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SPECTRAL PROPERTIES OF CONTINUOUS REFINEMENT OPERATORS

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ABSTRACT. This paper studies the spectrum of continuous refinement operators and relates their spectral properties with the solutions of the corresponding continuous refinement equations.

1. INTRODUCTION

This paper studies the spectral properties of the operator $W_c : L^p(\mathbf{R}) \to L^p(\mathbf{R})$, $1 \le p \le \infty$, defined by

$$(W_c f)(x) := \int_{\mathbf{R}} \alpha c(y) f(\alpha x - y)\, dy, \qquad f \in L^p(\mathbf{R}), \tag{1.1}$$

and its restriction to a certain invariant subspace. Here $\alpha > 1$ is a *dilation constant*. An eigenfunction of W_c with eigenvalue 1 is a solution in $L^p(\mathbf{R})$ of the functional equation

$$\phi(x) = \int_{\mathbf{R}} \alpha c(y) \phi(\alpha x - y)\, dy, \tag{1.2}$$

which is called a *continuous refinement equation*. The operator W_c will be called a *continuous refinement operator*.

The simplest continuous refinement equation is one in which the kernel $c = \frac{1}{2}\chi_{[-1,1]}$ and the dilation constant $\alpha = 2$. Its solution with compact support is known as the *up function*, denoted by $up(x)$, and is defined in terms of its Fourier transform

$$\widehat{up}(u) = \prod_{j=1}^{\infty} \frac{\sin(u2^{-j})}{u2^{-j}}. \tag{1.3}$$

The up function has interesting properties and important applications, and has been extensively studied by Russian mathematicians (see the survey articles [9], [10]). The up function corresponding to a dilation constant $\alpha > 1$ and the kernel

$$c = \frac{1}{2(\alpha - 1)} \chi_{[-\alpha+1, \alpha-1]} \tag{1.4}$$

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has also been studied independently by Kabaya and Iri [7], [8]. In [8] they gave a complete description of the spectrum of the corresponding refinement operator W_c restricted to the space $L^1[-1,1]$. Recently, more general continuous refinement equations have been studied in conjunction with nonstationary subdivision processes and nonstationary multiresolution analyses (see [3], [2], [4], [5]). In this connection, one is interested in the existence and properties of the solution of (1.2) for a compactly supported kernel $c \in L^p(\mathbf{R})$, and also in the convergence of the power iteration $W_c^n \phi_0$. The operator W_c is related to the adjoint of the continuous subdivision operator.

The convergence of the power iteration depends on the spectral radius of the refinement operator and is related to the existence and uniqueness of the solution of the continuous refinement equation. The object of this paper is to study the relationship between the spectrum of W_c on one hand and the existence and uniqueness of the solution of the continuous refinement equation (1.2) and its properties on the other. We shall assume throughout this paper that c is compactly supported with $\operatorname{supp}(c) \subset [-a,a]$, and, unless otherwise stated, $c \in L^1(\mathbf{R})$. In general a depends on the dilation constant α. We are interested in the compactly supported solution of the equation (1.2) and since it is an eigenfunction of W_c, we are led to consider the restriction of the operator to a subspace L_K^p of $L^p(\mathbf{R})$ consisting of functions with supports in an interval $[-K,K]$. Then L_K^p is an invariant subspace of W_c if and only if

$$K \geq \frac{a}{\alpha - 1}, \tag{1.5}$$

a condition which we shall assume throughout the paper. Note that, in general, a depends on α. The restriction of W_c to L_K^p will be denoted by $W_{c,K}$. Thus

$$(W_c f)(x) := \int_{-a}^{a} \alpha c(y) f(\alpha x - y)\, dy, \qquad f \in L^p(\mathbf{R}), \tag{1.6}$$

and

$$(W_{c,K} f)(x) := \int_{-K}^{K} \alpha c(\alpha x - y) f(y)\, dy, \qquad f \in L_K^p. \tag{1.7}$$

In §2 we show that these operators are bounded and that $W_{c,K}$ is compact. The compactness of the operator $W_{c,K}$ is exploited to obtain its spectral properties. The main results are contained in §3. It is shown that if the kernel $c \in L^1(\mathbf{R})$ is compactly supported and normalized so that $\hat{c}(0) = 1$, then the spectrum of $W_{c,K}$ is precisely the set

$$\{1/\alpha^k : k = 0, 1, \dots\} \cup \{0\},$$

and all the nonzero elements in the spectrum are simple eigenvalues of $W_{c,K}$. These results extend those in [8] to a general kernel c and to continuous refinement operators on L_K^p for $1 \leq p \leq \infty$. Further, the method here is simpler and more general, and using the same method the results can be further extended to the multivariate setting.

It follows from the spectral properties of $W_{c,K}$ that the continuous refinement equation (1.2) has a unique solution if and only if $\hat{c}(0) = 1$, and the power iteration $W_{c,K}^n \phi_0$ converges to the solution for any normalized initial function ϕ_0. Further, the solution is infinitely differentiable. The adjoint operator W_c^* is related to the continuous subdivision operator (see [4]). It is studied in §4.

We remark that our results on the solution of the continuous refinement equation and the convergence of the power iteration of the continuous refinement operator and the continuous subdivision operator are obtained via the spectral properties of $W_{c,K}$ without recourse to the Paley-Wiener theorem. In one-dimension they are more general than most of the results in [4] and [3]. We also remark that our approach depends on the fact that W_c has an invariant subspace consisting of functions with supports in a compact set. This property is not available for the continuous subdivision operator.

2. The operators W_c and $W_{c,K}$

The operator W_c, and hence $W_{c,K}$, can be expressed as a convolution-dilation operator,

$$(W_c f)(x) := \alpha(c * f)(\alpha x), \qquad f \in L^p(\mathbf{R}). \tag{2.1}$$

Taking the Fourier transforms of the functions in (2.1), it is equivalent to

$$(W_c f)\hat{}(u) := \hat{c}(u/\alpha)\hat{f}(u/\alpha), \qquad f \in L^p(\mathbf{R}). \tag{2.2}$$

Proposition 2.1. *Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. If $c \in L^1(\mathbf{R})$, the operator W_c is a bounded linear operator on $L^p(\mathbf{R})$, and*

$$\|W_c\| \le \alpha^{\frac{1}{q}} \|c\|_1. \tag{2.3}$$

Proof. For any $f \in L^p(\mathbf{R})$,

$$W_c f(x) = \alpha c * f(\alpha x), \qquad x \in \mathbf{R},$$

and

$$\begin{aligned} \|W_c f\|_p &= \left\{ \int_{\mathbf{R}} |\alpha(c * f)(\alpha x)|^p \, dx \right\}^{1/p} \\ &= \left\{ \alpha^{p-1} \int_{\mathbf{R}} |(c * f)(x)|^p \, dx \right\}^{1/p} \\ &= \alpha^{\frac{1}{q}} \|c * f\|_p. \end{aligned}$$

By Young's inequality,

$$\|W_c f\|_p \le \alpha^{\frac{1}{q}} \|c\|_1 \|f\|_p, \quad \text{for all } f \in L^p(\mathbf{R}). \tag{2.4}$$

Hence, W_c is bounded and (2.3) holds. ∎

It follows from Proposition 2.1 that $W_{c,K}$ is also a bounded linear operator on L^p_K. We shall show that $W_{c,K}$ is a compact operator.

Lemma 2.1. *For $f \in L^p_K$, $1 \le p < \infty$, and $h > 0$, define*

$$f^h(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(t) \, dt, \qquad x \in [-K, K], \tag{2.5}$$

and $f^h(x) = 0$ otherwise. Then

$$f^h \in L^p_K \quad \textit{and} \quad \|f^h\|_p \le \|f\|_p, \tag{2.6}$$

$$\|f^h - f\|_p \to 0 \quad \textit{as } h \to 0, \tag{2.7}$$

and

$$(W_{c,K} f)^h = W_{c^{\alpha h},K} f. \tag{2.8}$$

Proof. The relations (2.6) and (2.7) are well-known (see [1]). To prove (2.8), note that

$$(W_{c,K}f)^h(x) = \frac{1}{2h}\int_{x-h}^{x+h} W_{c,K}f(t)\,dt$$
$$= \frac{1}{2h}\int_{x-h}^{x+h}\int_{-K}^{K} \alpha c(\alpha t - y) f(y)\,dy\,dt.$$

Applying Fubini's Theorem a straightforward calculation gives (2.8). □

Proposition 2.2. *If $c \in L^1[-a,a]$, the operator $W_{c,K}$ is a compact operator on L^p_K, for $1 \le p \le \infty$.*

Proof. Let $\{f_n\}$ be a bounded sequence in L^p_K, and suppose that $\|f_n\|_p \le M$. By Young's inequality,

$$\|W_{c,K}f_n\|_p \le \alpha^{\frac{1}{q}}\|c\|_1\|f_n\|_p, \quad \text{for all } n.$$

Therefore, $\{W_{c,K}f_n\}$ is a bounded sequence in L^p_K.

We now show that $\{W_{c,K}f_n\}$ is equicontinuous in L^p_K for $1 \le p < \infty$. By Young's inequality and (2.8),

$$\|(W_{c,K}f_n)^h - W_{c,K}f_n\|_p \le \alpha^{\frac{1}{q}}\|c^{\alpha h} - c\|_1\|f_n\|_p \le \alpha^{\frac{1}{q}}M\|c^{\alpha h} - c\|_1, \quad \text{for all } n.$$

By (2.7), for any $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\|c^{\alpha h} - c\|_1 < \frac{\varepsilon}{\alpha^{\frac{1}{q}}M},$$

for $0 < h < \delta$. Hence,

$$\|(W_{c,K}f_n)^h - W_{c,K}f_n\|_p < \varepsilon,$$

for $0 < h < \delta$. By the Kolmogorov-Tulajkov Theorem ([1], p. 216), $\{W_{c,K}f_n\}$ has a subsequence which converges in L^p_K. Thus $W_{c,K}$ is a compact operator on L^p_K.

If $p = \infty$, then $W_{c,K}f_n$ is continuous for each n. Therefore $\{W_{c,K}f_n\}$ is a bounded sequence in $C([-K,K])$. A similar argument as above shows that $\{W_{c,K}f_n\}$ is equicontinuous. By Ascoli's Theorem, $\{W_{c,K}f_n\}$ is a relatively compact set in $C([-K,K])$. Therefore, $W_{c,K}$ is a compact operator on L^∞_K. □

3. Spectrum of $W_{c,K}$

In this section we give a complete description of the spectrum of the compact operator $W_{c,K}$, and deduce properties of the solutions of the continuous refinement equation from the spectral properties of $W_{c,K}$. Let $\sigma_p(W_{c,K})$ denote the spectrum of $W_{c,K} : L^p_K \to L^p_K$ and let $r_p(W_{c,K})$ denote its spectral radius.

Lemma 3.1. *Suppose $\hat{c}(0) = 1$. If $\lambda \ne 0$ is an eigenvalue of $W_{c,K}$, then $\lambda = 1/\alpha^k$ for some nonnegative integer k.*

Proof. Let f be an eigenfunction of $W_{c,K}$ corresponding to the eigenvalue λ. Taking the Fourier transforms of the functions on both sides of the equation $W_{c,K}f = \lambda f$, we have

$$\hat{c}(u/\alpha)\hat{f}(u/\alpha) = \lambda\hat{f}(u), \quad u \in \mathbf{R}. \tag{3.1}$$

Suppose $\hat{f}(0) = \cdots = \hat{f}^{(k-1)}(0) = 0$ but $\hat{f}^{(k)}(0) \ne 0$. Then $\hat{f}(u) = u^k g(u)$, $u \in \mathbf{R}$, for some analytic function g with $g(0) \ne 0$. Equation (3.1) becomes

$$\hat{c}(u/\alpha)u^k g(u/\alpha)/\alpha^k = \lambda u^k g(u).$$

It follows that

$$\hat{c}(u/\alpha)g(u/\alpha) = \alpha^k \lambda g(u).$$

Taking the limit as $u \to 0$ in the above equation, we obtain $g(0) = \alpha^k \lambda g(0)$. Therefore $\alpha^k \lambda = 1$, and hence $\lambda = 1/\alpha^k$, as desired. $\square$

Proposition 3.1. *Suppose that $\hat{c}(0) = 1$, and let k be a nonnegative integer. If ϕ is a nontrivial function in L_K^p such that $(W_{c,K} - 1/\alpha^k)^m \phi = 0$ for some positive integer m, then*

$$\hat{\phi}(0) = \cdots = \hat{\phi}^{(k-1)}(0) = 0 \quad \textit{and} \quad \hat{\phi}^{(k)}(0) \neq 0. \tag{3.2}$$

Consequently, any nonzero eigenvalue of $W_{c,K}$ is simple.

Proof. Let j be the smallest nonnegative integer such that $\hat{\phi}^{(j)}(0) \neq 0$. Then $\hat{\phi}$ can be written as $\hat{\phi}(u) = u^j g(u)$, $u \in \mathbf{R}$, where g is an analytic function with $g(0) \neq 0$. For $n = 1, 2, \ldots, m$, let $\phi_n := (W_{c,K} - 1/\alpha^k)^n \phi$. Taking the Fourier transforms of the functions on both sides of $\phi_1 = (W_{c,K} - 1/\alpha^k)\phi$ leads to

$$\begin{aligned} \hat{\phi}_1(u) &= \hat{c}(u/\alpha)\hat{\phi}(u/\alpha) - \hat{\phi}(u)/\alpha^k \\ &= u^j(\hat{c}(u/\alpha)g(u/\alpha)/\alpha^j - g(u)/\alpha^k), \qquad u \in \mathbf{R}. \end{aligned} \tag{3.3}$$

It follows that if $j \neq k$, then $\hat{\phi}_1^{(j)}(0) \neq 0$. Since $\phi_n = (W_{c,K} - 1/\alpha^k)\phi_{n-1}$, an inductive argument shows that if $j \neq k$, then $\hat{\phi}_n^{(j)}(0) \neq 0$ for all $n = 1, 2, \ldots, m$. Therefore, $(W_{c,K} - 1/\alpha^k)^m \phi = 0$ implies that $j = k$, i.e., (3.2) holds.

Now suppose that $\lambda \neq 0$ is an eigenvalue of $W_{c,K}$. Then by Lemma 3.1, $\lambda = 1/\alpha^k$ for some nonnegative integer k. Let $\phi \in L_K^p$ be the corresponding eigenfunction, and let $\varphi \in L_K^p$ be a nontrivial function such that $(W_{c,K} - 1/\alpha^k)^m \varphi = 0$ for some positive integer m. Then (3.2) also holds for φ. Let $\psi := \varphi - t\phi$, where $t := \hat{\varphi}^{(k)}(0)/\hat{\phi}^{(k)}(0)$. Then $(W_{c,K} - 1/\alpha^k)^m \psi = 0$. But $\hat{\psi}(0) = \cdots = \hat{\psi}^{(k-1)}(0) = \hat{\psi}^{(k)}(0) = 0$, and by what has been just proved, we must have $\psi = 0$. It follows that $\varphi = t\phi$. Therefore, the eigenvalue $1/\alpha^k$ is simple. $\square$

Theorem 3.1. *Suppose ϕ_0 is a function in L_K^p such that*

$$\hat{\phi}_0(0) = \cdots = \hat{\phi}_0^{(k-1)}(0) = 0 \quad \textit{and} \quad \hat{\phi}_0^{(k)}(0) \neq 0.$$

Then $(\alpha^k W_{c,K})^n \phi_0$ converges in the L_p-norm to an eigenfunction of $W_{c,K}$ corresponding to the eigenvalue $1/\alpha^k$.

Proof. For $k = 0, 1, 2, \ldots$, let

$$N_k := \{f \in L_K^p : \hat{f}(0) = \cdots = \hat{f}^{(k-1)}(0) = 0\}.$$

Then N_k is an invariant subspace of $W_{c,K}$. By Proposition 3.1, $\sigma_p(W_{c,K}|_{N_{k+1}})$ does not contain $1/\alpha^j$ for $j = 0, \ldots, k$. Hence

$$r_p(W_{c,K}|_{N_{k+1}}) \leq 1/\alpha^{k+1}. \tag{3.4}$$

Let $\psi := \alpha^k W_{c,K}\phi_0 - \phi_0$. By the assumption on ϕ_0, we have $\hat{\phi}_0(u) = u^k g(u)$, $u \in \mathbf{R}$, for some analytic function g with $g(0) \neq 0$. Hence

$$\hat{\psi}(u) = \alpha^k \hat{\phi}_0(u/\alpha)\hat{c}(u/\alpha) - \hat{\phi}_0(u) = u^k(g(u/\alpha)\hat{c}(u/\alpha) - g(u)).$$

It follows that $\hat{\psi}(0) = \cdots = \hat{\psi}^{(k)}(0) = 0$. In other words, $\psi \in N_{k+1}$. From (3.4) we see that there exist constants $C > 0$ and r, $0 < r < 1$, such that

$$\|(W_{c,K})^n \psi\|_p \le C(r/\alpha^k)^n, \qquad n = 1, 2, \dots .$$

Therefore we have

$$\|(\alpha^k W_{c,K})^{n+1} f - (\alpha^k W_{c,K})^n f\|_p = \|(\alpha^k W_{c,K})^n \psi\|_p \le C r^n.$$

This shows that $(\alpha^k W_{c,K})^n f$ converges in the L_p-norm to some function $\phi \in L_K^p$. Clearly, ϕ is an eigenfunction of $W_{c,K}$ corresponding to the eigenvalue $1/\alpha^k$. □

Corollary 3.1. *Suppose $c \in L^1(\mathbf{R})$ is compactly supported and $\hat{c}(0) = 1$. Then*

$$\sigma_p(W_{c,K}) = \{1/\alpha^k : k = 0, 1, \dots\} \cup \{0\}. \tag{3.5}$$

The nonzero elements of $\sigma_p(W_{c,K})$ are simple eigenvalues of $W_{c,K}$. Further, 0 is not an eigenvalue of $W_{c,K}$.

Proof. The result (3.5) and the simplicity of the eigenvalues follow from Lemma 3.1, Proposition 3.1 and Theorem 3.1.

To show that 0 is not an eigenvalue of $W_{c,K}$, suppose that $f \in L_K^p$ such that $W_{c,K} f = 0$. Then

$$\hat{c}(u)\hat{f}(u) = 0, \quad \text{for all } u \in \mathbf{R}.$$

Since $\hat{c}$ is entire and $\hat{c}(0) = 1$, there is a positive number δ such that $\hat{c}(z) \ne 0$ for all $|z| < \delta$. Hence, $\hat{f}(z) = 0$ for all $|z| < \delta$. It follows that $\hat{f} = 0$, and hence $f = 0$. Therefore, 0 is not an eigenvalue of $W_{c,K}$. □

Corollary 3.2. *Suppose that $c \in L^1(\mathbf{R})$ is compactly supported. The continuous refinement equation* (1.2) *has a unique nontrivial solution in L_K^p if and only if $\hat{c}(0) = 1$.*

Theorem 3.2. *Suppose that $c \in L^1(\mathbf{R})$ with* $\operatorname{supp}(c) \subset [-a, a]$ *and $\hat{c}(0) = 1$. Let $\phi \in L_K^p(\mathbf{R})$ be the solution of the continuous refinement equation* (1.2) *with $\hat{\phi}(0) = 1$. Then $\phi \in C^\infty(\mathbf{R})$ and, for each positive integer k, $\phi^{(k)}$ is an eigenfunction of $W_{c,K}$ corresponding to the eigenvalue $1/\alpha^k$.*

Further, $\operatorname{supp}(\phi) \subset [-a/(\alpha - 1), a/(\alpha - 1)]$.

Proof. Taking the Fourier transforms of the functions on both sides of the equation $W_{c,K}\phi = \phi$, we obtain $\hat{c}(u/\alpha) = \hat{\phi}(u)/\hat{\phi}(u/\alpha)$. It follows that

$$\prod_{j=1}^{n} \hat{c}(u/\alpha^j) = \hat{\phi}(u)/\hat{\phi}(u/\alpha^n), \qquad u \in \mathbf{R}.$$

Since $\hat{\phi}(0) = 1$, this shows that $\prod_{j=1}^{n} \hat{c}(u/\alpha^j)$ converges to $\hat{\phi}(u)$ for each $u \in \mathbf{R}$.

Let φ be an eigenfunction of $W_{c,K}$ corresponding to the eigenvalue $1/\alpha^k$. By Proposition 3.1, $\hat{\varphi}(u) = u^k g(u)$, $u \in \mathbf{R}$, for some analytic function g with $g(0) \ne 0$. Since $\varphi = \alpha^k W_{c,K}\varphi$, we have $\hat{\varphi}(u) = \alpha^k \hat{c}(u/\alpha)\hat{\varphi}(u/\alpha)$. Iterating n times gives

$$\hat{\varphi}(u) = \alpha^{nk}\hat{\varphi}(u/\alpha^n) \prod_{j=1}^{n} \hat{c}(u/\alpha^j) = u^k g(u/\alpha^n) \prod_{j=1}^{n} \hat{c}(u/\alpha^j).$$

Taking the limit as $n \to \infty$ gives

$$\hat{\varphi}(u) = g(0) u^k \hat{\varphi}(u), \qquad u \in \mathbf{R}.$$

Hence the k-order derivative $\phi^{(k)}$ of ϕ exists and $\varphi = t\varphi^{(k)}$ for some $t \in \mathbf{C}$. Since this holds for all $k = 0, 1, \ldots$, it follows that $\phi \in C^\infty(\mathbf{R})$.

Note that if ϕ_0 is continuous and $\operatorname{supp}(\phi_0) \subset [-K, K]$, then ϕ_n is continuous and $\operatorname{supp}(\phi_n) \subset [-l_n, l_n]$, for $n = 1, 2, \ldots$, where

$$l_n := \frac{a\alpha^{n-1} + a\alpha^{n-2} + \cdots + K}{\alpha^n} = \frac{a(\alpha^n - 1) + K(\alpha - 1)}{(\alpha - 1)\alpha^n}.$$

Since ϕ_n converges to ϕ in $L^p(\mathbf{R})$, $1 \le p \le \infty$, and $l_n \to a/(\alpha - 1)$, as $n \to \infty$, it follows that ϕ is supported in $[-a/(\alpha-1), a/(\alpha-1)]$. □

4. The adjoint W^*_{cK} and the continuous subdivision operator

For $1 \le p < \infty$, the adjoint of W_c is the operator $W_c^* : L^q(\mathbf{R}) \to L^q(\mathbf{R})$ defined by

$$(W_c^* g)(x) = \int_{\mathbf{R}} \alpha c(\alpha y - x) g(y)\, dy, \qquad g \in L^q(\mathbf{R}). \tag{4.1}$$

A straightforward calculation shows that

$$(W_c^* g)(x) = \int_{-\frac{a}{\alpha}}^{\frac{a}{\alpha}} \alpha c(-\alpha y) g(x/\alpha - y)\, dy, \qquad g \in L^q(\mathbf{R}). \tag{4.2}$$

Taking the Fourier transforms of the functions in (4.1) leads to

$$(W_c^* g)\widehat{\ }(u) = \alpha \hat{c}(-u) \hat{g}(\alpha u), \qquad g \in L^q(\mathbf{R}). \tag{4.3}$$

A spectral radius formula for $r_q(W_c^*)$ is derived in [6].

Note that L_K^q is not an invariant subspace of W_c^* since the mapping $W_c^* f$ expands the support of f. In order to describe the adjoint of the restricted operator $W_{c,K}$, let

$$\begin{aligned} G(x) &:= \int_{-K}^{K} \alpha c(\alpha y - x) g(y)\, dy \\ &= \int_{\frac{x-a}{\alpha}}^{\frac{x+a}{\alpha}} \alpha c(\alpha y - x) g(y)\, dy, \qquad x \in \mathbf{R}. \end{aligned} \tag{4.4}$$

The function G is compactly supported with support in $[-a - \alpha K, a + \alpha K]$. Then

$$(W_{c,K}^* g)(x) = \begin{cases} G(x), & x \in [-K, K], \\ 0, & \text{otherwise}. \end{cases} \tag{4.5}$$

In general, the simple relationship that exists between the Fourier transforms of $W_{c,K} f$ and f is not available anymore for the conjugate operator $W_{c,K}^*$, but we do have

$$\widehat{G}(u) = \alpha \hat{c}(-u) \hat{g}(\alpha u), \qquad g \in L_K^p. \tag{4.6}$$

The following results follow from Corollary 3.1 and the properties of adjoint operators.

Theorem 4.1. *Suppose $c \in L^1(\mathbf{R})$ with* $\operatorname{supp}(c) \subset [-a, a]$ *and $\hat{c}(0) = 1$. Then*

$$\sigma_q(W_{c,K}^*) = \{1/\alpha^k : k = 0, 1, \ldots\} \cup \{0\}. \tag{4.7}$$

The nonzero elements of the spectrum are simple eigenvalues of $W_{c,K}^*$.

Note that the eigenfunctions p_n of $W^*_{c,K}$ and the eigenfunctions $\phi^{(k)}$ of $W_{c,K}$ corresponding to the eigenvalues $1/\alpha^n$, $n = 0, 1, \ldots,$ form a biorthogonal pair of sequences. It is easy to see that $p_0(x) = 1$. It was shown by Kabaya and Iri [8] that for the kernel

$$c = \frac{1}{2(\alpha - 1)}\chi_{[-\alpha+1,\alpha-1]},$$

the eigenfunctions p_n are polynomials. In general if $c \in L^1(\mathbf{R})$, they are not.

The adjoint operator W^*_c is related to the *continuous subdivision operator*

$$(S_c g)(x) := \int_{\mathbf{R}} \alpha c(x - \alpha y) g(y)\, dy, \qquad g \in L^q(\mathbf{R}). \tag{4.8}$$

Clearly, $W^*_{\tilde{c}} = S_c$, where $\tilde{c}(x) = c(-x), x \in \mathbf{R}$. From an initial function f_0, the power iteration

$$f_n := S_c^n f_0, \qquad n = 0, 1, \ldots,$$

defines a *continuous subdivision process* (or *integral subdivision process*) studied in [4] and [3].

Let $D_\alpha : L^q(\mathbf{R}) \to L^q(\mathbf{R})$ be the dilation operator

$$D_\alpha f(x) := \sqrt{\alpha} f(\alpha x),$$

and define $\widetilde{W}^*_{\tilde{c},K} : L^q_K \to L^q_K$ by

$$\widetilde{W}^*_{\tilde{c},K} := D_\alpha W^*_{\tilde{c},K} D_\alpha. \tag{4.9}$$

Lemma 4.1. *For any compactly supported $c \in L^1(\mathbf{R})$,*

$$\widetilde{W}^*_{\tilde{c},K} = W_{c,K}. \tag{4.10}$$

Proof. A straightforward substitution using (4.4), (4.5) and (4.9) gives

$$(\widetilde{W}^*_{c,K})g(x) = \int_{-K/\alpha}^{K/\alpha} \alpha^2 c(\alpha y - \alpha x) g(\alpha y)\, dy, \qquad g \in L^q_K, \tag{4.11}$$

for $x \in [-K, K]$. The relation (4.10) follows by a change of variable. □

The following result on the convergence of the continuous subdivision process follows from Theorem 3.1.

Theorem 4.2. *Suppose ϕ_0 is a function in L^q_K such that*

$$\hat{\phi}_0(0) = \cdots = \hat{\phi}_0^{(k-1)}(0) = 0 \quad \text{and} \quad \hat{\phi}_0^{(k)}(0) \neq 0,$$

for some nonnegative integer k. Then $(\alpha^k \widetilde{W}^*_{\tilde{c},K})^n \phi_0$ converges in the L_q-norm to an eigenfunction of $W_{c,K}$ corresponding to the eigenvalue $1/\alpha^k$.

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