LIFTING UP AN INFINITE CHAIN OF PRIME IDEALS TO A VALUATION RING

BYUNG GYUN KANG AND DONG YEOL OH

(Communicated by Wolmer V. Vasconcelos)

Abstract. We prove that for an arbitrary chain \( \{P_\alpha\} \) of prime ideals in an integral domain, there exists a valuation domain which has a chain of prime ideals \( \{Q_\alpha\} \) lying over \( \{P_\alpha\} \).

It is well-known that for a domain \( D \) and a chain of prime ideals \( P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_l \), there exists a valuation domain \( V \) containing \( D \) and a chain of prime ideals \( Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_l \) lying over \( P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_l \) [3, Corollary 19.7]. This result is crucial in proving various results. The problem about a chain of prime ideals with an arbitrary length was posed by D. D. Anderson [1, #7, p. 364]. In this paper, we answer this question affirmatively and prove that the above theorem [3, Corollary 19.7] is true for a chain of prime ideals of an arbitrary ordinal type. Throughout this paper, let \( D \) be an integral domain with quotient field \( K \).

Theorem. Let \( D \) be an integral domain and let \( \{P_\alpha\} \) be a chain of prime ideals in \( D \). Then there exists a valuation overring \( V \) of \( D \) on \( K \) with a chain of prime ideals \( \{Q_\alpha\} \) such that \( \overline{Q_\alpha} \cap D = P_\alpha \).

Corollary. Let \( \{P_\alpha\} \) be a chain of prime ideals in an integral domain \( D \). Then there exists, in the integral closure of \( D \), a chain of prime ideals \( \{Q_\alpha\} \) lying over \( \{P_\alpha\} \).

In order to prove the theorem, we need some lemmas.

Lemma 1. Let \( \{P_\alpha\}_{\alpha \in \Lambda} \) be a chain of prime ideals of \( D \). Let \( S_\alpha = (D - P_\alpha)^{-1} P_\alpha \) and \( S = \bigcup_\alpha S_\alpha \). Then

1. \( S \) is closed under multiplication and \( S \) is closed with respect to the multiplication by elements of \( D \);
2. \( S_\alpha \) is closed under addition.

Proof. (1) Suppose that \( a, b \in S \) and \( c \in D \). Let \( a = \frac{g}{f}, b = \frac{k}{h} \), where \( a \in S_\alpha, b \in S_\beta \). We may assume that \( P_\alpha \subsetneq P_\beta \) by symmetry. Then \( fh \notin P_\alpha \) since \( f \notin P_\alpha \) and \( h \notin P_\beta \). Since \( g \in P_\alpha \), \( gk \in P_\alpha \). Hence \( ab \in S_\alpha \subseteq S \). Since \( cg \in P_\alpha \), \( ca \in S_\alpha \subseteq S \). Hence \( S \) is closed under multiplication and with respect to the multiplication by elements of \( D \).

(2) This follows because \( S_\alpha \) is the maximal ideal of the ring \( (D - P_\alpha)^{-1}D \). \( \square \)

Received by the editors May 16, 1996 and, in revised form, July 28, 1996.
1991 Mathematics Subject Classification. Primary 13A18; Secondary 13B02.
This research was supported by the research grant BSRI-95-1431.

©1998 American Mathematical Society
Lemma 2. The ideal $\langle S \rangle$ generated by $S$ is a proper ideal in $D[S]$.

Proof. Let $P = \bigcup_n P_n$ and suppose $\langle S \rangle$ is not a proper ideal in $D[S]$. Then $1 \in \langle S \rangle$ in $D[S]$. By Lemma 1,

$$1 = s_1 + \cdots + s_n,$$

where $s_k \in S_k$ for $k = 1, 2, \ldots, n$.

Let $s_k = \frac{g_k}{f_k}$, where $g_k \in P_k$ and $f_k \in D \setminus P_k$—that is, $s_k \in S_k$ for each $k$. We may assume that $P_n \subseteq P_{n-1} \subseteq \cdots \subseteq P_2 \subseteq P_1 \subseteq P$. For the chain of prime ideals $P_n \subseteq P_{n-1} \subseteq \cdots \subseteq P_2 \subseteq P_1 \subseteq P$, choose a valuation overring $V$ and a chain of prime ideals $Q_n \subseteq Q_{n-1} \subseteq \cdots \subseteq Q_2 \subseteq Q_1 \subseteq Q$ lying over the given chain [3, Corollary 19.7]. For each $1 \leq i \leq n$, $g_i \in Q_i$ and $f_i \notin Q_i$. Since $Q_1 \cap D = P_1$ and $f_i \notin P_i$. Note that $\frac{g_i}{f_i} \notin V$ for otherwise $f_i \in g_i V \subseteq Q_i$. So $s_i = \frac{g_i}{f_i} \notin V$. Therefore $g_i \in f_i V$, so $s_i = g_i/f_i \in V$. Since $s_i f_i = g_i \in Q_i$ and $f_i \notin Q_i$, it follows that $s_i \in Q_i$. This implies that $1 = \frac{g_1}{f_1} + \cdots + \frac{g_n}{f_n} \in Q$ contrary to the fact that $Q$ is a proper ideal. Hence $\langle S \rangle$ is a proper ideal in $D[S]$.

Proof of the main theorem. Since $D_P \cap D = D$ and $P D_P \cap D = P$ [2, Theorem 34], we may assume that $P$ is a maximal ideal of $D$. Let $Q$ be a prime ideal containing $\langle S \rangle$ in $D[S]$. Then there exists a valuation overring $V$ of $D[S]$ such that $M \cap D[S] = Q$, where $M$ is the maximal ideal of $V$ [3, Theorem 19.6]. Since $P$ is a maximal ideal in $D$, $M \cap D = P$. We have only to show that $\sqrt{P \cap V \cap D} = P \cap V$. Suppose that $S$ fails. Then $f_i \notin V$ for otherwise $\frac{g_i}{f_i} \in V$. So $s_i \notin V$. Choose an element $f_i \in D \setminus P\alpha$ and $f^n \in P_{\alpha V}$, $n \in \mathbb{N}$. So $\frac{f^n}{g} \in V$, $g \in P_{\alpha}$. This implies $\frac{g}{f^n} \in S$ is a unit in $V$. This contradicts the fact that $\langle S \rangle \subseteq M$. Hence $\sqrt{P \cap V \cap D} = P$. Furthermore, $\sqrt{P_{\alpha V}}$ is a prime ideal in $V$ [3, Theorem 17.1].

References


Department of Mathematics, Pohang University of Science & Technology, Pohang, 790-784, Korea

E-mail address: bgkang@euclid.postech.ac.kr