

## A GENERALIZATION OF BANCHOFF'S TRIPLE POINT THEOREM

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ABSTRACT. Consider an immersion of a surface into  $S^3$ . Banchoff's theorem states that the parity of the number of triple points and the parity of the Euler characteristic of the surface coincide. Here we generalize this theorem to codimension 1 immersions of arbitrary even dimensional manifolds in spheres. The proof is an analogue of a proof of Banchoff's theorem circulated in preprint form due to R. Fenn and P. Taylor in 1977.

Let us consider a codimension 1 smooth generic (i.e. self-transverse) immersion  $f$  of a closed manifold  $M^n$  in the sphere  $S^{n+1}$ . Let us recall how a neighborhood of an  $i$ -tuple point (in  $R^{n+1} \subset S^{n+1}$ ) looks like in such a self-transverse immersion. Consider the coordinate hyperplanes in  $R^i$  and take the direct product of this configuration with  $R^{n+1-i}$ . What is obtained is diffeomorphic to the neighborhood of an  $i$ -tuple point in the image of  $f$ .

For any natural number  $i$ ,  $1 \leq i \leq n+1$ , let us denote by  $\tilde{\Delta}_i$  the set of  $i$ -tuple points in  $S^{n+1}$ , i.e.

$$\tilde{\Delta}_i = \{y \in S^{n+1} \mid f^{-1}(y) \text{ consists of } i \text{ different points}\}.$$

As is well known,  $\dim \tilde{\Delta}_i = n+1-i$ , and  $\bigcup_{r=i}^{\infty} \tilde{\Delta}_r$  is an immersed manifold (although it is not in general position, i.e. it is the image of a *non*-self-transverse immersion). Let  $\Delta_i$  be a closed manifold such that  $\bigcup_{r=i}^{\infty} \tilde{\Delta}_r$  is the image of an immersion of  $\Delta_i$  in  $S^{n+1}$ .

*Remark.* Of course, many different manifolds can be immersed into  $S^{n+1}$  so that their images are  $\bigcup_{r=i}^{\infty} \tilde{\Delta}_r$ . For example if a possible  $\Delta_i$  is given, then any of its finite coverings serves as well. We make the choice of  $\Delta_i$  explicit by assuming that the  $i$ -tuple points of  $f$  are non-multiple points of the immersion  $\Delta_i \hookrightarrow S^{n+1}$ .

We shall call the manifold  $\Delta_i$  the  $i$ -tuple manifold of  $f$ . Our theorem claims that for  $n$  even the sum of the Euler characteristics of  $i$ -tuple manifolds is even. (For  $n=2$  this is exactly Banchoff's theorem.)

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**Theorem.** *If  $n > 0$  is even, then*

$$\sum_{i=1}^{n+1} \chi(\Delta_i) \equiv 0 \pmod{2}.$$

The following proof is an analogue of the proof in [FT] for Banchoff’s triple point theorem.

*Proof.* Since  $n$  is even, we can omit the terms corresponding to even  $i$ ’s, because in those cases the dimension of  $\Delta_i$  is odd. Now let us triangulate the *image* of  $f$  in such a way that for any  $i$  the set of points of multiplicity  $i$  or higher forms a subcomplex of  $f(M)$ .

Let  $\alpha_r^i$  denote the number of  $i$ -dimensional simplexes whose interiors lie in  $\tilde{\Delta}_r$ , and let

$$\beta_r = \alpha_r^0 - \alpha_r^1 + \dots \pm \alpha_r^{n+1-r}.$$

Observe that  $\beta_r$  is *not* the Euler characteristic of any complex. However, we have that

$$\chi(\Delta_i) = \sum_{r=i}^{n+1} \binom{r}{i} \beta_r.$$

The coefficient  $\binom{r}{i}$  counts the multiplicity of the self-intersection of  $\Delta_i$  at  $\tilde{\Delta}_r$ . So

$$\sum_{i=1}^{n+1} * \chi(\Delta_i) = \sum_{i=1}^{n+1} * \sum_{r=i}^{n+1} \binom{r}{i} \beta_r,$$

where  $*$  indicates that the sum is taken only for odd  $i$ ’s. After changing the order of the summations we get:

$$(1) \quad \sum_{r=1}^{n+1} \left( \sum_{i=1}^r * \binom{r}{i} \right) \beta_r = \sum_{r=1}^{n+1} 2^{r-1} \beta_r \equiv \beta_1 \pmod{2}.$$

Now let us paint the complement of  $f(M)$  in  $S^{n+1}$  in two colors in a chessboard-style, i.e. let any two neighboring domains have different colors (where “neighboring” means that they are separated by a component of  $\tilde{\Delta}_1$ ). This is possible, since  $H_n(S^{n+1}; Z_2) = 0$ .

Let  $N$  be the boundary of an  $\varepsilon$ -neighborhood of  $f(M)$  in the black subset of  $S^{n+1}$ . Notice that from the given triangulation of  $f(M)$  we can construct a triangulation of  $N$  by pushing the simplexes from  $f(M)$  to  $N$  in a reasonable way. Simplexes in  $\tilde{\Delta}_i$  will have  $2^{i-1}$  counterparts in  $N$  ( $i$  hyperplanes divide the Euclidean  $n$ -space into  $2^i$  parts, half of which are black). Thus:

$$\chi(N) = \sum_{i=1}^{n+1} 2^{i-1} \beta_i \equiv \beta_1 \pmod{2}.$$

But  $\chi(N)$  is even, because  $N$  is embedded in codimension 1 (and  $n > 0$ ), so the proof is complete. □

*Remark 1.* As is clear from the proof, the space  $S^{n+1}$  can be replaced by any manifold such that its  $n$ th  $Z_2$ -homology group is 0.

*Remark 2.* The above proof does not work for  $n$  odd, since the sum  $\sum_{i=1}^r * \binom{r}{i}$  (where the star this time means summation for even  $i$ 's) equals to  $2^{r-1} - 1$ , so the sum in formula (1) gives  $\sum_{r=2}^{n+1} \beta_r$  (which is clearly the Euler characteristic of the complex  $f(M)$ ).

The figure 8 immersion of the circle in the plane shows that the statement of the theorem is false for  $n = 1$ . A theorem of Freedman [F] (and its generalization to unoriented 3-manifolds given in [A]) shows that it is true for  $n = 3$ . We do not know whether it is true or not for  $n > 3$ .

*Remark 3.* If we consider only *oriented*  $n$ -manifolds and their codimension 1 immersions in  $S^{n+1}$ , and the  $n$ th stable homotopy group of spheres has no 2-primary torsion, then the Euler characteristics of the  $i$ -tuple manifolds are all even, for any  $i$ . (Indeed, for any  $i$ ,  $\chi(\Delta_i) \bmod 2$  defines a homomorphism from the stable homotopy group  $\pi_{n+N}(S^N)$ ,  $N \gg n$  to  $Z_2$ .)

In particular the statement of the theorem is true for  $n = 5$  or  $n = 13$  for oriented manifolds.

*Remark 4.* If the dimension  $n = 4$ , then more is true than is stated in the theorem, namely all  $\chi(\Delta_i)$ 's are even, since the stable homotopy group  $\pi_5^s(RP^\infty)$  vanishes (see [L]), and this group is isomorphic to the cobordism group of immersions of 4-manifolds into  $R^5$ .

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