PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 3, March 1998, Pages 943–947
S 0002-9939(98)04139-2

METRIZABILITY OF SEQUENTIAL TOPOLOGICAL GROUPS WITH POINT-COUNTABLE k-NETWORKS

ALEXANDER SHIBAKOV

(Communicated by Franklin D. Tall)

Abstract. We prove that a Hausdorff sequential topological group with a point-countable k-network is metrizable iff its sequential order is less than ω_1. In the non Hausdorff case metrizability may be replaced by σ-locally finite base.

1. Introduction

A family γ of subsets of a topological space X is said to be *a k-network* if for any compact $K \subseteq X$ and any open $U \supseteq K$ there is a finite $\gamma_K \subseteq \gamma$ such that $K \subseteq \cup\gamma_K \subseteq U$. This notion has found several applications in the generalized metric spaces theory (see [GMT] and references there). Usually the notion of k-network is studied in combination with another property, namely point-countability. Recall that a family γ is called *point-countable* if any point of X belongs to at most countably many elements of γ.

In the present paper we study sequential topological groups with point-countable k-networks. We obtain a metrization criterion for such groups in terms of sequential order. P. Nyikos in [N] asked if the sequential order of a sequential topological group is equal to ω_1 if the group is not Fréchet. For general topological groups the answer is consistently negative [S]. We show that in the case of groups with point-countable k-networks it is possible even to strengthen P. Nyikos' hypothesis. Namely we prove that a Hausdorff sequential topological group with a point-countable k-network is metrizable if its sequential order is not equal to ω_1. In [A] A. Arhangel′skii proved that a Fréchet topological group is metrizable provided it has a countable k-network.

Some useful constructions give topological groups which have point-countable k-networks. Thus it follows from [A] that that the free topological group over a metrizable compact has a countable k-network. In paper [ZP] on every infinite Abelian group a sequential non Fréchet topology is introduced. It may be shown that the resulting topological group has a point-countable k-network. So it follows from our criterion that every infinite Abelian group admits a sequential topology with the sequential order ω_1. We also consider the general non Hausdorff case where metrizability is replaced by σ-locally finite base. Non Hausdorff groups arise naturally as quotients via nonclosed subspaces.

Received by the editors May 10, 1995 and, in revised form, September 15, 1996.
1991 *Mathematics Subject Classification.* Primary 54H11; Secondary 54D55, 54A20, 54E35.
Key words and phrases. Topological group, sequential space, sequential order, Fréchet space, k-network.

Let us now give some definitions. If A is a subset of a space X, then $[A]^{seq}$ denotes the *sequential closure of* A, i.e. the set of limits of convergent sequences in A. Obviously $A \subseteq [A]^{seq}$. We define $[A]_\alpha$ by induction on $\alpha \in \omega_1 + 1$ as follows: $[A]_0 = A$, $[A]_{\alpha+1} = [[A]_\alpha]^{seq}$ and $[A]_\alpha = \cup\{\, [A]_\beta \mid \beta < \alpha \,\}$ for a limit α. One can easily see that $[A]_{\omega_1+1} = [A]_{\omega_1}$, and that a space X is sequential if and only if $\overline{A} = [A]_{\omega_1}$ for every $A \subset X$. For a sequential space X we define $so(X)$, the *sequential order* of X, by $so(X) = \min\{\, \alpha \in \omega_1 + 1 \mid \overline{A} = [A]_\alpha$ for every $A \subset X \,\}$. Note that a space is Fréchet if and only if it is a sequential space with sequential order 1. Consider the set $S = \omega^2 \cup \omega \cup \{\omega\}$. Define a topology on S as follows. Every point of ω^2 is isolated, a typical neighborhood of n is $\{n\} \cup (\{n\} \times \omega \setminus$ finitely many points), $U \ni \{\omega\}$ is open if $(U \cap \{n\} \times \omega) \cup \{n\}$ is a neighborhood of n for every $n \in U$ and $\omega \setminus U$ is finite. The resulting space is called *Arens' space* S_2. Identifying the limit points of the sum of countably many converging sequences and equipping the resulting space with the quotient topology we get another canonical space called *sequential fan* S_ω. A subset $\sigma \subseteq \omega^2$ is called *thin* if $|\sigma \cap \{n\} \times \omega| < \aleph_0$ for any $n \in \omega$. Define $\omega(n) = \{\, i \mid i > n \,\}$.

2. Sequential topological groups with point-countable k-networks

We shall prove some technical lemmas first.

Lemma 2.1. *A Hausdorff sequential non Fréchet space with a point-countable k-network contains a closed copy of S_2.*

Proof. Let X be a sequential non Fréchet space with a point-countable k-network γ. Then there exists an injection $t : \omega^2 \to X$ such that $t(n,k) \to x_n$ as $k \to \infty$, $x_i \neq x_j$ if $i \neq j$, $x_i \to x \in X$ as $i \to \infty$ and there is no sequence in $t(\omega^2)$ converging to x. Put $\{\, \xi_i \mid i \in \omega \,\} = \{\, \xi \in \gamma \mid \xi \cap t(\omega^2) \neq \varnothing, x \notin \overline{\xi} \,\}$. Then by induction choose $n_i \in \omega$ and $k_i \in \omega$ for every $i \in \omega$ so that $\{\, t(n_i, j) \mid j \geq k_i \,\} \subseteq X \setminus \bigcup_{j \leq i} \xi_j$. It is now easy to see that any set of the form $\{\, t(n_i, p_i) \mid i \in \omega \,\}$ where $p_i \geq k_i$ is a closed discrete subset of X. Indeed, were it not both it would have a convergent subsequence, so assume without loss of generality that $\langle t(n_i, p_i) : i \in \omega \rangle$ converges to p. Since $p \neq x$, there exists ξ_j containing infinitely many $t(n_i, p_i)$. But this contradicts the way the points $t(n_i, p_i)$ were chosen. It follows that $\{\, t(n_i, j) \mid i \in \omega, j \geq k_i \,\} \cup \{\, x_{n_i} \mid i \in \omega \,\} \cup \{x\}$ is a closed subset of X homeomorphic to S_2. □

Let us introduce a property stronger than the one described in Lemma 2.1:

$P(x,U)$ If $U \subseteq X$, $\{x_i\}_{i\in\omega} \subseteq U$, $x_i \to x$ as $i \to \infty$ and $x_i \neq x_j$ if $i \neq j$ then there is $\Gamma = \{\, x(n,k) \mid n,k \in \omega \,\} \subseteq U$ such that $x(n,k) \to x_n$ as $k \to \infty$, $t : \omega^2 \to \Gamma$ is a bijection where $t(n,k) = x(n,k)$ and $\Gamma \cup \{\, x_i \mid i \in \omega \,\} \cup \{x\}$ is a closed subset of G homeomorphic to S_2.

Lemma 2.2. *Let G be a Hausdorff non Fréchet sequential topological group with a point-countable k-network. Then for any $x \in G$ and any open $U \subseteq G$ the property $P(x,U)$ takes place.*

Proof. By non Fréchetness of G and Lemma 2.1 there is a closed subset of G homeomorphic to S_2. So by [NST2, Lemma 2.1] G contains a closed subset homeomorphic to S_ω. Let $y(n,k) \to e$ as $k \to \infty$ where e is the unit of G and $y(n,k) \neq y(l,m)$ if $(n,k) \neq (l,m)$ and the set $\{\, y(n,k) \mid n,k \in \omega \,\} \cup \{e\}$ be a closed subset of G naturally homeomorphic to S_ω. Let U be open in G, $\{x_i\}_{i\in\omega} \subseteq U$, $x_i \to x$ as $i \to \infty$ and

$x_i \neq x_j$ if $i \neq j$. Denote $x(i,k) = x_i \cdot y(i,k)$ for $i, k \in \omega$. Then $x(i,k) \to x_i$ as $k \to \infty$ so it is possible to choose $k_i \in \omega$ for every $i \in \omega$ so that $\{\, x(i,k) \mid i \in \omega, k \geq k_i \,\} \subseteq U$ and $x(i,k) \neq x(i',k')$ if $(i,k) \neq (i',k')$ and $k \geq k_i$, $k' \geq k_{i'}$. To prove $P(x,U)$ it is enough to show that $\{\, x_i \mid i \in \omega \,\} \cup \{\, x(i,k) \mid i \in \omega, k \geq k_i \,\} \cup \{x\}$ is a closed in G subset homeomorphic to S_2. Suppose the contrary. Then there is a sequence $\langle x(i_j,k_j) : j \in \omega \rangle$ converging to $p \in G$ such that $i_j \neq i_{j'}$ if $j \neq j'$. Now $y(i_j,k_j) = x_{i_j}^{-1} \cdot x(i_j,k_j)$ so $y(i_j,k_j) \to x^{-1} \cdot p$ as $j \to \infty$ contradicting the way the points $y(n,k)$ were chosen. □

Lemma 2.3. *Let X be a regular sequential space with a point-countable k-network such that for any $x \in X$ and $U \subseteq X$ the property $P(x,U)$ holds. Then for any $\alpha < \omega_1$, $x \in X$, $U \subseteq X$ open in X the following property holds:*

$Q(\alpha,x,U)$: If $\{x_i\}_{i\in\omega} \subseteq U$, $x_i \to x$ as $i \to \infty$ then there is $Q \subseteq U$ such that $\overline{Q}$ is countable, $\overline{Q} \setminus \{x\} \subseteq U$, $x \in [Q]_\alpha$, $x \notin [Q]_\beta$ for $\beta < \alpha$.

Proof. Suppose we have already proved $Q(\beta,x,U)$ for any $x \in X$, $U \subseteq X$ open in X and $\beta < \alpha$. First suppose that $\alpha = \beta + 1$ for some $\beta < \omega_1$. Let us note the following fact:

Fact. *Let $t : \omega^2 \to \Gamma = \{\, x(n,k) \mid x(n,k) = t((n,k)) \,\}$ be a bijection. Let $\delta = \{\, \xi_n \mid n \in \omega \,\}$ be a family of subsets of X. Then there exists a thin set $\sigma(\delta) \subseteq \omega^2$ such that for any $\xi_k \in \delta$ either $\overline{\xi_k} \cap t(\sigma(\delta)) \neq \varnothing$ or $\overline{\xi_k} \cap t(\omega(m) \times \omega) = \varnothing$ for some $m \in \omega$.*

Let $\{x_i\}_{i\in\omega} \subseteq U$. Using $P(x,U)$ choose $t : \omega^2 \to \Gamma \subseteq U \setminus \{x\}$ a bijection so that $t(n,k) \to x_n$ as $k \to \infty$, $x_n \neq x_k \neq x$ if $n \neq k$ and $\Gamma \cup \{\, x_n \mid n \in \omega \,\} \cup \{x\}$ is a closed subset of X homeomorphic to S_2. Let $S \subseteq X$ be countable. Let us denote

$$\gamma(S) = \{\, \xi \mid \xi \in \gamma, \xi \cap S \neq \varnothing, \overline{\xi} \cap t(\omega(m) \times \omega) = \varnothing \text{ for some } m \in \omega \,\}.$$

Now $\gamma(S)$ is countable so put $\gamma(S) = \{\, \xi_n(S) \mid i \in \omega \,\}$. Let us now construct the sets $Q_n \subseteq U$ such that:

(1) $\overline{Q_i}$ is countable, $\overline{Q_i} \cap (\{\, x_j \mid j \neq n_i \,\} \cup \{x\}) = \varnothing$, $x_{n_i} \in [Q_i]_\beta$, $x_{n_i} \notin [Q_i]_\gamma$ if $\gamma < \beta$, $n_{i+1} > n_i$,

(2) $\overline{Q_i} \setminus \{x_{n_i}\} \subseteq (X \cap U) \setminus \bigcup_{n,k<i} \overline{\xi_n(\overline{Q_k})}$.

Let us first show how to arrange for this. Suppose we have already chosen $n_i \in \omega$, $Q_i \subseteq X$ so that they satisfy (1)–(2) for all $i < k$. Consider the family

$$\Delta = \{\, \overline{\xi_i(\overline{Q_j})} \mid i,j < k \,\}.$$

Δ is finite. By the construction of $\xi_i(\overline{Q_j})$ for every $i < k$, $j < k$ there is $m(i,j)$ such that

$$\overline{\xi_i(\overline{Q_j})} \cap t(\omega(m(i,j)) \times \omega) = \varnothing.$$

Put $m = \max(\{\, m(i,j) \mid i < k, j < k \,\} \cup \{\, n_j \mid j < k \,\})$. Then it is easy to see that $(\cup\Delta) \cap t(\omega(m) \times \omega) = \varnothing$. So we have that $x(m+1,i) \to x_{m+1}$ as $i \to \infty$, $\{\, x(m+1,i) \mid i \in \omega \,\} \subseteq (X \cap U) \setminus \cup\Delta$. Thus by $Q(\beta, x_{m+1}, ((X \cap U) \setminus \cup\Delta))$ there is Q_k such that if $x_{n_k} = x_{m+1}$ then (1) takes place and $\overline{Q_k} \setminus \{x_{n_k}\} \subseteq (X \cap U) \setminus \cup\Delta$ that is (2) takes place.

Now consider the set $Q' = \bigcup_{i\in\omega} \overline{Q_i} \cup \{x\}$. Using the fact find the set $D = t(\sigma(\gamma(Q')))$.

Claim. *If $z_i \in \overline{Q_i} \setminus \{x_{n_i}\}$ and $z_{k_i} \to z$ as $i \to \infty$ then $z \in D$; in particular $Q' \cup D$ is closed in X.*

Proof of claim. Suppose the contrary; then $z \in V = X \setminus D$ and V is open in X because D is a closed discrete subset of X by the construction of t and Γ. Using the regularity of X choose $\xi \in \gamma$ such that $\overline{\xi} \subseteq V$ and $\xi \cap \{ z_{n_i} \mid i \in \omega \}$ is infinite. Then $\overline{\xi} \cap t(\sigma(\gamma(Q'))) = \varnothing$ and thus $\overline{\xi} \cap t(\omega(m) \times \omega) = \varnothing$ for some $m \in \omega$ by the choice of $\sigma(\gamma(Q'))$. So $\xi = \xi_n(\overline{Q_k})$ for some $n \in \omega$, $k \in \omega$. But then $z_{k_i} \notin \xi$ if $k_i > \max\{n+1, k+1\}$ by (2). A contradiction. □

Consider the set $Q = \bigcup_{i\in\omega} Q_i$. It follows from the claim that $\overline{Q} \setminus \{x\} \subseteq U$. Also $x \in [Q]_{\beta+1} = [Q]_\alpha$. Let $z_i \to x$ as $i \to \infty$ and $z_i \in [Q]_{\gamma_i}$ for some $\gamma_i \leq \beta$. Using the claim and the fact that D is a closed discrete subset of X we may assume that $z_i \in [Q_{k_i}]_{\gamma_i}$ and $k_{i+1} > k_i$. Since $x \notin D \subseteq \Gamma$ it follows from the claim that for some $k \in \omega$ $\{ z_i \mid i \geq k \} \subseteq \{ x_{n_i} \mid i \in \omega \}$. It implies that $\gamma_i \geq \beta$ by (1). So $Q(\alpha, x, U)$ holds. The case of a limit α may be considered similarly. □

Theorem 2.4. *Let G be a Hausdorff sequential topological group with a point-countable k-network. If $so(G) < \omega_1$ then G is metrizable.*

Proof. Suppose $so(G) = \alpha$. Let us first show that G is Fréchet. Suppose not. Then by Lemmas 2.2–2.3 property $Q(\alpha+1, e, G)$ takes place. Obviously $Q(\alpha+1, e, G)$ implies $so(G) \geq \alpha + 1 > \alpha$. A contradiction. Now G is a Fréchet topological group with a point-countable k-network. By [N] it is an α_4-space and thus is a countably bi-k-space in the sense of [M2]. Now by [GMT, Corollary 3.6] G is first countable. Thus by the classical Birkhoff-Kakutani theorem G is metrizable. □

Now let us consider the general case.

Theorem 2.5. *Let G be a (possibly non Hausdorff) sequential topological group with a point-countable k-network. If $so(G) < \omega_1$ then G has a σ-locally finite base.*

Proof. Let $E = \overline{e} \subseteq G$ be the closure of the unit in G. Then E is a closed normal subgroup of G. Let $f : G \to G/E$ be the natural quotient map. Then G/E is a sequential Hausdorff topological group. The map f has the property: for any $x \in G/E$ the subspace $f^{-1}(x)$ has antidiscrete topology. Let $g_x \in G$ be chosen so that $f(g_x) = x$. Then the property of f mentioned above gives that for any $\{x_i\}_{i\in\omega} \subseteq G/E$ such that $x_n \to x \in G/E$ as $n \to \infty$ holds $g_{x_n} \to g_x$ as $n \to \infty$. This yields that the sequential order of G/E is less than or equal to the sequential order of G. Put $G' = \{ g_x \mid x \in G/E \}$. If γ is a point-countable k-network for G then the family $\{ f(\xi \cap G') \mid \xi \in \gamma \}$ is point-countable. Using the property of f again one has that for any convergent sequence $S \subseteq G/E$ and its neighborhood U in G/E there is $\xi \in \gamma$ such that $f(\xi \cap G') \subseteq U$ and the set $f(\xi \cap G') \cap S$ is infinite. This gives that the family is a k-network.

If G has sequential order less than ω_1 then by Theorem 2.4 G/E is metrizable and thus has a σ-locally-finite base $\mathcal{B}$ by the Bing-Nagata-Smirnoff theorem. Now since the fibres of f are antidiscrete, then the preimage of $\mathcal{B}$ under f is a σ-locally-finite base for G. □

The author wishes to thank Professor T. Nogura for advice and moral support and the referee for the suggestion to include the non Hausdorff case and many other useful remarks.

References

[A] A. Arkhangel′skii, *Topological properties in topological groups*, XVIII All Union Algebraic Conference, Kishinev, 1985 (Russian).

[GMT] G. Gruenhage, E. Michael, Y. Tanaka, *Spaces determined by point-countable covers*, Pacif. J. Math. **113** (1984), 303–332. MR **85m:**54018

[M1] E. Michael, $\aleph_0$-*spaces*, J. of Math. and Mech. **15** (1966), 983–1002. MR **34:**6723

[M2] E. Michael, *A quintuple quotient quest*, Gen. Topology Appl. **2** (1972), 91–138. MR **46:**8156

[N] P. J. Nyikos, *Metrizability and Fréchet-Urysohn property in topological groups*, Proc. Amer. Math. Soc. **83** (1981), 793–801. MR **82k:**54049

[NST1] T. Nogura, D. Shakhmatov, Y. Tanaka, *Metrizability of topological groups having weak topologies with respect to good covers*, Topology Appl. **54** (1993), 203–212. MR **95b:**54046

[NST2] T. Nogura, D. Shakhmatov, Y. Tanaka, α_4*-property versus A-property in topological spaces and groups* (to appear).

[S] A. Shibakov, *Sequential group topology on rationals with intermediate sequential order*, Proc. Amer. Math. Soc. **124** (1996), 2599–2607. MR **96j:**54032

[ZP] Zelenyuk, E., and Protasov, I., *Topologies on Abelian groups*, Math. USSR Izvestia **37** (1991), 445-460. MR **93b:**22001

Department of Mathematics, Auburn University, Auburn University, Alabama 36849

E-mail address: `shobaay@mallard.duc.auburn.edu`