PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 3, March 1998, Pages 645–646
S 0002-9939(98)04063-5

LIFTING UP AN INFINITE CHAIN OF PRIME IDEALS TO A VALUATION RING

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(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. We prove that for an arbitrary chain $\{P_\alpha\}$ of prime ideals in an integral domain, there exists a valuation domain which has a chain of prime ideals $\{Q_\alpha\}$ lying over $\{P_\alpha\}$.

It is well-known that for a domain D and a chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_l$, there exists a valuation domain V containing D and a chain of prime ideals $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_l$ lying over $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_l$ [3, Corollary 19.7]. This result is crucial in proving various results. The problem about a chain of prime ideals with an arbitrary length was posed by D. D. Anderson [1, #7, p. 364]. In this paper, we answer this question affirmatively and prove that the above theorem [3, Corollary 19.7] is true for a chain of prime ideals of an arbitrary ordinal type. Throughout this paper, let D be an integral domain with quotient field K.

Theorem. *Let D be an integral domain and let $\{P_\alpha\}$ be a chain of prime ideals in D. Then there exists a valuation overring V of D on K with a chain of prime ideals $\{Q_\alpha\}$ such that $Q_\alpha \cap D = P_\alpha$.*

Corollary. *Let $\{P_\alpha\}$ be a chain of prime ideals in an integral domain D. Then there exists, in the integral closure of D, a chain of prime ideals $\{Q_\alpha\}$ lying over $\{P_\alpha\}$.*

In order to prove the theorem, we need some lemmas.

Lemma 1. *Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be a chain of prime ideals of D. Let $S_\alpha = (D - P_\alpha)^{-1} P_\alpha$ and $S = \bigcup_\alpha S_\alpha$. Then*

(1) *S is closed under multiplication and S is closed with respect to the multiplication by elements of D;*
(2) *S_α is closed under addition.*

Proof. (1) Suppose that $a, b \in S$ and $c \in D$. Let $a = \frac{g}{f}$, $b = \frac{k}{h}$, where $a \in S_\alpha, b \in S_\beta$. We may assume that $P_\alpha \subsetneq P_\beta$ by symmetry. Then $fh \notin P_\alpha$ since $f \notin P_\alpha$ and $h \notin P_\beta$. Since $g \in P_\alpha$, $gk \in P_\alpha$. Hence $ab \in S_\alpha \subseteq S$. Since $cg \in P_\alpha$, $ca \in S_\alpha \subseteq S$. Hence S is closed under multiplication and with respect to the multiplication by elements of D.

(2) This follows because S_α is the maximal ideal of the ring $(D - P_\alpha)^{-1}D$. □

Received by the editors May 16, 1996 and, in revised form, July 28, 1996.
1991 *Mathematics Subject Classification.* Primary 13A18; Secondary 13B02.
This research was supported by the research grant BSRI-95-1431.

Lemma 2. *The ideal $\langle S\rangle$ generated by S is a proper ideal in $D[S]$.*

Proof. Let $P = \bigcup_\alpha P_\alpha$ and suppose $\langle S\rangle$ is not a proper ideal in $D[S]$. Then $1 \in \langle S\rangle$ in $D[S]$. By Lemma 1,

$$1 = s_1 + \cdots + s_n, \quad \text{where } s_k \in S_k \text{ for } k = 1, 2, \ldots, n.$$

Let $s_k = \frac{g_k}{f_k}$, where $g_k \in P_k$ and $f_k \in D\backslash P_k$— that is, $s_k \in S_k$ for each k. We may assume that $P_n \subsetneq P_{n-1} \subsetneq \cdots \subsetneq P_2 \subsetneq P_1$. For the chain of prime ideals $P_n \subsetneq P_{n-1} \subsetneq \cdots \subsetneq P_2 \subsetneq P_1 \subsetneq P$, choose a valuation overring V and a chain of prime ideals $Q_n \subsetneq Q_{n-1} \subsetneq \cdots \subsetneq Q_2 \subsetneq Q_1 \subsetneq Q$ lying over the given chain [3, Corollary 19.7]. For each $1 \le i \le n$, $g_i \in Q_i$ and $f_i \notin Q_i$ since $Q_i \cap D = P_i$ and $f_i \notin P_i$. Note that $\frac{f_i}{g_i} \notin V$ for otherwise $f_i \in g_iV \subseteq Q_i$. So $\frac{g_i}{f_i} \in V$. Therefore $g_i \in f_iV$, so $s_i = g_i/f_i \in V$. Since $s_if_i = g_i \in Q_i$ and $f_i \notin Q_i$, it follows that $s_i \in Q_i$. This implies that $1 = \frac{g_1}{f_1} + \cdots + \frac{g_n}{f_n} \in Q$ contrary to the fact that Q is a proper ideal. Hence $\langle S\rangle$ is a proper ideal in $D[S]$. □

Proof of the main theorem. Since $D_P \cap D = D$ and $PD_P \cap D = P$ [2, Theorem 34], we may assume that P is a maximal ideal of D. Let Q be a prime ideal containing $\langle S\rangle$ in $D[S]$. Then there exists a valuation overring V of $D[S]$ such that $M \cap D[S] = Q$, where M is the maximal ideal of V [3, Theorem 19.6]. Since P is a maximal ideal in D, $M \cap D = P$. We have only to show that $\sqrt{P_\alpha V} \cap D = P_\alpha$. Suppose that it fails. We choose an element $f \in \sqrt{P_\alpha V} \cap D\backslash P_\alpha$. Then $f \in D\backslash P_\alpha$ and $f^n \in P_\alpha V$, $n \in \mathbb{N}$. So $\frac{f^n}{g} \in V$, $g \in P_\alpha$. This implies $\frac{g}{f^n} \in S$ is a unit in V. This contradicts the fact that $\langle S\rangle \subseteq M$. Hence $\sqrt{P_\alpha V} \cap D = P_\alpha$. Furthermore, $\sqrt{P_\alpha V}$ is a prime ideal in V [3, Theorem 17.1]. □

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