A NOTE ON KAMENEV TYPE THEOREMS
FOR SECOND ORDER MATRIX DIFFERENTIAL SYSTEMS

FANWEI MENG, JIZHONG WANG, AND ZHAOWEN ZHENG

(Communicated by Hal L. Smith)

Abstract. Some oscillation criteria are given for the second order matrix
differential system $Y'' + Q(t)Y = 0$, where $Y$ and $Q$ are $n \times n$ real continuous
matrix functions with $Q(t)$ symmetric, $t \in [t_0, \infty)$. These results improve
oscillation criteria recently discovered by Erbe, Kong and Ruan by using a
generalized Riccati transformation $V(t) = a(t)(Y'(t)Y^{-1}(t) + f(t)I)$, where $I$
is the $n \times n$ identity matrix, $f \in C^1$ is a given function on $[t_0, \infty)$ and
$a(t) = \exp(-2\int_t^\infty f(s)ds)$.

Consider the second order linear differential system

\begin{equation}
Y'' + Q(t)Y = 0, \quad t \in [t_0, \infty),
\end{equation}

where $Y$ and $Q$ are $n \times n$ real continuous matrix functions with $Q(t)$ symmetric.
A solution $Y(t)$ of (1) is said to be a nontrivial solution if $\det Y(t) \neq 0$ for at least
one $t \in [t_0, \infty)$, and a nontrivial solution $Y(t)$ of (1) is said to be prepared if

\begin{equation}
Y^*(t)Y'(t) - (Y^*(t))'Y(t) \equiv 0, \quad t \in [t_0, \infty),
\end{equation}

where for any matrix $A$, the transpose of $A$ is denoted by $A^*$. System (1) is said
to be oscillatory on $[t_0, \infty)$ in case the determinant of every nontrivial prepared
solution vanishes on $[T, \infty)$ for each $T > t_0$.

For matrix system (1), many authors have given some important simple oscil-
lation criteria (see [1], [2], [3], [6]). We particularly mention the results of Erbe,
Kong and Ruan [3] who proved the following theorem.

Erbe, Kong and Ruan’s Theorem. Let $H(t,s)$ and $h(t,s)$ be continuous on
$D = \{(t,s) : t \geq s \geq t_0\}$ such that $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for
$t > s \geq t_0$. We assume further that the partial derivative $\frac{\partial}{\partial s}H(t,s) = H_s(t,s)$ is
nonpositive and continuous for $t \geq s \geq t_0$ and $h(t,s)$ is defined by

\[ H_s(t,s) = -h(t,s)[H(t,s)]^{1/2}, \quad (t,s) \in D. \]

Finally, we assume that

\begin{equation}
\lim_{t \to \infty} \sup_{t_0} \frac{1}{H(t,t_0)} \lambda_1 \left[ \int_{t_0}^t \left( H(t,s)Q(s) - \frac{1}{4}h^2(t,s)I \right) ds \right] = \infty,
\end{equation}

Received by the editors May 25, 1996.
1991 Mathematics Subject Classification. Primary 34C10.
Key words and phrases. Matrix differential system, oscillatory theory, Riccati equation.
The research is supported by the Natural Science Foundation of Shandong Province, P.R.
China.

©1998 American Mathematical Society
where \(\lambda_1[A] \geq \lambda_2[A] \geq \cdots \geq \lambda_n[A]\) denotes the usual ordering of the eigenvalues of the symmetric matrix \(A, I\) is the \(n \times n\) identity matrix. Then system (1) is oscillatory.

However, if \(Q(t) = \text{diag}(\frac{\gamma}{t^2}, \frac{\alpha}{t^2})\) in (1), where \(\gamma \geq \alpha > 0\) are constants, then

\[
\lim_{t \to \infty} \sup_{s \leq t} \frac{1}{H(t, 1)} \lambda_1 \left[ \int_1^t \left( H(t, s)Q(s) - \frac{1}{4} h^2(t, s)I \right) ds \right] \\
\leq \lim_{t \to \infty} \sup_{s \leq t} \int_1^t \frac{\gamma}{s^2} ds = \gamma < \infty,
\]

where \(H(t, s), h(t, s)\) are defined as in Erbe, Kong and Ruan’s theorem. Thus the above mentioned criteria of Erbe, Kong and Ruan cannot be applied to the Euler differential system

\[
Y'' + \text{diag} \left( \frac{\gamma}{t^2}, \frac{\alpha}{t^2} \right) Y = 0,
\]

where \(Y\) is a \(2 \times 2\) matrix, \(\gamma \geq \alpha > 0\) are constants. In fact, the Euler differential system (4) is oscillatory if \(\gamma > \frac{1}{4}\) and nonoscillatory if \(\gamma \leq \frac{1}{4}\).

The purpose of this note is to improve Erbe, Kong and Ruan’s oscillation criteria by using a generalized Riccati transformation. Our main results are the following theorems.

**Theorem 1.** Let \(H(t, s)\) and \(h(t, s)\) be continuous on \(D = \{(t, s) : t \geq s \geq t_0\}\) such that \(H(t, t) = 0\) for \(t \geq t_0\) and \(H(t, s) > 0\) for \(t > s \geq t_0\). We assume further that the partial derivative \(\frac{\partial}{\partial s} H(t, s) = H_s(t, s)\) is nonpositive and is continuous for \(t \geq s \geq t_0\) and \(h(t, s)\) defined by

\[
H_s(t, s) = -h(t, s)[H(t, s)]^{1/2}, \quad (t, s) \in D.
\]

If there exists a function \(f \in C^1[t_0, \infty)\) such that

\[
\lim_{t \to \infty} \sup_{t \geq t_0} \frac{1}{H(t, t_0)} \lambda_1 \left[ \int_{t_0}^t \left( H(t, s)R(s) - \frac{1}{4} a(s)h^2(t, s)I \right) ds \right] = \infty,
\]

where \(a(t) = \exp\{-2 \int_{t_0}^t f(s)ds\}, R(t) = a(t)\{Q(t) + f^2(t)I - f'(t)I\},\) then equation (1) is oscillatory.

**Proof.** Suppose to the contrary that there exists a prepared solution \(Y(t)\) of (1) which is not oscillatory. Without loss of generality, we may suppose that \(\det Y(t) \neq 0\) for \(t \geq t_0\). Define

\[
V(t) = a(t)(Y'(t)Y^{-1}(t) + f(t)I), \quad t \geq t_0.
\]

This and (1) imply

\[
V'(t) = -2f(t)V(t) + a(t)(Y''(t)Y^{-1}(t) - [Y'(t)Y^{-1}(t)]^2 + f'(t)I)
\]

\[
= -\frac{1}{a(t)}V^2(t) - R(t), \quad t \geq t_0.
\]
Multiplying (6) with \( t \) replaced by \( s \), by \( H(t, s) \) and integrating from \( t_0 \) to \( t \), we obtain
\[
\int_{t_0}^{t} H(t, s)R(s) \, ds
\]
\[
= - \int_{t_0}^{t} H(t, s)V'(s)ds - \int_{t_0}^{t} \frac{H(t, s)}{a(s)}V^2(s)ds
\]
\[
= -H(t, s)V(s)|_{t_0}^{t} - \int_{t_0}^{t} \left( -H_s(t, s)V(s) + \frac{H(t, s)}{a(s)}V^2(s) \right) ds
\]
\[
= H(t, t_0)V(t_0) - \int_{t_0}^{t} \left( -H_s(t, s)V(s) + \frac{H(t, s)}{a(s)}V^2(s) \right) ds
\]
\[
= H(t, t_0)V(t_0) - \int_{t_0}^{t} \left( h(t, s)\sqrt{H(t, s)V(s)} + \frac{H(t, s)}{a(s)}V^2(s) \right) ds
\]
\[
= H(t, t_0)V(t_0) + \frac{1}{4} \int_{t_0}^{t} a(s)h^2(t, s)I \, ds
\]
\[
- \int_{t_0}^{t} \left[ \sqrt{\frac{H(t, s)}{a(s)}V(s)} + \frac{1}{2}\sqrt{a(s)}h(t, s)I \right]^2 \, ds.
\]
Hence we have
\[
\int_{t_0}^{t} \left( H(t, s)R(s) - \frac{1}{4}a(s)h^2(t, s)I \right) \, ds \leq H(t, t_0)V(t_0), \quad t \geq t_0.
\]
It follows that
\[
\lambda_1 \left[ \int_{t_0}^{t} \left( H(t, s)R(s) - \frac{1}{4}a(s)h^2(t, s)I \right) \, ds \right] \leq \lambda_1[H(t, t_0)V(t_0)].
\]
Since \( H(t, t_0) > 0 \) for \( t > t_0 \), dividing (7) by \( H(t, t_0) \), we get
\[
\frac{1}{H(t, t_0)} \lambda_1 \left[ \int_{t_0}^{t} \left( H(t, s)R(s) - \frac{1}{4}a(s)h^2(t, s)I \right) \, ds \right]
\leq \frac{1}{H(t, t_0)} \lambda_1[H(t, t_0)V(t_0)] = \lambda_1[V(t_0)].
\]
Taking the upper limit in both sides of (8) as \( t \to \infty \), the right-hand side is always bounded, which contradicts condition (5). This completes the proof.

Under a modification of the hypotheses of Theorem 1, we can obtain the following result.

**Theorem 2.** In Theorem 1, if condition (5) is replaced by the conditions
\[
\lim_{t \to \infty} \sup_{t_0} \frac{1}{H(t, t_0)} \int_{t_0}^{t} a(s)h^2(t, s) \, ds < \infty
\]
and
\[
\lim_{t \to \infty} \sup_{t_0} \frac{1}{H(t, t_0)} \lambda_1 \left[ \int_{t_0}^{t} H(t, s)R(s) \, ds \right] = \infty,
\]
then system (1) is oscillatory.
If $f(t) = 0$, then Theorems 1 and 2 reduce to the Erbe, Kong and Ruan criterion [3].

Let $H(t, s) = (t - s)^\alpha$, $t \geq s \geq t_0$, where $\alpha > 1$ is an integer. By Theorem 1, we have the following result.

**Corollary 1.** Let $\alpha > 1$ be an integer, suppose that there exists a function $f \in C^1(t_0, \infty)$ satisfying

$$\limsup_{t \to \infty} \frac{1}{t^\alpha} \lambda_1 \left[ \int_{t_0}^{t} \left( (t - s)^\alpha R(s) - \frac{\alpha^2}{4} (t - s)^{\alpha-2} a(s) I \right) \, ds \right] = \infty,$$

where $a(t) = \exp \{-2 \int f(s) \, ds\}$, $R(t) = a(t) \{Q(t) + f^2(t)I - f'(t)I\}$. Then Eq. (1) is oscillatory.

If $Q(t) = q(t)$, a scalar function, take $f(t) = 0$; then Corollary 1 reduces to the Kamenev criterion [5].

**Example.** Consider the Euler differential system (4) for $\gamma > \frac{1}{4}$ and let $f(t) = -\frac{1}{2t}$, then $a(t) = t$ and $f'(t) = \frac{1}{2t^2}$. This implies that

$$\int_{t}^{t_0} (t - s)^{\alpha-2} s \, ds = \left( \frac{t}{\alpha(\alpha - 1)} + \frac{1}{\alpha} \right) (t - 1)^{\alpha-1}.$$

Since $\alpha > 1$ is an integer, it follows from the book of Hardy, Littlewood and Pólya [4, Theorem 41], that

$$(t - s)^\alpha \geq t^\alpha - \alpha s t^{\alpha-1}, \quad \text{for } t \geq s \geq 1.$$

Then

$$\limsup_{t \to \infty} \frac{1}{t^\alpha} \lambda_1 \left[ \int_{t_0}^{t} \left( (t - s)^\alpha R(s) - \frac{\alpha^2}{4} (t - s)^{\alpha-2} a(s) I \right) \, ds \right]$$

$$= \limsup_{t \to \infty} \frac{1}{t^\alpha} \int_{t_0}^{t} \left( (t - s)^\alpha \left[ \frac{\gamma}{s^2} + \frac{1}{4s^2} - \frac{1}{2s^2} \right] - \frac{\alpha^2}{4} (t - s)^{\alpha-2} s \right) \, ds$$

$$= \frac{4\gamma - 1}{4} \limsup_{t \to \infty} \frac{1}{t^\alpha} \int_{t_0}^{t} \left( \frac{t^\alpha - \alpha s t^{\alpha-1}}{s^2} \right) \, ds - \frac{\alpha}{4(\alpha - 1)}$$

$$= \frac{4\gamma - 1}{4} \limsup_{t \to \infty} \left[ \log t - \frac{\alpha(t - 1)}{t} \right] - \frac{\alpha}{4(\alpha - 1)} = \infty.$$

It follows from Corollary 1 that the Euler differential system (4) is oscillatory if $\gamma > \frac{1}{4}$. Now, let us consider the function defined by

$$H(t, s) = \left( \int_{s}^{t} \frac{d\tau}{\theta(\tau)} \right)^{\alpha}, \quad \text{for } t \geq s \geq t_0,$$

where $\alpha > 1$ is an integer and $\theta$ is a positive continuous function on $[t_0, \infty)$ such that $\int_{t_0}^{\infty} (1/\theta(\tau)) \, d\tau = \infty$. Clearly

$$H(t, t) = 0 \quad \text{for } t \geq t_0, \quad \text{and} \quad H(t, s) > 0 \quad \text{for } t > s \geq t_0,$$

$$\frac{\partial}{\partial s} H(t, s) = \frac{\alpha}{\theta(s)} \left( \int_{s}^{t} \frac{d\tau}{\theta(\tau)} \right)^{\alpha-1},$$
and

\[ h(t, s) = \frac{\alpha}{\theta(s)} \left( \int_s^t \frac{d\tau}{\theta(\tau)} \right)^{\alpha/2 - 1}, \quad t \geq s \geq t_0. \]

One of the important cases is to consider \( \theta(\tau) = \tau^\beta \), where \( \beta \leq 1 \) is a real number. Here

\[ H(t, s) = \frac{1}{(1 - \beta)^\alpha} \left| t^{1 - \beta} - s^{1 - \beta} \right|^\alpha, \quad \beta < 1, \]

\[ = \left( \log \frac{t}{s} \right)^\alpha, \quad \beta = 1, \]

and

\[ h(t, s) = \frac{\alpha}{s^\beta} \left( \frac{1}{1 - \beta} \right)^{\alpha/2 - 1} \left| t^{1 - \beta} - s^{1 - \beta} \right|, \quad \beta < 1, \]

\[ = \frac{\alpha}{s} \left( \log \frac{t}{s} \right)^{\alpha/2 - 1}, \quad \beta = 1. \]

Therefore, by applying Theorems 1 and 2 in the special case considered, we derive many new criteria for the oscillation of system (1)

ACKNOWLEDGMENT

The authors express their heartfelt gratitude to the referee for corrections to the original manuscript and for helpful comments.

REFERENCES


(F. Meng and Z. Zheng) DEPARTMENT OF MATHEMATICS, QUFU NORMAL UNIVERSITY, QUFU, SHANDONG, 273165, PEOPLE’S REPUBLIC OF CHINA

(J. Wang) DEPARTMENT OF MATHEMATICS, LINYI TEACHER’S COLLEGE, LINYI, SHANDONG, 276005, PEOPLE’S REPUBLIC OF CHINA