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ZERO DIVISORS AND $L^p(G)$

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Abstract. Let G be a discrete group, $\mathbb{C}G$ the group ring of G over $\mathbb{C}$ and $L^p(G)$ the Lebesgue space of G with respect to Haar measure. It is known that if G is torsion free elementary amenable, $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^2(G)$, then $\alpha * \beta \neq 0$. We will give a sufficient condition for this to be true when $p > 2$, and in the case $G = \mathbb{Z}^n$ we will give sufficient conditions for this to be false when $p > 2$.

1. Introduction

Let G be a discrete group, and let f be a complex valued function on G. We may represent f as a formal sum $\sum_{g\in G} a_g g$ where $a_g \in \mathbb{C}$ and $f(g) = a_g$. With respect to the counting measure on G the Lebesgue spaces $L^\infty(G), C_0(G)$ and $L^p(G)$, $1 \leq p < \infty$, may be thought of in the following ways. $L^\infty(G)$ will consist of all formal sums where $\sup_{g\in G} |a_g| < \infty$, $C_0(G)$ will consist of all formal sums for which the set $\{g | |a_g| > \varepsilon\}$ is finite for all $\varepsilon > 0$, and $L^p(G)$ will consist of all formal sums where $\sum_{g\in G} |a_g|^p < \infty$. Let $\mathbb{C}G$ be the group ring of G over $\mathbb{C}$, so $\mathbb{C}G$ consists of all formal sums $\sum_{g\in G} a_g g$ where $a_g = 0$ for all but finitely many g. Then $\mathbb{C}G$ can also be thought of as the complex valued functions on G with compact support. The following inclusions are clear:

$$\mathbb{C}G \subseteq L^p(G) \subseteq C_0(G) \subseteq L^\infty(G).$$

For $\alpha = \sum_{g\in G} a_g g \in \mathbb{C}G$ and $\beta = \sum_{g\in G} b_g g \in L^p(G)$, $1 \leq p \leq \infty$, we define a multiplication $*$: $L^1(G) \times L^p(G) \to L^p(G)$ by

$$\alpha * \beta = \sum_{g,h} a_g b_h gh = \sum_{g\in G} \left(\sum_{h\in G} a_{gh^{-1}} b_h \right) g.$$

In [4] it is shown that if G is torsion free elementary amenable, $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^2(G)$, then $\alpha * \beta \neq 0$. A natural question to ask is does this remain true if 2 is replaced by any $p < \infty$.

Let α be an element of $L^1(G)$, and let $1 \leq p < \infty$. If there exists a nonzero β in $L^p(G)$ such that $\alpha * \beta = 0$, then we shall say that α is a *p-zero divisor*. If $\alpha * \beta \neq 0$ for all $\beta \in C_0(G) \setminus 0$, then we shall say that α is a *uniform nonzero divisor*. We shall see that if $\alpha \in \mathbb{C}\mathbb{Z}$, then α is a uniform nonzero divisor; however, this is not

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true in general. In this paper we will give sufficient conditions for $\alpha \in \mathbb{C}\mathbb{Z}^n$, $n \geq 2$, to be a p-zero divisor. Also sufficient conditions will be given for α to be a uniform nonzero divisor. The proofs of these results will reveal a connection between p-zero divisors and the concepts of sets of uniqueness and spectral synthesis from Fourier analysis. Examples of p-zero divisors will also be given, along with an application to the theory of sets of uniqueness.

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2. Statement of main results

For the rest of this paper, unless otherwise stated, assume that G is abelian. Let Γ be the group of homomorphisms from G to the unit circle in $\mathbb{C}$. Γ is known as the dual group of G. Also a topology can be defined on Γ that makes it a topological group. If $f = \sum_{g \in G} a_g g \in L^1(G)$, the Fourier transform of f is defined by

$$\hat{f}(\gamma) = \sum_{g \in G} a_g (g^{-1}, \gamma)$$

where $\gamma \in \Gamma$ and $(g^{-1}, \gamma) = \gamma(g^{-1})$. Let $Z(f)$ denote the elements γ in Γ such that $\hat{f}(\gamma) = 0$. We shall prove

Theorem 1. *Let G be torsion free and let $\alpha \in L^1(G)$. If $Z(\alpha)$ is a proper subgroup of Γ, then α is a uniform nonzero divisor.*

It is well known that $\mathbb{T}^n$, where $\mathbb{T} = [-\pi, \pi]/\{-\pi \sim \pi\}$, is the dual group of $\mathbb{Z}^n$. Let $V = (-\pi, \pi)^n$ and identify V with an open subspace of $\mathbb{T}^n$. Using Theorem 1 we will then show that if $\alpha \in L^1(\mathbb{Z}^n)$ and $Z(\alpha)$ is contained in a finite union of proper closed cosets on $\mathbb{T}^n$, then α is a uniform nonzero divisor. Let $\alpha \in L^1(\mathbb{Z}^n)$ and $x_0 \in E := Z(\alpha) \cap V$. We shall say that x_0 is a *regular point* if there exists an open neighborhood U of x_0 such that $F_{x_0} = U \cap E$ is a smooth m-dimensional submanifold of V, where m is a natural number. Consider F_{x_0} in a sufficiently small neighborhood of x_0 and write F_{x_0} as the image of a smooth mapping $\phi : W \to V$, where W is a neighborhood of the origin in $\mathbb{R}^m$. Also assume that the vectors $\frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_m}$ are linearly independent for each $x \in W$. Now suppose for each $y_0 \in W$ and each unit vector $\eta \in V$, there is a multi-index j, with $|j| \geq 1$, so that

$$\partial_x^j [\phi(x) \cdot \eta]|_{x=y_0} \neq 0$$

(where of course $\phi(x) \cdot \eta$ is the usual Euclidean inner product). The smallest k so that, for each unit vector η, there exists a j with $|j| \leq k$ for which

$$\partial_x^j [\phi(x) \cdot \eta]|_{x=y_0} \neq 0,$$

is called the *type* of ϕ (and the type of F_{x_0}) at y_0. Also, if $W_1 \subset W$ is a compact set, the type of ϕ in W_1 is defined to be the maximum of the type of the $y_0 \in W_1$. We shall prove

Theorem 2. *Let $\alpha \in L^1(\mathbb{Z}^n)$, $n \geq 2$, and suppose that x_0 is a regular point in $Z(\alpha)$. If F_{x_0} is of finite type k, then α is a p-zero divisor for $p > nk$.*

3. Preliminaries

Let p be a real number that is greater than or equal to one, and let q denote the conjugate index of p, i.e., if $p > 1$ then $\frac{1}{p} + \frac{1}{q} = 1$, and $q = \infty$ if $p = 1$. Let $y \in G$, let $f = \sum_{g \in G} a_g g \in L^p(G)$, and denote the right translation of f by y with f_y, where $f_y(x) = f(xy^{-1})$. Let $h = \sum_{g \in G} b_g g \in L^q(G)$ and define a map $\langle \cdot, \cdot \rangle : L^p(G) \times L^q(G) \to \mathbb{C}$ by $\langle f, h \rangle = \sum_{g \in G} a_g \overline{b_g}$. Fix $h \in L^q(G)$. Then $\langle \cdot, h \rangle$ is a continuous linear functional on $L^p(G)$. By the Riesz representation theorem every continuous linear functional on $L^p(G)$ is of this form. Let $T^p[f]$ be the closure in $L^p(G)$ of the set of linear combinations of translates of f. By the Hahn-Banach theorem, $T^p[f] = L^p(G)$ if and only if no nonzero continuous linear functional on $L^p(G)$ vanishes on all translates of f. For $\beta = \sum_{g \in G} a_g g \in L^p(G)$, set $\beta^* = \sum_{g \in G} \overline{a_g} g^{-1}$.

Lemma 1. *Let $\alpha = \sum_{g \in G} a_g g$ be an element of $L^1(G)$. Then α is a p-zero divisor if and only of $T^q[\alpha]$ is not equal to $L^q(G)$.*

Proof. Let $\beta = \sum_{g \in G} b_g g \in L^p(G)$; then

$$\langle \alpha_y, \beta^* \rangle = \sum_{g \in G} a_{gy^{-1}} b_{g^{-1}} = \sum_{g \in G} a_{y^{-1}g^{-1}} b_g = (\alpha * \beta^*)(y^{-1}).$$

So $\alpha * \beta^* = 0$ if and only if $\langle \alpha_y, \beta^* \rangle = 0$ for all $y \in G$. □

Remark. If $\alpha \in \mathbb{C}G$, then α is a uniform nonzero divisor if and only if $T^q[\alpha] = L^q(G)$ for all q, $1 < q < \infty$. If $Z(\alpha)$ is finite, then α is a uniform nonzero divisor by Theorem 4.2 of [2]. Hence, nonzero elements of $\mathbb{C}\mathbb{Z}$ are uniform nonzero divisors.

Recall that Γ denotes the dual group of G. Let $M(\Gamma)$ be the set of bounded regular Borel measures on Γ. Let E be a closed subset of Γ. Denote by $M(E)$ the elements in $M(\Gamma)$ that are concentrated on E. For $\mu \in M(\Gamma)$, the Fourier-Stieltjes transform of μ is defined by

$$\hat{\mu}(x) = \int_\Gamma (x, \gamma)\, d\mu(\gamma)$$

where $x \in G$. Clearly $\hat{\mu} \in L^\infty(G)$, so $\hat{\mu}$ is a continuous linear functional on $L^1(G)$. Set $\hat{\mu}^*(g) = \overline{\hat{\mu}(g^{-1})}$.

Lemma 2. *If $f = \sum_{g \in G} a_g g \in L^1(G)$ and $\mu \in M(Z(f))$, then $\langle f_y, \hat{\mu}^* \rangle = 0$ for all $y \in G$.*

Proof.

$$\langle f, \hat{\mu}^* \rangle = \sum_{g \in G} a_g \overline{\hat{\mu}^*(g)} = \sum_{g \in G} a_g \int_\Gamma (g^{-1}, \gamma)\, d\mu(\gamma)$$
$$= \int_\Gamma \sum_{g \in G} a_g (g^{-1}, \gamma)\, d\mu(\gamma) = 0$$

since $\sum_{g \in G} a_g (g^{-1}, \gamma) = \hat{f}(\gamma)$. The lemma now follows after we make the observation that $Z(f_y) = Z(f)$. □

Remark. If $\alpha \in L^1(G)$ and $\mu \in M(Z(\alpha))$, then $\alpha * \hat{\mu}^* = 0$.

Let E be a closed subset of Γ, $I(E)$ the set of all $f \in L^1(G)$ such that $E \subseteq Z(f)$ and $j(E)$ the set of all $f \in L^1(G)$ such that $E \subseteq O \subseteq Z(f)$, where O is an open set in Γ. Denote by $J(E)$ the closure of $j(E)$ in the $L^1(G)$-norm. $I(E)$ and $J(E)$ are translation-invariant subspaces of $L^1(G)$ and $J(E) \subseteq I(E)$. Let $\Phi(E) = \{h \in L^\infty(G) | \langle f, h \rangle = 0, f \in I(E)\}$ and $\Psi(E) = \{h \in L^\infty(G) | \langle f, h \rangle = 0, f \in J(E)\}$. The following are easily deduced from Chapter 7 in [6], and will be used in the sequel.

1. $\Phi(E)$ is the weak*-closed subspace of $L^\infty(G)$ generated by E.
2. $\Phi(E)$ is the weak*-closure of $\{\hat{\mu} | \mu \in M(E)\}$.
3. If $\alpha * h = 0$, where $\alpha \in \mathbb{C}G$ and $h \in L^\infty(G)$, then $h \in \Psi(Z(\alpha))$.

E is said to be a set of *spectral synthesis* (S-set) if $I(E) = J(E)$. E is a set of *uniqueness* if $\Psi(E) \cap C_0(G) = 0$. If E is not a set of uniqueness, then E is a set of *multiplicity*.

4. A result on sets of uniqueness

In this section we will prove a result on the union of sets of uniqueness. This result will be used to show certain elements of $L^1(\mathbb{Z}^n)$ are uniform nonzero divisors. Our result may or may not be new, but we record it here for completeness. For more information about sets of uniqueness see [1].

We will begin by showing that given a closed set E on $\mathbb{T}^n$, there exists an $f \in L^1(\mathbb{Z}^n)$ such that $Z(f) = E$. Before we do this we need to prove a technical lemma. If $f \in L^1(\mathbb{Z}^n)$, the Fourier transform of f is

$$\hat{f}(t) = \sum_{m \in \mathbb{Z}^n} f(m) e^{-i(m \cdot t)}$$

where $t \in \mathbb{T}^n$ and $m \cdot t$ is the usual Euclidean inner product. The Fourier transform induces an isometry between $L^2(\mathbb{Z}^n)$ and $L^2(\mathbb{T}^n)$. For $g \in L^2(\mathbb{T}^n)$, the inverse map to the Fourier transform is given by

$$\check{g}(m) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} g(t_1, \ldots, t_n) e^{i(\sum_{k=1}^n m_k t_k)} dt_1 \cdots dt_n$$

where $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Recall that $C^\infty(\mathbb{T}^n)$ denotes the infinitely differentiable functions on $\mathbb{T}^n$. The next lemma is a generalization of exercise 4 from page 30 of [3].

Lemma 3. *If $g \in C^\infty(\mathbb{T}^n)$, then*

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} |\check{g}(m)| \leq \|g\|_{L^1(\mathbb{T}^n)} &+ \sqrt{\left(\sum_{k=1}^\infty \frac{2}{k^2}\right)^n} \left\| \frac{\partial^n g}{\partial x_1 \cdots \partial x_n} \right\|_{L^2(\mathbb{T}^n)} \\ &+ \sqrt{\left(\sum_{k=1}^\infty \frac{2}{k^2}\right)^{n-1}} \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} \left\| \frac{\partial^{n-1} g}{\partial x_{i_1} \cdots \partial x_{i_{n-1}}} \right\|_{L^2(\mathbb{T}^n)} \\ &+ \sqrt{\left(\sum_{k=1}^\infty \frac{2}{k^2}\right)^{n-2}} \sum_{1 \leq i_1 < \cdots < i_{n-2} \leq n} \left\| \frac{\partial^{n-2} g}{\partial x_{i_1} \cdots \partial x_{i_{n-2}}} \right\|_{L^2(\mathbb{T}^n)} \\ &+ \cdots + \sqrt{\sum_{k=1}^\infty \frac{2}{k^2}} \sum_{1 \leq i \leq n} \left\| \frac{\partial g}{\partial x_i} \right\|_{L^2(\mathbb{T}^n)}. \end{aligned}$$

Proof. Since our proof is easy to generalize, we will prove the lemma for the case $n = 2$. To begin with

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} |\check{g}(n_1,n_2)| = |\check{g}(0,0)| + \sum_{n_1\neq 0} |\check{g}(n_1,0)| + \sum_{n_2\neq 0} |\check{g}(0,n_2)| + \sum_{n_1\neq 0, n_2\neq 0} |\check{g}(n_1,n_2)|.$$

Using integration by parts we obtain

$$\begin{aligned}\frac{\partial^2 \check{g}}{\partial x_1 \partial x_2}(n_1,n_2) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial^2 g}{\partial x_1 \partial x_2}(x_1,x_2)e^{i(n_1x_1+n_2x_2)}\,dx_1\,dx_2 \\ &= \frac{-n_1n_2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} g(x_1,x_2)e^{i(n_1x_1+n_2x_2)}\,dx_1\,dx_2 \\ &= -n_1n_2\check{g}(n_1,n_2).\end{aligned}$$

With the help of the Cauchy-Schwarz inequality and Parseval's relation we obtain

$$\begin{aligned}\sum_{n_1\neq 0, n_2\neq 0} |\check{g}(n_1,n_2)| &= \sum_{n_1\neq 0, n_2\neq 0} \left|\frac{1}{n_1n_2}\frac{\partial^2 \check{g}}{\partial x_1 \partial x_2}(n_1,n_2)\right| \\ &\leq \sqrt{\left(\sum_{k=1}^{\infty}\frac{2}{k^2}\right)^2}\left\|\frac{\partial^2 g}{\partial x_1 \partial x_2}\right\|_{L^2(\mathbb{T}^2)}.\end{aligned}$$

It can be shown by similar calculations that

$$\sum_{n_1\neq 0} |\check{g}(n_1,0)| \leq \sqrt{\sum_{k=1}^{\infty}\left(\frac{2}{k^2}\right)^2}\left\|\frac{\partial g}{\partial x_1}\right\|_{L^2(\mathbb{T}^2)}$$

and

$$\sum_{n_2\neq 0} |\check{g}(0,n_2)| \leq \sqrt{\sum_{k=1}^{\infty}\left(\frac{2}{k^2}\right)^2}\left\|\frac{\partial g}{\partial x_2}\right\|_{L^2(\mathbb{T}^2)}.$$

The lemma follows after we make the observation that $|\check{g}(0,0)| \leq \|g\|_{L^1(\mathbb{T}^2)}$. □

Corollary 1. *If $g \in C^\infty(\mathbb{T}^n)$, then there exists an $f \in L^1(\mathbb{Z}^n)$ such that $\hat{f} = g$.*

Let E be a closed subset of $V = (-\pi,\pi)^n$ and set d equal to the Euclidean distance from E to $\mathbb{R}^n \setminus V$. Let U_r be the set of points of $\mathbb{R}^n$ at a distance less than $\frac{d}{r}$ from E, where r is a natural number. Edwards constructs a function F on $\mathbb{R}^n$ [2, pp. 229–230] such that $Z(F) = E$ and $F \in C^\infty(\mathbb{R}^n)$, the infinitely differentiable functions on $\mathbb{R}^n$. By using the U_r's defined above in place of the U_r's used in Edward's argument, we construct a function g such that $Z(g) = E$ and g is constant on the frontier of V, so $g \in C^\infty(\mathbb{T}^n)$. By the above corollary we see that given a closed set $E \subseteq V$ there exists an $f \in L^1(\mathbb{Z}^n)$ such that $Z(f) = E$. Now suppose that E is any closed set on $\mathbb{T}^n$. Write E as the union of two nonempty closed sets E_1, E_2 and let $f_1, f_2 \in L^1(\mathbb{Z}^n)$ such that $Z(f_i) = E_i$. Since $L^1(\mathbb{Z}^n)$ is a ring and $Z(f_1 * f_2) = Z(f_1) \cup Z(f_2) = E$, we can conclude that given a closed set E on $\mathbb{T}^n$ there exists an $f \in L^1(\mathbb{Z}^n)$ such that $Z(f) = E$. We are now ready to give our result.

Proposition 1. *Suppose that E_1, E_2 are closed sets of uniqueness on $\mathbb{T}^n$. If $E_1 \cup E_2$ is an S-set, then $E_1 \cup E_2$ is a set of uniqueness.*

Proof. From the discussion above we know that there exist functions f_1, f_2 in $L^1(\mathbb{Z}^n)$ such that $E_i = Z(f_i)$. Let $h \in C_0(\mathbb{Z}^n)$. Since the E_i's are sets of uniqueness and $f_1 * f_2 \in L^1(\mathbb{Z}^n)$, it follows that $(f_1 * f_2) * h = 0$ if and only if $h = 0$. Since $E_1 \cup E_2 = Z(f_1 * f_2)$ and $E_1 \cup E_2$ is an S-set we see that $E_1 \cup E_2$ is a set of uniqueness. □

5. Proof of Theorem 1 and related results

Let $A = \{a \in G | \gamma(a) = 1 \text{ for all } \gamma \in Z(\alpha)\}$. A is known as the annihilator subgroup of $Z(\alpha)$ and $A \neq 0$ since $Z(\alpha) \neq \Gamma$. Since G is torsion free, A is infinite. Let $h \in \Phi(Z(\alpha))$ and fix $g \in G$ such that $h(g) \neq 0$; now $|h(ga)| = |h(g)|$ for all $a \in A$ since $\gamma(ga) = \gamma(g)\gamma(a) = \gamma(g)$. Thus h is not in $C_0(G)$. $Z(\alpha)$ is an S-set (Theorem 7.5.2(d), [6]), so $\Psi(Z(\alpha)) \cap C_0(G) = 0$ and the theorem follows.

Note that the theorem is still true if the subgroup generated by $Z(\alpha)$ is not all of Γ. The following corollary is undoubtedly well known. However, we cannot find a suitable reference, so we record it here for completeness.

Corollary 2. *If $Z(\alpha)$ is a proper subgroup of Γ, then $Z(\alpha)$ is a set of uniqueness.*

Corollary 3. *Let $\alpha \in L^1(\mathbb{Z}^n)$, $n \geq 2$. If $Z(\alpha)$ is contained in a finite union of proper closed cosets, then α is a uniform nonzero divisor.*

Proof. Let $\beta(x_1, \ldots, x_n) = x_1 - x_n$, so $\hat{\beta}(\theta_1, \ldots, \theta_n) = e^{-i\theta_1} - e^{-i\theta_n}$ and $Z(\beta) = \{(\theta_1, \theta_2, \ldots, \theta_{n-1}, \theta_1) | -\pi \leq \theta_k \leq \pi\}$. $Z(\beta)$ is a proper subgroup of $\mathbb{T}^n$, so is a set of uniqueness (note that β is a uniform nonzero divisor). Assume for now that $Z(\alpha)$ is a coset in $\mathbb{T}^n$. By translation we may assume that $Z(\alpha) \subseteq Z(\beta)$, so $\Psi(Z(\alpha)) \subseteq \Psi(Z(\beta))$, thus $Z(\alpha)$ is a set of uniqueness and α is a uniform nonzero divisor. The corollary now follows from Theorem 7.5.2 in [6] and Proposition 1. □

6. Proof of Theorem 2 and related results

Let x_0 be a regular point of $Z(\alpha)$, such that F_{x_0} is a submanifold of finite type k. Let μ be a smooth nonzero mass density on the closure of F_{x_0} in $\mathbb{R}^n$, so $\mu \in M(Z(\alpha))$ and $\alpha * \hat{\mu}^* = 0$. The theorem will be proved once we show that $\hat{\mu}^* \in L^p(\mathbb{Z}^n)$ for $p > nk$. Let $0 \neq \eta \in \mathbb{Z}^n$; it is shown in Chapter 8 of [7] that

$$|\hat{\mu}(\eta)| \leq C|\eta|^{-\frac{1}{k}}$$

where $|\cdot|$ is the usual Euclidean norm and C is some constant. Set $\eta = (m_1, \ldots, m_n)$, $\eta' = (m_1+1, \ldots, m_n+1)$ and let j be the least integer greater than $n^{\frac{1}{2}}$. The triangle inequality implies that $|\eta'| - n^{\frac{1}{2}} \leq |\eta|$, so for $|\eta| > n^{\frac{1}{2}} - 1$ we obtain

$$\begin{aligned} |\hat{\mu}(\eta)|^p &\leq (C(|\eta|)^{-\frac{1}{k}})^p \\ &\leq (C(|\eta'| - n^{\frac{1}{2}})^{-\frac{1}{k}})^p \\ &\leq \int_{m_n}^{m_n+1} \cdots \int_{m_1}^{m_1+1} (Cf(x))^p \, dx_1 \cdots dx_n \end{aligned}$$

where $f(x) = ((x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} - n^{\frac{1}{2}})^{-\frac{1}{k}}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $N \in \mathbb{N}$ we have

$$
\begin{aligned}
\sum_{\eta\in\mathbb{Z}^n, |\eta|\le N} |\hat{\mu}(\eta)|^p \le & \sum_{|\eta|<n^{\frac{1}{2}}+1} |\hat{\mu}(\eta)|^p \\
& + \sum_{1+n^{\frac{1}{2}}\le|\eta|\le N+j} \int_{m_n}^{m_n+1} \cdots \int_{m_1}^{m_1+1} (Cf(x))^p \, dx_1 \cdots dx_n.
\end{aligned} \tag{1}
$$

Let p be a real number strictly greater than nk; so

$$\int_{|x|\ge 1+n^{\frac{1}{2}}} (f(x))^p \, dx_1 \cdots dx_n$$

is finite, hence

$$\sum_{\eta\in\mathbb{Z}^n, |\eta|\ge 1+n^{\frac{1}{2}}} \int_{m_n}^{m_n+1} \cdots \int_{m_1}^{m_1+1} (f(x))^p \, dx_1 \cdots dx_n$$

converges. Letting $N \to \infty$ in (1) we obtain $\sum_{\eta\in\mathbb{Z}^n} |\hat{\mu}(\eta)|^p < \infty$; therefore, $\hat{\mu}^* \in L^p(\mathbb{Z}^n)$ for $p > nk$. The proof is complete.

Corollary 4. *If α is as in Theorem 2, then $T^q[\alpha]$ does not equal $L^q(\mathbb{Z}^n)$ for q such that $1 \le q < \frac{nk}{nk-1}$.*

Proof. Use Lemma 1. □

Corollary 5. *Let $\alpha \in L^1(\mathbb{Z}^n)$, $n \ge 2$, and suppose that there exists an $x_0 \in Z(\alpha)$ such that F_{x_0} is an $n-1$ dimensional submanifold of V. If F_{x_0} has strictly positive Gaussian curvature, then α is a p-zero divisor for $p > \frac{2n}{n-1}$.*

Proof. Let $0 \ne \eta \in \mathbb{Z}^n$; then by [5]

$$|\hat{\mu}(\eta)| \le C|\eta|^{-\frac{n-1}{2}}.$$

Now proceed as in the theorem. □

7. Examples

In [4] it is shown that if G is torsion free elementary amenable, $0 \ne \alpha \in \mathbb{C}G$, then α is not a p-zero divisor for $p \le 2$. We will give an example to show that this cannot be improved. Let $p > 2$ be given and pick an integer n such that $2 < \frac{2n}{n-1} < p$. Let

$$\alpha(x_1, \ldots, x_n) = \frac{2n-1}{2} - \frac{1}{2}\left(\sum_{k=1}^{n}(x_k + x_k^{-1})\right),$$

so $\alpha(x_1, \ldots, x_n) \in \mathbb{C}\mathbb{Z}^n, \hat{\alpha}(t_1, \ldots, t_n) = \frac{2n-1}{2} - \sum_{k=1}^n \cos t_k$. Near $(0, \ldots, 0, \frac{\pi}{3})$, $Z(\alpha)$ is of the form $\{(t, g(t)) | t \in U\}$, where U is a bounded open set containing 0 in $\mathbb{R}^{n-1}$, $t = (t_1, \ldots, t_{n-1})$ and $g(t) = \cos^{-1}(\frac{2n-1}{2} - \sum_{k=1}^{n-1} \cos t_k)$. A computation shows that the rank of the matrix $(\frac{\partial^2 g(t)}{\partial t_i \partial t_k})$ is $n-1$ at $t = 0$, hence $Z(\alpha)$ has strictly positive curvature. Therefore, $\alpha(x_1, \ldots, x_n)$ is an r-zero divisor for $r > \frac{2n}{n-1}$.

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