

## SMORODINSKY'S CONJECTURE ON RANK-ONE MIXING

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ABSTRACT. We prove Smorodinsky's conjecture: the rank-one transformation, obtained by adding staircases whose heights increase consecutively by one, is mixing.

### 0. INTRODUCTION

The first rank-one mixing transformation was constructed by Ornstein [O] using “random” spacers on each column. We refer to [F1] for a description of rank-one constructions. Recently, the first rank-one mixing transformation was constructed with an explicit formula for adding spacers [AF]. In [AF] a method for adding staircases was given which produced mixing. However, Smorodinsky's conjecture remained open. M. Smorodinsky conjectured that by adding staircases whose heights increase consecutively by one, the resulting transformation (classical staircase construction) is mixing.

In this paper, we will prove that an infinite staircase construction, whose sequence  $r_n$  of cuts and  $h_n$  of heights satisfy the condition  $\lim_{n \rightarrow \infty} \frac{r_n^2}{h_n} = 0$ , is mixing. Thus Smorodinsky's conjecture follows as a corollary.

### 1. STAIRCASE CONSTRUCTIONS

A rank-one transformation  $T$  is called a **staircase construction** if there exists a sequence  $(r_n)_{n=1}^{\infty}$  of natural numbers such that each column  $C_{n+1}$  is obtained by cutting  $C_n$  into  $r_n$  subcolumns of equal width, placing  $i - 1$  spacers on the  $i^{\text{th}}$  subcolumn for  $1 \leq i \leq r_n$ , and then stacking the  $(i + 1)^{\text{st}}$  subcolumn on top of the  $i^{\text{th}}$  subcolumn for  $1 \leq i \leq r_n$ . Denote  $T = T_{(r_n)}$ . Let  $h_n$  be the height of column  $C_n$  for  $n \geq 1$ . From now on, we assume the sequence  $(h_n)$  is derived from the sequence  $(r_n)$  in this manner. If the sequence  $r_n$  is bounded then  $T$  is called a **finite staircase construction**. The **classical staircase construction** is given by  $r_n = n$ . If  $r_n \rightarrow \infty$ , we call  $T$  an **infinite staircase construction**. In this case  $T$  may not be finite measure preserving since we may be adding measure too quickly. Assume  $T$  is finite measure preserving. The following question remains open:

**Question.** *Is every infinite staircase construction mixing?*

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Recently, much has been proved about rank-one mixing transformations. They are mixing of order 3 [Ka], and by the same argument, mixing of all orders. King [Ki] proved they have minimal self-joinings of all orders. In [R], Ryzhikov proved finite rank mixing transformations are mixing of all orders. In [Kl], it is proved that the classical staircase construction has singular spectrum, and in [KR] it is proved that any rank-one transformation satisfying  $\sum_{n=1}^{\infty} (1/r_n^2) = \infty$  has singular spectrum, where  $r_n$  is still the number of cuts at the  $n^{\text{th}}$  stage.

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## 2. MIXING STAIRCASE CONSTRUCTIONS

Staircase constructions are known to be weakly mixing (see [F2] for infinite staircase constructions and [AF] for finite staircase constructions), hence totally ergodic. This means each power  $T^j$  is ergodic. This is a necessary fact and is used repeatedly. Now we formulate von Neumann's mean ergodic theorem to suit our needs. Note

$$\begin{aligned} \left| \frac{1}{N} \sum_{i=0}^{N-1} \mu(T^{-i}A \cap B) - \mu(A)\mu(B) \right| &= \left| \frac{1}{N} \sum_{i=0}^{N-1} \mu(A \cap T^i B) - \mu(A)\mu(B) \right| \\ &\leq \int_A \left| \frac{1}{N} \sum_{i=0}^{N-1} \chi_B(T^{-i}x) - \mu(B) \right| d\mu \\ &\leq \left\| \frac{1}{N} \sum_{i=0}^{N-1} \chi_B(T^{-i}x) - \mu(B) \right\|_1. \end{aligned}$$

Hence  $T$  ergodic implies

$$\left| \frac{1}{N} \sum_{i=0}^{N-1} \mu(T^{-i}A \cap B) - \mu(A)\mu(B) \right| \rightarrow 0$$

uniformly over sets  $A$ , as  $N \rightarrow \infty$ . This is used in Lemma 2.2 and our main theorem.

Lemma 2.1 is a basic fact about measure preserving transformations which is utilized in Theorem 3.1. We leave the proof to the reader.

In the argument of Lemma 2.2 we establish that  $\rho_n$  satisfying  $h_n \leq \rho_n \leq 2h_n$  is a mixing sequence. This is used to prove convergence in mean of the transformations  $T^{\rho_n}$  (uniformly over  $n$ ).

**Lemma 2.1.** *Let  $T$  be a finite measure preserving transformation and let  $B$  be any measurable set. For any positive integers  $R$ ,  $L$  and  $\rho$  we have*

$$\int \left| \frac{1}{R} \sum_{i=0}^{R-1} \chi_B(T^{-i}x) - \mu(B) \right| d\mu(x) \leq \int \left| \frac{1}{L} \sum_{i=0}^{L-1} \chi_B(T^{-i\rho}x) - \mu(B) \right| d\mu(x) + \frac{\rho L}{R}.$$

**Lemma 2.2.** *Let  $T$  be an infinite staircase construction and let  $B$  be a union of levels in some column. If  $\ell_n$  and  $\rho_n$  are sequences of positive integers which converge*

to infinity and such that  $h_n \leq \rho_n \leq 2h_n$ , then

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{\ell_n} \sum_{i=0}^{\ell_n-1} \mathcal{X}_B(T^{-i\rho_n}x) - \mu(B) \right| d\mu(x) = 0.$$

*Proof.* Let  $\epsilon > 0$ . Let  $j$  be a positive integer such that  $i\rho_n = jh_n + t$  where  $0 \leq t \leq h_n$ . Consider the disjoint union  $C_n = D_1 \cup D_2$  where  $D_1$  is the union of the top  $t$  levels of  $C_n$ . Let  $B_1 = B \cap D_1$  and  $B_2 = B \cap D_2$ . Let

$$B'_1 = B_1 \setminus \{\text{bottom } (j+1)r_n \text{ levels of } D_1\} \setminus \{(j+1) \text{ rightmost subcolumns of } C_n\}.$$

We have not thrown away much:

$$\mu(B'_1 \triangle B_1) \leq \frac{(j+1)r_n}{h_n} + \frac{j+1}{r_n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

For each level  $I$  of  $C_n$  let  $I' = I \cap B'_1$ . Each  $I'$  passes completely through the staircase on  $C_n$ ,  $(j+1)$  times, under  $T^{i\rho_n}$ . Thus  $T^{i\rho_n} I'$  intersects  $(r_n - j - 1)$  levels of  $C_n$  with precisely  $\frac{\mu(I)}{r_n}$  measure (on each level of intersection). Moreover the levels for which  $T^{i\rho_n} I'$  intersect appear in an arithmetic progression in  $C_n$ . Let  $I^*$  denote the first (top) of such levels, and let  $B_1^* = \bigcup_{I' \subset B'_1} I^*$ . Thus

$$\mu(T^{i\rho_n} B'_1 \cap B) = \frac{1}{r_n} \sum_{\eta=0}^{r_n-j-1} \mu(T^{-\eta(j+1)} B_1^* \cap B).$$

Similarly we define  $B'_2 \subset B_2$  and  $B_2^*$  so that

$$\mu(T^{i\rho_n} B'_2 \cap B) = \frac{1}{r_n} \sum_{\eta=0}^{r_n-j-1} \mu(T^{-\eta j} B_2^* \cap B).$$

Since  $T^{-j}$  and  $T^{-(j+1)}$  are ergodic we get  $\mu(T^{i\rho_n} B \cap B) \rightarrow \mu(B)^2$  as  $n \rightarrow \infty$ . Hence for  $i_1 \neq i_2$ , we have

$$\mu(T^{i_1\rho_n} B \cap T^{i_2\rho_n} B) \rightarrow \mu(B)^2 \text{ as } n \rightarrow \infty.$$

By a technique of Blum-Hansen [BH], there exists a positive integer  $L$  such that for sufficiently large  $n$ ,

$$\int \left| \frac{1}{L} \sum_{i=0}^{L-1} \mathcal{X}_B(T^{-i\rho_n}x) - \mu(B) \right| d\mu(x) < \epsilon.$$

Therefore

$$\limsup_{n \rightarrow \infty} \int \left| \frac{1}{\ell_n} \sum_{i=0}^{\ell_n-1} \mathcal{X}_B(T^{-i\rho_n}x) - \mu(B) \right| d\mu(x) \leq \epsilon.$$

□

Before proceeding with the main theorem, we show that any staircase construction satisfies

$$(H) \quad \frac{h_{p-1}^2}{h_p} \rightarrow \infty \quad \text{as } p \rightarrow \infty.$$

First we write

$$\frac{h_{p-1}^2}{h_p} = \frac{r_{p-1} h_{p-1}^2}{r_{p-1} h_p} = \left( \frac{r_{p-1} h_{p-1}}{h_p} \right) \left( \frac{h_{p-1}}{r_{p-1}} \right).$$

While the first factor  $\frac{r_{p-1}h_{p-1}}{h_p} \rightarrow 1$ , our second factor  $\frac{h_{p-1}}{r_{p-1}} \rightarrow \infty$  as  $p \rightarrow \infty$  since  $T$  is finite measure preserving.

**Theorem 2.3.** *Let  $r_n$  be a divergent sequence of positive integers. The staircase construction  $T = T_{(r_n)}$  is mixing, if  $\lim_{n \rightarrow \infty} \frac{r_n^2}{h_n} = 0$ .*

*Proof.* Let  $m_n$  be a sequence of positive integers such that  $h_n \leq m_n < h_{n+1}$ . We may choose positive integers  $k_n$  and  $t_n$  so that

$$m_n = k_n h_n + t_n$$

where  $1 \leq k_n \leq r_n$  and  $0 \leq t_n < h_n$ .

Let  $A$  and  $B$  be sets which form a union of levels from some column. Since we let  $n \rightarrow \infty$  we may assume each of  $A$  or  $B$  is a union of levels in  $C_n$ . We need only show that  $\mu(T^{m_n} A \cap B) - \mu(A)\mu(B) \rightarrow 0$  as  $n \rightarrow \infty$ . We will consider the mixing in three different parts of the column  $C_n$ . Partition  $C_n$  into the  $r_n$  subcolumns of equal width. Let  $D_1$  be the set consisting of the  $(k_n + 1)$  rightmost subcolumns. Let  $D_2 = \{\text{union of the top } t_n \text{ levels of } C_n\} \cap D_1^c$ . Finally let  $D_3$  be the remaining part of  $C_n$ . Thus  $D_1, D_2$  and  $D_3$  are disjoint, and  $C_n = \bigcup_{i=1}^3 D_i$ . Denote  $A_i = A \cap D_i$  for  $i = 1, 2, 3$ . Figure 2.4 shows the partition of  $C_n$  into the three sets  $D_1, D_2$  and  $D_3$  where  $r_n = 18, k_n = 4$  and  $t_n = 3$ .

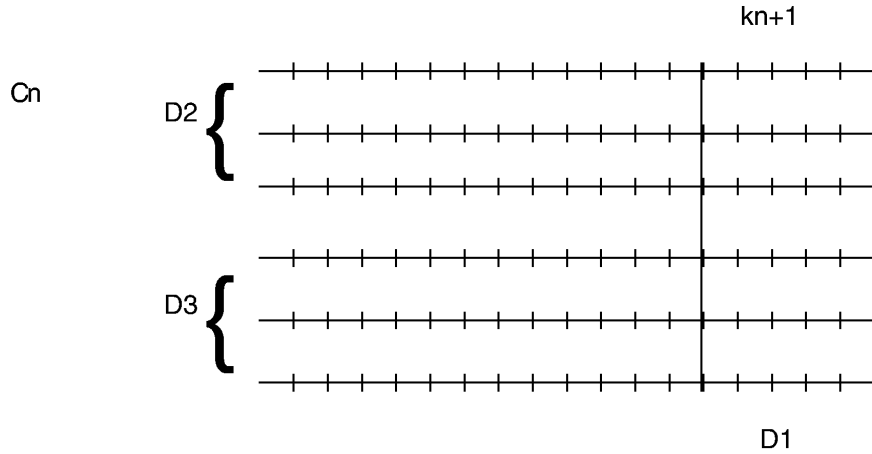


FIGURE 2.4

**Mixing on  $D_1$ .** The set  $D_1$  sits as whole levels in the top  $((k_n + 1)h_n + r_n^2)$  levels of  $C_{n+1}$ . Let  $\bar{D}_1 = D_1 - \{\text{bottom } (h_n + r_{n+1}) \text{ levels in } D_1\} - \{\text{rightmost subcolumn of } C_{n+1}\}$ . Thus each level of  $C_{n+1}$ , which is in  $D_1$ , gets pushed completely through the staircase on  $C_{n+1}$  (under  $T^{m_n}$ ). Let  $\bar{A}_1 = A_1 \cap \bar{D}_1$ . Then

$$\mu(\bar{A}_1) \geq \mu(A_1) - \frac{h_n + r_{n+1}}{h_{n+1}} - \frac{1}{r_{n+1}} \geq \mu(A_1) - \frac{1}{r_n} - \frac{r_{n+1}}{h_{n+1}} - \frac{1}{r_{n+1}}.$$

Hence  $\mu(A_1 \triangle \bar{A}_1) \rightarrow 0$  as  $n \rightarrow \infty$ .

Let  $I \subset \bar{A}_1$  be a level of  $C_{n+1}$ .  $T^{m_n} I$  intersects  $(r_{n+1} - 1)$  consecutive levels of  $C_{n+1}$  with  $\frac{\mu(I)}{r_{n+1}}$  measure (on each level in the intersection). Let  $I^*$  denote the

first(top) of such levels, and let  $A^* = \bigcup_{I \subset \bar{A}_1} I^*$ . Thus we have

$$|\mu(T^{m_n} \bar{A}_1 \cap B) - \mu(\bar{A}_1)\mu(B)| = \left| \frac{\mu(I)}{r_{n+1}} \sum_{i=0}^{r_{n+1}-2} \mu(T^{-i} A^* \cap B) - \mu(A^*)\mu(B) \right|.$$

Since  $T$  is ergodic and  $r_n \rightarrow \infty$ , the second expression tends to 0 as  $n \rightarrow \infty$ . Therefore we have that  $|\mu(T^{m_n} A_1 \cap B) - \mu(A_1)\mu(B)| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Mixing on  $D_2$ .** Let  $D^* = \{\text{levels } \mathcal{L} \text{ of } C_n \mid \mathcal{L} \cap D_2 \neq \emptyset\}$ . Let  $A' = A_2 - \{\text{bottom } r_n^2 \text{ levels of } D^*\}$ . We have not thrown away much:

$$\mu(A') \geq \mu(A_2) - \frac{r_n^2}{h_n}.$$

For each level  $I$  of  $C_n$  let  $I' = I \cap A'$ . Each  $I' \subset A'$  passes completely through the staircase on  $C_n$ ,  $(k_n + 1)$  times, under  $T^{m_n}$ . Thus  $T^{m_n} I'$  intersects  $(r_n - k - 1)$  levels of  $C_n$  with precisely  $\frac{\mu(I)}{r_n}$  measure (on each level of intersection). Moreover the levels for which  $T^{m_n} I'$  intersect, appear in an arithmetic progression in  $C_n$ . Let  $I^*$  denote the first (top) of such levels, and let  $A^* = \bigcup_{I' \subset A'} I^*$ . (Note that  $A^*$  is a union of whole levels in  $C_n$ .) Thus  $\mu(A') = \left(\frac{r_n - k_n - 1}{r_n}\right)\mu(A^*)$  and

$$\begin{aligned} \mu(T^{m_n} A' \cap B) &= \frac{1}{r_n} \sum_{i=0}^{r_n - k_n - 2} \mu(T^{-i(k_n+1)} A^* \cap B) \\ &= \frac{r_n - k_n - 1}{r_n} \left( \frac{1}{r_n - k_n - 1} \sum_{i=0}^{r_n - k_n - 2} \mu(T^{-i(k_n+1)} A^* \cap B) \right). \end{aligned}$$

Hence

$$\begin{aligned} &\mu(T^{m_n} A' \cap B) - \mu(A')\mu(B) \\ &= \frac{r_n - k_n - 1}{r_n} \left( \frac{1}{r_n - k_n - 1} \sum_{i=0}^{r_n - k_n - 2} \mu(T^{-i(k_n+1)} A^* \cap B) - \mu(A^*)\mu(B) \right). \end{aligned}$$

The previous expression will converge to zero, if

$$\int \left| \frac{1}{r_n - k_n - 1} \sum_{i=0}^{r_n - k_n - 2} \mathcal{X}_B(T^{-i(k_n+1)} x) - \mu(B) \right| d\mu(x) \rightarrow 0.$$

Since  $\mu(A') = \left(\frac{r_n - k_n - 1}{r_n}\right)\mu(A^*)$  we may assume  $\frac{k_n}{r_n}$  is bounded away from 1. So if we choose  $p$  so that  $h_{p-1} \leq (k_n + 1) \leq h_p$ , then property (H) implies  $\frac{(r_n - k_n - 2)(k_n + 1)}{h_p} \rightarrow \infty$  as  $n \rightarrow \infty$ .

Now choose  $k'_n = \inf\{i \in \mathbb{Z}^+ : i(k_n + 1) \geq h_p\}$ . Thus  $\frac{r_n - k_n - 2}{k'_n} \rightarrow \infty$  as  $n \rightarrow \infty$ . By Lemma 2.2 we can choose  $\ell_n \rightarrow \infty$  so that

$$\int \left| \frac{1}{\ell_n} \sum_{i=0}^{\ell_n - 1} \mathcal{X}_B(T^{-i(k'_n(k_n+1))} x) - \mu(B) \right| d\mu(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and  $\frac{r_n - k_n - 2}{\ell_n k'_n} \rightarrow \infty$ . Therefore by Lemma 2.1

$$\int \left| \frac{1}{r_n - k_n - 1} \sum_{i=0}^{r_n - k_n - 2} \mathcal{X}_B(T^{-i(k_n+1)} x) - \mu(B) \right| d\mu(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence  $T$  is mixing on  $D_2$ .

**Mixing on  $D_3$ .** This can be handled in the same manner as  $D_2$  with  $(k_n + 1)$  replaced by  $k_n$ .

Therefore  $T$  is mixing.  $\square$

**Corollary.** *The classical staircase construction is mixing.*

*Proof.* For the classical staircase construction our sequence of cuts  $r_n = n$ . Thus  $h_n \geq n!$ . Therefore  $\frac{r_n^2}{h_n} \leq \frac{n^2}{n!} \rightarrow 0$ .  $\square$

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