A GENERALIZATION OF BANCHOFF’S TRIPLE POINT THEOREM

P. AKHMETIEV, R. RIMÁNYI, AND A. SZŰCS

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Abstract. Consider an immersion of a surface into \( S^3 \). Banchoff’s theorem states that the parity of the number of triple points and the parity of the Euler characteristic of the surface coincide. Here we generalize this theorem to codimension 1 immersions of arbitrary even dimensional manifolds in spheres. The proof is an analogue of a proof of Banchoff’s theorem circulated in preprint form due to R. Fenn and P. Taylor in 1977.

Let us consider a codimension 1 smooth generic (i.e. self-transverse) immersion \( f \) of a closed manifold \( M^n \) in the sphere \( S^{n+1} \). Let us recall how a neighborhood of an \( i \)-tuple point (in \( R^{n+1} \subset S^{n+1} \)) looks like in such a self-transverse immersion. Consider the coordinate hyperplanes in \( R^n \) and take the direct product of this configuration with \( R^{n+1-i} \). What is obtained is diffeomorphic to the neighborhood of an \( i \)-tuple point in the image of \( f \).

For any natural number \( i \), \( 1 \leq i \leq n+1 \), let us denote by \( \tilde{\Delta}_i \) the set of \( i \)-tuple points in \( S^{n+1} \), i.e.

\[
\tilde{\Delta}_i = \{ y \in S^{n+1} \mid f^{-1}(y) \text{ consists of } i \text{ different points}\}.
\]

As is well known, \( \dim \tilde{\Delta}_i = n+1-i \), and \( \bigcup_{r=i}^{\infty} \tilde{\Delta}_r \) is an immersed manifold (although it is not in general position, i.e. it is the image of a non-self-transverse immersion). Let \( \Delta_i \) be a closed manifold such that \( \bigcup_{r=i}^{\infty} \tilde{\Delta}_r \) is the image of an immersion of \( \Delta_i \) in \( S^{n+1} \).

Remark. Of course, many different manifolds can be immersed into \( S^{n+1} \) so that their images are \( \bigcup_{r=i}^{\infty} \tilde{\Delta}_r \). For example if a possible \( \Delta_i \) is given, then any of its finite coverings serves as well. We make the choice of \( \Delta_i \) explicit by assuming that the \( i \)-tuple points of \( f \) are non-multiple points of the immersion \( \Delta_i \to S^{n+1} \).

We shall call the manifold \( \Delta_i \) the \( i \)-tuple manifold of \( f \). Our theorem claims that for \( n \) even the sum of the Euler characteristics of \( i \)-tuple manifolds is even. (For \( n = 2 \) this is exactly Banchoff’s theorem.)
Theorem. If \( n > 0 \) is even, then
\[
\sum_{i=1}^{n+1} \chi(\Delta_i) \equiv 0 \mod 2.
\]

The following proof is an analogue of the proof in [FT] for Banchoff’s triple point theorem.

Proof. Since \( n \) is even, we can omit the terms corresponding to even \( i \)'s, because in those cases the dimension of \( \Delta_i \) is odd. Now let us triangulate the image of \( f \) in such a way that for any \( i \) the set of points of multiplicity \( i \) or higher forms a subcomplex of \( f(M) \).

Let \( \alpha_i^r \) denote the number of \( i \)-dimensional simplexes whose interiors lie in \( \tilde{\Delta}_r \), and let
\[
\beta_r = \alpha_0^r - \alpha_1^r + \ldots \pm \alpha_{n+1-r}^r.
\]
Observe that \( \beta_r \) is not the Euler characteristic of any complex. However, we have that
\[
\chi(\Delta_i) = \sum_{r=1}^{n+1} \binom{r}{i} \beta_r.
\]
The coefficient \( \binom{r}{i} \) counts the multiplicity of the self-intersection of \( \Delta_i \) at \( \tilde{\Delta}_r \). So
\[
\sum_{i=1}^{n+1} \chi(\Delta_i) = \sum_{i=1}^{n+1} \sum_{r=1}^{n+1} \binom{r}{i} \beta_r,
\]
where \( * \) indicates that the sum is taken only for odd \( i \)'s. After changing the order of the summations we get:
\[
\sum_{r=1}^{n+1} \left( \sum_{i=1}^{r} \binom{r}{i} \right) \beta_r = \sum_{r=1}^{n+1} 2^{r-1} \beta_r \equiv \beta_1 \mod 2. \tag{1}
\]

Now let us paint the complement of \( f(M) \) in \( S^{n+1} \) in two colors in a chessboard-style, i.e. let any two neighboring domains have different colors (where “neighboring” means that they are separated by a component of \( \tilde{\Delta}_1 \)). This is possible, since \( H_n(S^{n+1}; \mathbb{Z}_2) = 0 \).

Let \( N \) be the boundary of an \( \varepsilon \)-neighborhood of \( f(M) \) in the black subset of \( S^{n+1} \). Notice that from the given triangulation of \( f(M) \) we can construct a triangulation of \( N \) by pushing the simplexes from \( f(M) \) to \( N \) in a reasonable way. Simplexes in \( \tilde{\Delta}_i \) will have \( 2^{i-1} \) counterparts in \( N \) (\( i \) hyperplanes divide the Euclidean \( n \)-space into \( 2^i \) parts, half of which are black). Thus:
\[
\chi(N) = \sum_{i=1}^{n+1} 2^{i-1} \beta_i \equiv \beta_1 \mod 2.
\]
But \( \chi(N) \) is even, because \( N \) is embedded in codimension 1 (and \( n > 0 \)), so the proof is complete. \( \square \)

Remark 1. As is clear from the proof, the space \( S^{n+1} \) can be replaced by any manifold such that its \( n \)th \( \mathbb{Z}_2 \)-homology group is 0.
Remark 2. The above proof does not work for \( n \) odd, since the sum \( \sum_{i=1}^{r} * \binom{r}{i} \) (where the star this time means summation for even \( i \)'s) equals to \( 2^{r-1} - 1 \), so the sum in formula (1) gives \( \sum_{r=2}^{n+1} \beta_r \) (which is clearly the Euler characteristic of the complex \( f(M) \)).

The figure 8 immersion of the circle in the plane shows that the statement of the theorem is false for \( n = 1 \). A theorem of Freedman [F] (and its generalization to unoriented 3-manifolds given in [A]) shows that it is true for \( n = 3 \). We do not know whether it is true or not for \( n > 3 \).

Remark 3. If we consider only oriented \( n \)-manifolds and their codimension 1 immersions in \( S^{n+1} \), and the \( n \)th stable homotopy group of spheres has no 2-primary torsion, then the Euler characteristics of the \( i \)-tuple manifolds are all even, for any \( i \). (Indeed, for any \( i \), \( \chi(\Delta_i) \mod 2 \) defines a homomorphism from the stable homotopy group \( \pi_{n+N}(S^N) \), \( N > n \) to \( \mathbb{Z}_2 \).)

In particular the statement of the theorem is true for \( n = 5 \) or \( n = 13 \) for oriented manifolds.

Remark 4. If the dimension \( n = 4 \), then more is true than is stated in the theorem, namely all \( \chi(\Delta_i)'s \) are even, since the stable homotopy group \( \pi_6(RP^\infty) \) vanishes (see [L]), and this group is isomorphic to the cobordism group of immersions of 4-manifolds into \( R^5 \).

REFERENCES


