s \right) ds \\
&\quad = \frac{4\gamma-1}{4} \lim_{t\to\infty} \sup \frac{1}{t^\alpha} \int_1^t \frac{(t-s)^\alpha}{s} ds - \frac{\alpha}{4(\alpha-1)} \\
&\quad \geq \frac{4\gamma-1}{4} \lim_{t\to\infty} \sup \frac{1}{t^\alpha} \int_1^t \frac{t^\alpha - \alpha s t^{\alpha-1}}{s} ds - \frac{\alpha}{4(\alpha-1)} \\
&\quad = \frac{4\gamma-1}{4} \lim_{t\to\infty} \sup \left[\log t - \frac{\alpha(t-1)}{t} \right] - \frac{\alpha}{4(\alpha-1)} = \infty.
\end{aligned}$$

It follows from Corollary 1 that the Euler differential system (4) is oscillatory if $\gamma > \frac{1}{4}$. Now, let us consider the function defined by

$$H(t,s) = \left(\int_s^t \frac{d\tau}{\theta(\tau)} \right)^\alpha, \quad \text{for } t \geq s \geq t_0,$$

where $\alpha > 1$ is an integer and θ is a positive continuous function on $[t_0, \infty)$ such that $\int_{t_0}^\infty (1/\theta(\tau)) d\tau = \infty$. Clearly

$$H(t,t) = 0 \quad \text{for } t \geq t_0, \qquad \text{and} \qquad H(t,s) > 0 \quad \text{for } t > s \geq t_0,$$

$$-\frac{\partial}{\partial s} H(t,s) = \frac{\alpha}{\theta(s)} \left(\int_s^t \frac{d\tau}{\theta(\tau)} \right)^{\alpha-1},$$

and

$$h(t,s) = \frac{\alpha}{\theta(s)} \left(\int_s^t \frac{d\tau}{\theta(\tau)} \right)^{\alpha/2-1}, \qquad t \geq s \geq t_0.$$

One of the important cases is to consider $\theta(\tau) = \tau^\beta$, where $\beta \leq 1$ is a real number. Here

$$\begin{aligned} H(t,s) &= \frac{1}{(1-\beta)^\alpha} [t^{1-\beta} - s^{1-\beta}]^\alpha, \qquad \beta < 1, \\ &= \left(\log \frac{t}{s} \right)^\alpha, \qquad \beta = 1, \end{aligned}$$

and

$$\begin{aligned} h(t,s) &= \frac{\alpha}{s^\beta} \left(\frac{t^{1-\beta} - s^{1-\beta}}{1-\beta} \right)^{\alpha/2-1}, \qquad \beta < 1, \\ &= \frac{\alpha}{s} \left(\log \frac{t}{s} \right)^{\alpha/2-1}, \qquad \beta = 1. \end{aligned}$$

Therefore, by applying Theorems 1 and 2 in the special case considered, we derive many new criteria for the oscillation of system (1)

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