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NIL SUBSETS OF GRADED ALGEBRAS

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Abstract. We prove that if A is a Noetherian $\mathbf{Z}$-graded algebra, then the Jacobson radical of A is nilpotent under mild hypotheses on A_0. We also consider affine PI-algebras graded by torsion groups. Finally we prove a Nullstellensatz-type theorem for enveloping algebras of Lie color algebras.

0. Introduction

In this paper we study various subsets of a graded algebra A by using Jacobson's notion of weakly-closed subsets. When A is Noetherian, a result of Jacobson-Goldie says that weakly-closed nil subsets are nilpotent; as a consequence we prove that when A is $\mathbf{Z}$-graded, the Jacobson radical $J(A)$ of A is nilpotent provided that $J(A) \cap A_0$ is nil. We also show that if A is an affine PI-algebra graded by a torsion group G, then A is a finite module over its identity component A_e whenever A_e is central.

We then consider some applications to the enveloping algebra of a Lie color algebra L. We show that when L is finite-dimensional, then the endomorphism ring of an L-module of finite length is finite-dimensional; this is a kind of generalized Hilbert Nullstellensatz, and extends the analogous result for ordinary Lie algebras. We also give a short proof of an Engel-type result of [BG].

Throughout, A denotes an algebra over the commutative ring k. Usually A is graded by the group G; thus $A = \oplus_{g\in G} A_g$, where the A_g are k-submodules of A and $A_g A_h \subseteq A_{gh}$. Following [J], we say that a subset S of A is *weakly-closed* if for any pair $s, t \in S$, there exists $\gamma = \gamma(s,t) \in k$ such that $st + \gamma(s,t)ts \in S$. Jacobson proves that when A is right Artinian, the (associative) subalgebra of A generated by a nil weakly-closed subset is nilpotent [J, p. 201].

1. Jacobson radicals

In this section we show that the Jacobson radicals of certain graded rings are nilpotent. Our first result is a strengthening of Jacobson's result due to Goldie.

Proposition 1.1 ([G]). *A nil weakly closed subset of a Noetherian algebra generates a nilpotent subalgebra.*

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We note that 1.1 follows by standard arguments: if N is the nilpotent radical of the algebra A, then one may apply Jacobson's result to the ring of fractions of the semiprime Noetherian ring A/N.

Corollary 1.2. *Let A be a Noetherian algebra, graded by the group G, and let B be a graded subalgebra of A such that the homogeneous elements of B are nilpotent. Then B is nilpotent.*

Proof. Let S be the set of homogeneous elements in B. Then S is a nil weakly closed subset of A, and so is nilpotent by Proposition 1.1.

But then B is nilpotent. □

We note an easy consequence of the corollary: if A is Noetherian and graded-semiprime, then any non-zero graded left or right ideal contains a non-nilpotent homogeneous element. This condition is used in [NvO] in proving a graded version of Goldie's theorem.

We also note that Corollary 1.2 is really special to the graded case; it is not sufficient to assume that every element in B is a sum of nilpotent elements. For consider the ring R of $n \times n$ matrices ($n > 1$) over a division ring D such that every element of D is a sum of additive commutators; then every element of R is a sum of nilpotent elements, but R is certainly not nilpotent [H].

We can now prove the main result of this section. We let $J(A)$ denote the Jacobson radical of A.

Theorem 1.3. *Let A be a Noetherian algebra, graded by $\mathbf{Z}$, and assume that $J(A) \cap A_0$ is nil. Then $J(A)$ is nilpotent.*

Proof. By a result of Bergman (see [P, p. 225]), $J(A)$ is a graded ideal and each $J(A)_m$ is nil, for $m \neq 0$. By hypothesis, also $J(A)_0$ is nil. Thus $B = J(A)$ satisfies the hypotheses of Corollary 1.2, and so is nilpotent. □

Since $J(A) \cap A_0 \subseteq J(A_0)$, the hypothesis of the theorem will be satisfied whenever $J(A_0)$ is nil.

If an algebra A is graded by a finite group G, a similar result is shown in [CR, Lemma 1] with no finiteness hypotheses on A: if B is a graded subalgebra such that B_e is nilpotent, then B is nilpotent.

2. Affine PI algebras

In this section we give some sufficient conditions for an algebra A, graded by a torsion group G, to be a finite module over A_e.

Theorem 2.1. *Let A be a k-affine PI algebra, graded by the torsion group G, such that A_e is central. Then A is a finite module over A_e and consequently A_e is k-affine. If also k is Noetherian, then both A and A_e are Noetherian.*

Proof. We may assume that A is generated by homogeneous elements. Since G is torsion, for each $g \in G$, there is some $n > 0$ such that $g^n = e$, and thus if $a \in A_g$, we have $a^n \in A_e$.

Now define $B_0 := A_e$. Then $A = B_0[a_1, ..., a_t]$, where the a_i are homogeneous; we may assume that $a_1 = 1$. We consider A as a filtered algebra, as follows: define $B_1 := a_1 B_0 + \cdots + a_t B_0$ and $B_m := {B_1}^m$, for all $m > 0$ (we assume ${B_1}^0 = B_0$). Then $A = \bigcup_{m \geq 0} B_m$. We may then form the associated graded algebra

$$Gr(A) = \oplus_{m \geq 0} B_m / B_{m-1},$$

where $B_{-1} := 0$. In $Gr(A)$, let S be the semigroup generated by those $\overline{a}_i = a_i + B_0 \in \overline{B}_1 = B_1/B_0$ such that $a_i \notin B_0 = A_e$. Then S is weakly closed (as it is closed under ordinary products) and S is nil. For, consider any monomial $\overline{a}_{i_1} \cdots \overline{a}_{i_k}$ in S. Now in A, $b = a_{i_1} \cdots a_{i_k}$ is homogeneous, and thus $b^n \in A_e$ for some n, as noted above. Thus in $Gr(A)$, $(\overline{a}_{i_1} \cdots \overline{a}_{i_k})^n = \overline{0}$.

Now $Gr(A)$ is also a PI ring, and so it follows that S is nilpotent, by standard arguments (if not, there exists a prime ideal P so that $S^k \not\subseteq P$ for all k; then pass to $Gr(A)/P$).

But then $\overline{B}_m = 0$ for all $m \geq M$, for some M. Thus $Gr(A)$ is a finite module over B_0, and so A itself is a finite module over B_0. We are now done by the Artin-Tate lemma. □

A related result was shown in [MS]: if G is finite and A is k-affine and Noetherian, then A is finite over A_e, and A_e is also affine and Noetherian.

3. Enveloping algebras of Lie color algebras

In this last section, we apply some of our previous methods to the case of Lie color algebras; a basic reference on Lie color algebras is [BMPZ]. Throughout, k will be a field and G will be an abelian group with a fixed bicharacter $\langle\cdot|\cdot\rangle : G \times G \longrightarrow k^*$. Then a G-Lie color algebra is a G-graded k-space $L = \oplus_{g \in G} L_g$ with a bilinear map $[\cdot,\cdot] : L \otimes L \longrightarrow L$ satisfying

(i) $[x, y] = -\langle h|g\rangle [y, x]$,

(ii) $\langle g|l\rangle [x, [y, z]] + \langle l|h\rangle [z, [x, y]] + \langle h|g\rangle [y, [z, x]] = 0$

for all homogeneous elements $x \in L_g$, $y \in L_h$, and $z \in L_l$. $U(L)$ will denote the universal enveloping algebra of L.

We first note that a graded version of Engel's theorem is known for Lie color algebras: if $L \subset End(V)$, where V is a finite-dimensional vector space, such that the homogeneous elements of L act nilpotently on V, then L is nilpotent on V. For, simply apply Jacobson's result to the set S of homogeneous elements of L. This result was pointed out by Scheunert for the case of Lie superalgebras [S]: he says that "the usual proof works", and one assumes that he means to use weakly-closed sets. Recently Bergen and Grzeszczuk have extended Engel's theorem to the case when V is not necessarily finite-dimensional. We give here a short proof of their result.

Theorem 3.1 ([BG])**.** *Let L be a finite-dimensional Lie color algebra and let V be an L-module. If every homogeneous element of L acts nilpotently on V, then L acts nilpotently on V.*

Proof. Since L is finite-dimensional , $U(L)$ is Noetherian, and thus so is its image $\overline{U(L)}$ in $End(V)$. If S is the set of homogeneous elements in L, then $\overline{S}$ is weakly closed in $\overline{U(L)}$, and each $\overline{s} \in \overline{S}$ is nilpotent. Thus by Proposition 1.1, the associative algebra generated by $\overline{S}$ is nilpotent. Thus the image of L in $End(V)$ is certainly nilpotent. □

We next prove a "generalized Nullstellensatz" for enveloping algebras of Lie color algebras, which extends known results for ordinary Lie algebras.

Theorem 3.2. *Let L be a finite-dimensional Lie color algebra and let V be a $U(L)$-module of finite length. Then $End_{U(L)}(V)$ is finite-dimensional over k.*

Proof. The proof follows the outline in [McR, Section 9.5] for ordinary Lie algebras. In fact the argument works for any Noetherian algebra A such that

(i) $A \otimes_k K$ is also Noetherian, for any field extension $K \supset k$;

(ii) $End_A(V)$ is algebraic, for any simple A-module V.

For, $End_A(V) = D$, a division ring, since V is simple. If C is a maximal subfield of D, then C is a finitely-generated field extension of k by [McR, Proposition 9.5.4]; this uses property (i). Then C is finite-dimensional since D is algebraic over k by property (ii). Thus D itself is finite-dimensional . To go from the case of V simple to the case of V having finite length, one may follow exactly the diagram in the proof of [McR, Theorem 9.5.5(ii)].

Thus it suffices to show (i) and (ii) for $U(L)$. Now (i) is true, since $U(L) \otimes K \cong U(L \otimes K)$, which is still Noetherian. Also (ii) holds: this is known as Quillen's Lemma for ordinary Lie algebras, and the extension to Lie color algebras is shown in [BMPZ, p. 51]. □

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