THE COHOMOLOGY OF THE MORAVA STABILIZER GROUP
$S_2$ AT THE PRIME 3

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Abstract. We compute the cohomology of the Morava stabilizer group $S_2$ at the prime 3 by resolving it by a free product $\mathbb{Z}/3 \ast \mathbb{Z}/3$ and analyzing the “relation module.”

1. Introduction and statement of the main result

The applications of the main theorem of this paper in homotopy theory are due to the Morava Change of Rings Theorem [7]. Let $p$ be a prime, and denote by $S_n$ the group of units in the maximal order of a cyclic division algebra over $\mathbb{Q}_p$ of index $n$ and Hasse invariant $1/n$. The Morava Theorem says essentially that the cohomology of $S_n$ with coefficients in a certain representation describes the Bousfield localization functor $L_{K(n)}$. This is the localization of stable homotopy theory with respect to the spectrum of the $n$-th Morava $K$-theory, $K(n)$ [2]. The functors $L_{K(n)}$ play an important role in homotopy theory [10]. At present the case $n = 1$ is completely understood for all primes $p$. The next case, $n = 2$, has been partially investigated for primes $p \geq 5$ (see for example [15], [16]). The functor $L_{K(2)}$ for small primes is harder to study because the group $S_2$ is of infinite cohomological dimension.

In this paper we deal with the prime 3 only. As the first step of the analysis of $L_{K(2)}$ one needs to compute the continuous cohomology of a certain canonical subgroup $S_0$ of $S_2$ with trivial coefficients. We compute these cohomology groups in the course of the proof of Theorem 1.1 below. This “almost” computes, according to the Morava Change of Ring Theorem, the homotopy groups of the localization of the Toda Smith complex $V(1)$; for more details see [9]. The rest of the calculation of $\pi_\ast L_{K(2)}S^0$ consists of two Bockstein spectral sequences. We will not pursue this here.

Theorem 1.1. $H_\ast^c(S_2; \mathbb{F}_3)$ is freely generated as a graded-commutative $\mathbb{F}_3$-algebra by elements $Z$ (in degree 1), $C, E$ (in degree 3), and $X$ (in degree 4). Its Poincaré series is $\frac{(1+t)(1+t^3)}{1-t^4}$.

We obtain this by first calculating the cohomology of the canonical subgroup, $S_0$, of $S_2$.

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Theorem 1.2. $H^*_c(Sl; F_3)$ has generators $c_1, c_2$ (in degree 1), $x_1, x_2$, $a$ (in degree 2), $c_1, c_2$ (in degree 3). The product of any two generators with different subscripts is 0 and in addition there are relations
\[
a^2 = a e_1 = a e_2 = a c_1 = a c_2 = 0,\quad c_1 e_1 = a x_1,\quad c_2 e_2 = a x_2.
\]
Its Poincaré series is \(\frac{1 + t + t^2 + t^3}{1 - t}\).

A computation of $H^*_c(Sl; F_3)$ was sketched in [9], but the multiplicative structure given there does not agree with the one above. The question of the cohomology of $Sl$ was first reopened by Henn in connection with a deep theorem of his on the cohomology of profinite groups [4], and he also obtained the result stated here. Our calculation proceeds by more classical methods.

The structure of $H^*_c(Sl; F_3)$ can also be described as follows: it was shown in [3] that if $j$ is the quotient map

\[ Sl \rightarrow Sl/Sl' \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3, \]

(where $Sl'$ denotes the commutator subgroup $[Sl, Sl]$ of $Sl$), then there exists a homomorphism $\mathbb{Z}/3 \ast \mathbb{Z}/3 \rightarrow Sl$ such that $ji$ is onto. The image

\[ R = j^*H^*(\mathbb{Z}/3 \oplus \mathbb{Z}/3) \subset H^*_c(Sl) \]

is mapped isomorphically onto $H^*(\mathbb{Z}/3 \ast \mathbb{Z}/3)$ by $i^*$ ($R$ is the subring generated by $1, e_1, e_2, x_1, x_2$). The kernel $R' = \text{Ker } i^*$ (generated by $a, c_1, c_2$, as an $R$-module) is additively like $R$ but with the degrees increased by 2. The structure of $R'$ as an $R$-module is as given above and $R'^2 = 0$: this determines the ring $H^*_c(Sl) \cong R \oplus R'$.

2. Background information

We briefly recall some facts about the group of units of a maximal order in a division algebra. A full account can be found in [11], for example. Consider a cyclic algebra $D$ over $Q_p$ of index $n$ and Hasse invariant $\frac{1}{n}$. It can be constructed as follows. Let $\mathbb{W}$ be the totally unramified extension of $Q_p$ of degree $n$ (so $\mathbb{W} \cong Q_p(\zeta)$ where $\zeta$ is a $(p^n - 1)$-th root of unity). The Galois group of the extension $\mathbb{W}/Q_p$ is a cyclic group of order $n$: it is generated by the Frobenius homomorphism $\sigma$. We form the crossed product algebra of $\mathbb{W}$ and $\text{Gal}(\mathbb{W}/Q_p)$. This amounts to introducing a variable $S$ which commutes with $\mathbb{W}$ according to the formula $wS = Sw^\sigma$ and satisfies $S^n = a \in Q_p^\times$. To define $D$ we set $S^n = p$. $D$ is a division algebra over $Q_p$ of rank $n^2$.

Let $\mathcal{O}$ be the maximal order in $D$: it is generated by $S$ and the integers of $\mathbb{W}$. Its maximal ideal is $\mathcal{O}S$ and $\mathcal{O}/\mathcal{O}S \cong F_{p^n}$. We are interested in three groups contained in $\mathcal{O}$. The first is the group of units of $\mathcal{O}$, which we denote by $S_n$. The second is the subgroup of strict units in $S_n$. It consists of the elements $a \in S_n$ such that $a \equiv 1 \mod S$. We denote it by $S_n^0$. The third is the kernel of the reduced norm restricted to $S_n^0$. We denote this subgroup by $Sl$. It is a pro-$p$ group because it is $p$-filtered and compact ([6], II, 2.1.3).

Let $H_r$ denote the subgroup of $Sl$ consisting of elements congruent to 1 modulo $S^r$. By definition $H_1 = Sl$. According to [12], there are injective maps $\rho_r : H_j/H_{j+1} \rightarrow \mathcal{O}/\mathcal{O}S \cong F_{p^n}$ given by $\rho_r(1 + aS^r) \equiv a \mod S$ ($a \in \mathcal{O}$). They are also surjective unless $n|\rho$, in which case the image consists of those elements of trace 0 over the prime field.
From now on we shall only consider \( p = 3 \) and \( n = 2 \). Then \( |H_2/H_1| = 3 \), \( |H_2/H_3| = 3 \), \( |H_3/H_4| = 9 \), and according to [12], \([H_1, H_1] = H_2 \), \([H_1, H_2] = H_3 \), \([H_2, H_2] = H_4 \). Now \( Z \) contains an 8-th root of unity \( \zeta \), and so \( z = \frac{\zeta^{S-1}}{3} \in \mathbb{D} \) is a cube root of unity. Thus \( \mathcal{X} = z \) and \( \mathcal{Y} = z^3 \) are two elements of \( Sl \) of order 3. Their images \( x = j(\mathcal{X}) \), \( y = j(\mathcal{Y}) \) generate \( Sl/Sl' \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \) (where \( Sl' = [Sl, Sl] = H_2 \)). We now define \( i : \mathbb{Z}/3 \ast \mathbb{Z}/3 \to Sl \) to be the map which takes the generators \( X \) and \( Y \) to \( \mathcal{X} \) and \( \mathcal{Y} \). This map is in fact injective [3], but we do not need to know that here.

There is a group \( D \) of automorphisms of \( Sl \) which has order 8 and is generated by (i) conjugation by \( S \), which interchanges \( X \) and \( Y \), and (ii) conjugation by \( S \zeta \), which interchanges \( \mathcal{X} \) and \( \mathcal{X}^2 \) and fixes \( \mathcal{Y} \).

The natural cohomology theory for a profinite group is the cohomology on continuous cochains [13], denoted by \( H^*_c \), and that is what we use here. It agrees with the usual cohomology on a finite group.

Any maximal finite subgroup of \( Sl \) is cyclic of order 3, so the Krull dimension of \( H^*_c(Sl; \mathbb{F}_3) \) is one [8], [7].

### 3. Resolutions

The fact that \( ji \) is onto implies that \( Im \ i \) is dense in the pro-3 topology. So we have an epimorphism of pro-3 groups

\[
\mathbb{Z}/3 \ast \mathbb{Z}/3 \to Sl \quad (\sim \text{ denotes pro-3 completion}).
\]

Let \( K \) denote the kernel of this map. The kernel of \( \mathbb{Z}/3 \ast \mathbb{Z}/3 \to \mathbb{Z}/3 \oplus \mathbb{Z}/3 \) is free on the four generators \([X, Y, Y'] \) \((1 \leq i, j \leq 2) \) [14]. The completion of the corresponding short exact sequence remains exact, since \( \mathbb{Z}/3 \oplus \mathbb{Z}/3 \) is finite. This leads to a diagram with exact rows and columns:

\[
\begin{array}{ccc}
K & \rightarrow & K' \\
\downarrow & & \downarrow \\
K & \rightarrow & \mathbb{Z}/3 \ast \mathbb{Z}/3 \\
\downarrow & & \downarrow \\
\mathbb{Z}/3 \oplus \mathbb{Z}/3 & \rightarrow & \mathbb{Z}/3 \oplus \mathbb{Z}/3.
\end{array}
\]

Now \( K \) is a closed subgroup of a free pro-3 group (i.e. the pro-3 completion of a free group), so is itself a free pro-3 group ([13], Cor. 2 to I, Prop. 24).

According to the theory in ([6], V 2.5.7), \( Sl' \) is equi-3-valued (with the usual 3-adic valuation), so \( H^*_c(Sl') \cong \Lambda^*H^*_c(Sl') \). \( Sl'/[Sl', Sl'] = H_2/H_4 \) has order 27 and is easily seen to have exponent 3, so it has rank 3, and thus so does \( H^*_c(Sl') \).

Let \( N \) denote \( Sl'/[Sl', Sl'] \) as an \( \mathbb{F}_3 \) module where \( E = Sl/Sl' \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \). Then \( H^*_c(Sl') \cong N \) by duality.

We want to be able to describe modules such as \( N \) explicitly. For this purpose note that if we set \( X = x - 1 \) and \( Y = y - 1 \), then \( \mathbb{F}_3 E \cong \mathbb{F}_3 [X, Y] \), \( X^3 = Y^3 = 0 \). The augmentation ideal \( I \) is generated by \( X \) and \( Y \), and, because \( E \) is a \( p \)-group, \( I \) is the radical.
Since \([H_1, H_2] = H_3\) and \([H_2/H_3] = 3\), we must have \(N/IN \cong \mathbb{F}_3\). If we consider the image of \(\text{ann}N\) in \(I/I^2 \cong E\), we see that it cannot be 1-dimensional, since then it could not be invariant under the group of automorphisms \(D\). Hence \(\text{ann}N = I^2\) and \(N \cong \mathbb{F}_3E/I^2\).

Now consider the spectral sequence
\[
H^p_c(Sl'; H^q_c(K)) \Rightarrow H^{p+q}_c(\hat{F}_3).
\]
There are only two rows, since \(K\) is a free pro-3 group, and we deduce that
\[
H^r_c(Sl'; H^1_c(K)) = 0, \quad r \geq 2,
\]
and there are short exact sequences
\[
\begin{align*}
0 & \to H^1_c(Sl') \to H^1_c(\hat{F}_3) \to E^{0,1}_\infty \to 0, \\
0 & \to E^{0,1}_\infty \to H^1_c(K)^{Sl'} \to N \to 0.
\end{align*}
\]
Sequence (3.2) shows that \(E^{0,1}_\infty \cong \mathbb{F}_3\). Let \(M\) denote \(H^1_c(K)^{Sl'}\) as an \(\mathbb{F}_3\)E-module, so we have
\[
0 \to \mathbb{F}_3 \to M \to N \to 0.
\]

4. The structure of \(M\)

The short exact sequence \(0 \to \mathbb{F}_3 \to M \to N \to 0\) shows that either \(M \cong N \oplus \mathbb{F}_3\), or \(M\) is generated by one element. In the latter case, \(M \cong \mathbb{F}_3E/\text{ann} M\), and we shall assume that this holds for the rest of this section. The image of \(\text{ann} M\) in \(I^2/I^3\) is a subspace \(S\) of codimension 1, which must be invariant under the action of the group of automorphisms, \(D\). Since \(I^2/I^3\) has basis \(\{X^2, XY, Y^2\}\), it is easy to check that the only possibilities for \(S\) are \(S_1 = \langle X^2, Y^2 \rangle, S_2 = \langle XY, X^2 + Y^2 \rangle,\) and \(S_3 = \langle XY, X^2 - Y^2 \rangle\). Let \(M_1 = \mathbb{F}_3E/S_1\). We claim that, in fact, \(M \cong M_1\), but this will only become apparent later. Notice, however, that if we extend the field to \(\mathbb{F}_9\), then all three modules differ only by an automorphism of the group algebra, for if
\[
\phi_2 : X \mapsto X + Y, \quad Y \mapsto X - Y,
\]
then \(\phi_2(S_1) = S_2\), and so \(M_2^{\phi_2} \cong M_1\). Similarly, if
\[
\phi_3 : X \mapsto X + iY, \quad Y \mapsto X - iY \quad \text{(where } i^2 = 1),
\]
then \(\phi_3(S_1) = S_3\), and so \(M_3^{\phi_3} \cong M_1\).

5. The cohomology of \(M_1\)

Let \(p\) be an odd prime, \(k\) a field of characterisitc \(p\), and \(E = \langle x, y \rangle\) an elementary abelian \(p\)-group of order \(p^2\). (We will only need the case where \(p = 3\) and \(k = \mathbb{F}_3\), but it is just as easy to prove the result of this section more generally.) The group algebra \(kE\) is the truncated polynomial algebra \(k[X, Y | X^p = Y^p = 0]\), where \(X = x - 1\) and \(Y = y - 1\). A minimal projective resolution \(P \overset{\delta}{\to} k\) of the trivial \(kE\)-module \(k\) may be constructed as follows. First let \(P_n\) be the free left \(kE\)-module on the \(n + 1\) symbols \(e_{r,s}\) where \(r + s = n\) and \(r, s \geq 0\). For notational convenience we set \(e_{r,s} = 0\) if \(r < 0\) or \(s < 0\). Then define
\[
\partial(e_{r,s}) = X^{1+(p-2)r+(r+1)s}e_{r-1,s} + (-1)^rY^{1+(p-2)s+(s+1)}e_{r,s-1},
\]
where \( \nu(n) \) is defined to be 0 if \( n \) is even and 1 if \( n \) is odd, and set \( \epsilon(e_{0,0}) = 1 \). We have

\[
H^*(E; k) = \text{Hom}_{kE}(P, k) = k[x_1, x_2] \otimes_k A_k[e_1, e_2]
\]

where \( x_1 = e_{0,2}^* \) (by which we mean \( x_1(e_{0,2}) = 1 \) and \( x_1(e_{r,s}) = 0 \) for \( (r, s) \neq (0, 2) \)), \( x_2 = e_{2,0}^* \), \( e_1 = e_{0,1}^* \), and \( e_2 = e_{1,0}^* \). The generators \( e_1 \) and \( e_2 \) in \( H^1(E, k) \cong \text{Hom}(E, k) \) correspond to maps \( E \to k \) with kernels \( (x) \) and \( (y) \) respectively, and \( x_1 \) and \( x_2 \) are their Bocksteins.

Now let \( M_1 = kE/(X^{p-1}, Y^{p-1}) \). (If \( p = 3 \) and \( k = \mathbb{F}_3 \), this is consistent with the definition of \( M_1 \) given above.)

**Proposition 5.1.** As a right \( H^*(E; k) \)-module, \( H^*(E; M_1) \) is generated by elements \( \alpha \) (degree 0), \( \delta_1, \delta_2 \) (degree 1), and \( \beta \) (degree 2), subject to the relations

\[
\alpha e_1 = \alpha e_2 = \delta_2 e_1 = \delta_1 e_2 = 0,
\]

\[
\delta_1 e_1 = \delta_1 x_1, \quad \delta_2 e_2 = \alpha x_2,
\]

\[
\beta e_1 = -\delta_2 x_1, \quad \beta e_2 = \delta_1 x_2.
\]

In particular, \( H^*(E; M_1) \) is a free \( k[x_1, x_2] \)-module on \( \alpha, \delta_1, \delta_2, \beta \).

**Proof.** We compute an explicit basis for \( H^n(E; M_1) = Z^n(E, M_1)/B^n(E, M_1) \). First note that \( M_1 \) is a commutative ring, and \( \text{Hom}_{kE}(P_n, M_1) \) is a free left \( M_1 \)-module on the generators \( f^n_i = e_{i,n-i}^* \) \( (0 \leq i \leq n) \). Let \( \bar{a} \) denote the image in \( M_1 \) of an element \( a \) in \( kE \). Then \( \bar{X}^{p-1} = \bar{Y}^{p-1} = 0 \), so for \( f \in \text{Hom}_{kE}(P_n, M_1) \) and \( r, s \geq 0 \) we have

\[
f \partial(e_{r,s}) = \nu(r) \bar{X} f(e_{r-1,s}) + (-1)^r \nu(s) \bar{Y} f(e_{r,s-1}).
\]

Suppose first that \( n = 2m \) is even. Let \( f = \sum_{i=0}^n a_i f^n_i \) \( (a_i \in M_1) \) and for notational convenience set \( \alpha_{-1} = \alpha_{n+1} = 0 \). Then equation (5.1) implies that \( f \in Z^n(E, M_1) \) if and only if

\[
\nu(j) \alpha_{j-1} \bar{X} + (-1)^j \nu(n+1) \alpha_j \bar{Y} = 0 \quad \text{whenever} \quad 0 \leq j \leq n+1.
\]

But this occurs if and only if \( \alpha_0 \bar{X} = 0 = \alpha_j \bar{Y} \) for all even \( j \), or equivalently \( \alpha_j \in \langle \bar{X}^{p-2} \bar{Y}^{p-2} \rangle \) for all even \( j \). Hence

\[
Z^n(E, M_1) = \{ \sum_{j=0}^n \alpha_j f^n_j \mid \alpha_j \in M_1 \text{ if } j \text{ is odd}, \alpha_j \in \langle \bar{X}^{p-2} \bar{Y}^{p-2} \rangle \text{ if } j \text{ is even} \}.
\]

A similar calculation yields

\[
B^n(E, M_1) = \{ \sum_{j=0}^n \alpha_j f^n_j \mid \alpha_j \in \langle \bar{X}, \bar{Y} \rangle \text{ if } j \text{ is odd}, \alpha_j = 0 \text{ if } j \text{ is even} \}.
\]

Hence the images of the \( n + 1 \) elements \( \alpha_k^n = \bar{X}^{p-2} \bar{Y}^{p-2} e_{2k+2m-2k}^* \) \( (0 \leq k \leq m) \) and \( \beta_k^n = e_{2k+2m-2k}^* \) \( (0 \leq k < m) \) of \( Z^n(E, M_1) \) form a basis in \( H^n(E, M_1) \).

Now suppose that \( n = 2m+1 \) is odd. Let \( S = \{ (\zeta, \eta) \in M_1 \oplus M_1 \mid \zeta \bar{X} = \eta \bar{Y} \} \) and \( T = \{ (\zeta \bar{Y}, \zeta \bar{X}) \mid \zeta \in M_1 \} \). Working as in the even case, we get

\[
Z^n(E, M_1) = \{ \sum_{i=0}^m (\alpha_i f^n_{2i} + \beta_i f^n_{2i+1}) \mid (\alpha_i, \beta_i) \in S \},
\]

\[
B^n(E, M_1) = \{ \sum_{i=0}^m (\alpha_i f^n_{2i} + \beta_i f^n_{2i+1}) \mid (\alpha_i, \beta_i) \in T \}.
\]
Now $S/T$ is 2-dimensional and is spanned by the images of $(\tilde{X}^{p-2}, 0)$ and $(0, \tilde{Y}^{p-2})$. Hence the images of the $n+1$ elements

$$\gamma^m_k = \tilde{X}^{p-2}e_{2k, 2m+1-2k}, \quad \delta^m_k = \tilde{Y}^{p-2}e_{2k+1, 2m-2k} \quad (0 \leq k \leq m)$$

of $Z^n(E; M_1)$ form a basis in $H^n(E; M_1)$.

We now turn to the module structure. Recall that composition with $\epsilon$ is a chain map $\text{Hom}_{kE}(P, P) \to \text{Hom}_{kE}(P, k)$ and this map induces an isomorphism in cohomology $H^* \text{Hom}_{kE}(P, P) \cong H^*(E; k)$. The action

$$H^*(E; M_1) \otimes_k H^*(E; k) \to H^*(E; M_1)$$

is induced by the map on the cochain level

$$\text{Hom}_{kE}(P, M_1) \otimes_k \text{Hom}_{kE}(P, P) \to \text{Hom}_{kE}(P, M_1)$$

given by composition. So we must first lift $x_1, e_i$ to maps $\tilde{x}_1 \in Z^2 \text{Hom}_{kE}(P, P)$, $\tilde{e}_i \in Z^1 \text{Hom}_{kE}(P, P)$. This is accomplished by setting

$$\tilde{x}_1(e_{r,s}) = e_{r,s-2}, \quad \tilde{x}_2(e_{r,s}) = e_{r-2,s}, \quad \tilde{e}_1(e_{r,s}) = (-1)^{r+s+1}Y^{(p-2)\nu(s+1)}e_{r,s-1}, \quad \tilde{e}_2(e_{r,s}) = (-1)^{r+1}X^{(p-2)\nu(r+1)}e_{r-1,s}.$$

Now let $\alpha = [\alpha_0^0], \, \delta_1 = -[\delta_0^0], \, \delta_2 = -[\delta_0^0], \, \beta = [\beta_0^0]$. We have $\alpha x_1x_2 = [\alpha_{ij}^0], \delta_1x_1^2x_2^2 = -[\delta_{ij}^0], \delta_2x_1^2x_2^2 = -[\delta_{ij}^0], \beta x_1^2x_2^2 = [\beta_{ij}^0]$. These are easily verified; the first, for example, just follows from the fact that

$$\alpha_0^0\tilde{x}_1\tilde{x}_2^j(e_{r,s}) = \alpha_0^0(e_{r-2,s-2i}) = \tilde{X}^{p-2}\tilde{Y}^{p-2}((r, s) = (2j, 2i)) = \alpha_{ij}^0(e_{r,s}).$$

(Here we are using the computer science notation where, for a proposition $P$, $(P) = 1$ if $P$ and $(P) = 0$ otherwise.) This proves that $\alpha, \delta_1, \delta_2, \beta$ generate $H^*(G; M_1)$ as a $k[x_1, x_2]$-module and that $H^*(G, M_1)$ is a free $k[x_1, x_2]$-module on those four generators.

We now turn to the relations. It is routine to check that the generators satisfy these relations; for example, the last of these follows from the fact that

$$\beta_0^0\tilde{x}_1(e_{r,s}) = (-1)^{r+s+1}\tilde{Y}^{(p-2)\nu(s+1)}\beta_0^0(e_{r,s-1}) = \tilde{Y}^{p-2}((r, s) = (1, 2)) = \delta_{0}^0\tilde{x}_1(e_{r,s}).$$

Now let $A^*$ denote the graded $H^*(E; k)$-ring defined abstractly by these 4 generators and 8 relations. Since $H^*(E; M_1)$ satisfies the relations, there is a surjective $H^*(E; k)$-homomorphism $A^* \to H^*(E; M_1)$. We wish to show that this homomorphism is an isomorphism, and to do this it suffices to show that $\dim_k H^n(E; M_1)$ for all $n$. This will follow if we can show that $A^*$ is generated as a $k[x_1, x_2]$-module by the four generators, because we know that $H^*(G, M_1)$ is a free $k[x_1, x_2]$-module on those generators. Hence for each $\theta \in \{\alpha, \delta_1, \delta_2, \beta\}$ and $i \in \{1, 2\}$ we must show that $\theta e_i$ is in the $k[x_1, x_2]$-submodule of $A^*$ generated by $\alpha, \delta_1, \delta_2, \beta$. But this is exactly what the 8 relations tell us.

The following lemma is easy to check, but will be useful.

**Lemma 5.2.** We can replace $\beta$ by any other element of $H^2(E; M_1)$ linearly independent of $\alpha x_1$ and $\alpha x_2$ and get the same relations.
6. The final calculation for $\text{Sl}$

We assume for now that $M = M_1$: this will be justified later. Consider the spectral sequence

$$H^p(E; H^q_c(\text{Sl}'; H^1_c(K))) \Rightarrow H^{p+q}(\text{Sl}; H^1_c(K))$$

Again it has only two rows.

$$H^{*-2}(E) \xrightarrow{d_2} H^*(E; M).$$

$H^*(E)$ is a torsion-free $\mathbb{F}_3[x_1, x_2]$ module; hence so is $\text{Ker} d_2$. But this spectral sequence shows that $H^*(\text{Sl}; H^1_c(K))$ is finitely generated over $H^*(E)$, and hence over $H^*_c(\text{Sl})$, which has Krull dimension 1. It maps onto $\text{Ker} d_2$, forcing a common bound on $\dim \text{Ker} d_2$ in each degree. Thus $\dim \text{Ker} d_2 = 2$ and we have:

$$\dim \text{Ker}(d_2(1)) = 1.$$ (6.1)

Now $d_2(1)$ is not a linear combination of $\alpha x_1$ and $\alpha x_2$; otherwise it would be annihilated by $e_1$ and $e_2$. By Lemma 5.2 we may assume that $d_2(1) = \beta$.

We have proved:

**Proposition 6.2.** $H^*_c(\text{Sl}; H^1_c(K))$ is an $H^*(E)$-module on generators $\alpha$ (degree 0), $\delta_1$, $\delta_2$ (degree 1) with relations

$$\alpha e_1 = \alpha e_2 = \delta_1 e_2 = \delta_2 e_1 = \delta_1 x_2 = \delta_2 x_1 = 0,$$

$$\delta_1 e_1 = \alpha x_1, \quad \delta_2 e_2 = \alpha x_2.$$

Now consider the spectral sequence

$$H^*_c(\text{Sl}; H^*(K)) \Rightarrow H^*_c(\mathbb{Z}/3 * \mathbb{Z}/3).$$

One has $H^*_c(\mathbb{Z}/3 * \mathbb{Z}/3) \cong H^*(\mathbb{Z}/3 * \mathbb{Z}/3)$. (Consider the short exact sequence $F_3 \to \mathbb{Z}/3 * \mathbb{Z}/3 \to \mathbb{Z}/3 \oplus \mathbb{Z}/3$ and its pro-3 completion, and use the Comparison Theorem.) The map $i^*: H^*_c(\text{Sl}) \to H^*(\mathbb{Z}/3 * \mathbb{Z}/3)$ is an isomorphism in degree 1, by construction. As the right-hand side is generated by elements of degree 1 and their Bocksteins, $i^*$ is onto in all degrees and the spectral sequence becomes the short exact sequence

$$0 \to H^{*-2}_c(\text{Sl}; H^1_c(K)) \xrightarrow{d_2} H^*_c(\text{Sl}) \to H^*(\mathbb{Z}/3 * \mathbb{Z}/3) \to 0,$$

of right $H^*(E)$-modules. Set $a = d_2(\alpha)$, $c_i = d_2(\delta_i)$. Identify $e_i$ and $x_i$ with their images under $j^*$. All that remains is to check that $\text{Im} j^* = R \cong H^*(\mathbb{Z}/3 * \mathbb{Z}/3)$, i.e. that $e_1 e_2 = e_1 x_2 = e_2 x_1 = x_1 x_2 = 0$.

The spectral sequence

$$H^*(E; H^*_c(\text{Sl}')) \Rightarrow H^*_c(\text{Sl})$$

has four rows. We know that $d_2(E^2_{-1, 1}) \subset E^*_2 \cong H^*(E)$ is contained in $\text{Ker}(ji)^*$, which is generated as an $H^*(E)$-module by $e_1 e_2$, $e_1 x_2$, $e_2 x_1$, and $x_1 x_2$. Since $H^*_c(\text{Sl}')$ is dual to $N$ as an $E$-module it is isomorphic to $I^3$, and we can calculate $\dim E^0_{1, 1} = 1$ and $\dim E^1_{1, 1} = 3$. (Use dimension shifting: $F_3 E/I^3$ has invariants of dimension 3.) But $E^0_{2, 1}$ yields all of $H^1_c(\text{Sl})$ and thus $\dim d_2(E^0_{2, 1}) = 1$. This accounts for $e_1 e_2$. Also $\dim E^0_{\infty} = 2$ and $\dim H^2_c(\text{Sl}) = 3$, so

$$\dim \text{Ker}(d_2; E^1_{1, 1} \to E^0_{2, 1}) \leq 1.$$ 

The image of this map must have dimension $\geq 2$, which accounts for $e_1 x_2$ and $e_2 x_1$. Finally $x_1 x_2$ is the Bockstein of $e_1 x_2$. 

All that remains is to justify our assertion that $M \cong M_1$. We do this by carrying out the above calculation for each of the other possibilities and obtaining a contradiction.

If $M \cong \mathbb{F}_3 \oplus N$, then the short exact sequence (6.1) shows that $\dim H^n(S; M) = \dim H^n(E; N) + 2$. This is impossible because $N$ has complexity 2, yet $H_c^*(S)$ has Krull dimension 1.

If $M \cong M_2$, then the structure of $H^*(E; M_2)$ as an $H^*(E)$-module is like that of $H^*(E; M_1)$, but twisted by $\phi_2$. Let us denote the new module structures by $\ast$. Then

$$u \ast v = u(\phi_2^*v), \quad u \in H^*(E; M), v \in H^*(E).$$

Thus $a \ast x_1x_2 = a(x_1 + x_2)(x_1 - x_2) = ax_1^2 - ax_2^2 \neq 0$. But the argument that $x_1x_2 = 0$ is still valid since it only depends on the additive structure of $H_c^*(S; M)$. The case $M \cong M_3$ is similar.

7. THE COHOMOLOGY OF $S_2$

Remark 7.1 ([9]). If $p > 3$ and $n = 2$ then $S$ is torsion-free so $H^*(S)$ has finite cohomological dimension by [8]. It contains an open subgroup $H_2$, which is a Poincaré duality group of dimension 3. So by ([13], V.4.7) $S$ is also a Poincaré duality group of dimension 3. Since $\dim H^2(S) = 2$, there is only one possible multiplicative structure, namely that with generators $e_1, e_2$ (degree 1), $f_1, f_2$ (degree 2), and relations $e_1e_2 = 0, e_1f_1 = e_2f_2, f_1^2 = f_2^2 = f_1f_2 = 0$.

Remark 7.2. The map

$$S_2^0 \rightarrow (1 + 3\mathbb{Z}_3)^\times \xrightarrow{\log} 3\mathbb{Z}_3^+$$

is split by $x \mapsto \exp(\frac{1}{3}x)$, the image being central in $S_2^0$. Thus $S_2^0 \cong S \times \mathbb{Z}_3^+$ and $H^*(S_2^0) \cong H^*_c(S) \otimes \Lambda(z)$, deg $z = 1$.

The group $S_2$ is the semi-direct product of $S_2^0$ and $\mathbb{Z}/8$ (the splitting is obtained by lifting the elements of $S_2^0$ to roots of unity in $\mathbb{W}$). Therefore $H^*_c(S_2; \mathbb{F}_3)$ is isomorphic to the subring of $H^*_c(S; \mathbb{F}_3)$ invariant under conjugation by the eighth root of unity $z \in \mathbb{W}$.

We must track the action of $z$ through our calculation. Notice that conjugation by $z$ has the effect $X \mapsto Y \mapsto X^2 \mapsto Y^2 \mapsto X$. Thus $e_1 \zeta = -e_2, e_2 \zeta = e_1$, and the same for their Bocksteins, $x_1 \zeta = -x_2, x_2 \zeta = x_1$.

Regarding all groups as $(\zeta)$-modules now, (3.3) shows that $H^0(E; M) \cong F_3 \cong F_3$ from short exact sequence (3.2) we see that $\zeta$ acts on $H^0(E; M)$ as multiplication by $\det H_1^1(F_1) \zeta / \det N(\zeta)$. But $\zeta$ permutes the explicit generators of $F_4$ transitively; hence $\det H_1^1(F_1) \zeta = -1$. To calculate $\det N(\zeta)$, note that under the map $\rho_2$ of Section 2, conjugation by $z$ acts as the identity, whilst under $\rho_3$ it corresponds to multiplication by $-\zeta$ on $F_9$, which has determinant 1 over $F_3$. This proves that $\det N(\zeta) = -1$, and consequently from (6.1), $\alpha \zeta = -\alpha$. Sequence (6.3) then shows that $\alpha \zeta = -\alpha$.

The relations involving the elements $ci$ now force $c_1 \zeta = -c_2, c_2 \zeta = -c_1$. Clearly $z \zeta = z$, and this completes the calculation of the action of $\zeta$.

It is now fairly straightforward to calculate the invariants of this action, especially if one notes that, if $X = x_1^2 + x_2^2$, then $X$ is invariant and $H^*_c(S; F_3)$ is free over $F_3[X]$. We obtain Theorem 1.1 by setting $Z = z, C = c_1 - c_2, E = e_1x_1 + e_2x_2, X = x_1^2 + x_2^2$.
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