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A VISIT TO THE ERDŐS PROBLEM

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Abstract. Erdős asked if for every infinite set, A, of real numbers there exists a measurable subset of the reals having positive measure that does not contain a subset similar to A. In this note we transform this question to a finite combinatorial problem. Using this translation we extend some results of Eigen and Falconer concerning slow sequences for which the answer to Erdős' question is positive.

We first establish some notation. Let $\mathbb{N}_n = \{1, 2, \ldots, n\}$. The integer part of y is denoted by $[y]$. For $A, B \subset \mathbb{R}$ the notation $A \sim B$ means that there are $u, v \in \mathbb{R}$ with $v \neq 0$ such that $A = u + v \cdot B = \{u + vb : b \in B\}$ and in this case we say that A is similar to B. If $A \sim B$ with $v > 0$, then A is said to be homothetic to B and this is denoted by $A \approx B$.

Let $A \subset [0, 1]$ be given. Define Λ_n as the cardinality of the smallest set $B \subset \mathbb{N}_n$ that intersects each subset of $\mathbb{N}_n$ of the form

$$A_{x,y} = \{x + [ay] : a \in A\}, \text{ where } x, y, x + y \in \mathbb{N}_n \text{ and } y \geq [n/2].$$

Theorem 1. *Let $A \subset [0, 1]$ be such that $\inf(A) = 0$ and $\sup(A) = 1$. Then the following are equivalent.*

1. *There is a measurable set of positive measure which contains no subset similar to A.*
2. $\lim_{n\to\infty} \frac{\Lambda_n}{n} = 0$.
3. $\liminf_{n\to\infty} \frac{\Lambda_n}{n} = 0$.

Proof. We first prove that 1 implies 2. Let $\varepsilon > 0$ be fixed and suppose there is a set of positive measure which contains no subset similar to A. Then there is a closed set, $F = F(\varepsilon) \subset [0, 1]$ such that $\lambda(F) > 1 - \varepsilon$ and F contains no subset which is similar to A. Set

$$B_n = \left\{ i \in \mathbb{N}_n : \left[\frac{i-1}{n}, \frac{i}{n}\right] \cap F = \emptyset \right\}.$$

Then $|B_n| < \varepsilon \cdot n$ for all n. We show that for n large enough, $B_n \cap A_{x,y} \neq \emptyset$ whenever $x, y, x + y \in \mathbb{N}_n$ and $y \in [n/2, n]$. This will prove that $\Lambda_n \leq |B_n| < \varepsilon n$ for n large enough; that is, 2 holds.

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If this is not the case, then there is a sequence $\{(n_k, x_k, y_k)\}$ such that for every $k \in \mathbb{N}$, $B_{n_k} \cap A_{x_k,y_k} = \emptyset$, and $x_k, y_k, x_k + y_k \in \mathbb{N}_{n_k}$ and $[\frac{n_k}{2}] \leq y_k$. Selecting a subsequence, we may assume $(x_k/n_k) \to x_0$ and $(y_k/n_k) \to y_0 \geq \frac{1}{2}$.

Now, let $a \in A$. As $x_k + [ay_k] \notin B_{n_k}$, $F \cap I_k \neq \emptyset$, where

$$I_k = \left[\frac{x_k + [ay_k] - 1}{n_k}, \ \frac{x_k + [ay_k]}{n_k}\right].$$

But, $(x_k/n_k) + (ay_k/n_k) - (1/n_k) \in I_k$ for each $k \in \mathbb{N}$ and hence, $x_0 + ay_0 \in F$ since F is closed. As this is true for every $a \in A$, it follows that $x_0 + A \cdot y_0 \subset F$ which contradicts the choice of F. This completes the first part of the proof of the theorem. As the proof that 2 implies 3 is obvious, we turn to the proof that 3 implies 1.

Fix A and suppose that $\liminf_{n\to\infty} \frac{\Lambda_n}{n} = 0$. It is enough to show that there is a measurable set, $F \subset [0,1]$, of measure greater than $1/2$ containing no subset which is homothetic to A, since in this case $F \cap (1-F)$ is a measurable set of positive measure which contains no subset similar to A.

Let $\varepsilon > 0$ be given. We first construct an open set G_ε of measure less than ε such that if $A \approx A' \subset [0,1]$ and $\operatorname{diam} A' \geq \frac{1}{2}$, then $A' \cap G_\varepsilon \neq \emptyset$.

As $\liminf_{n\to\infty} \frac{\Lambda_n}{n} = 0$, for each $k = 1, 2, \ldots$ there are an n_k and $B_{n_k} \subset \mathbb{N}_{n_k}$ with the required property such that $|B_{n_k}| < \varepsilon n_k / 8 \cdot 2^k$. Let

$$G_\varepsilon = \bigcup_{k=1}^{\infty} \bigcup_{m \in B_{n_k}} \left(\frac{m-4}{n_k}, \frac{m+4}{n_k}\right).$$

Then $\lambda(G_\varepsilon) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$. Suppose $A' \subset [0,1]$, $A' \approx A$, $\operatorname{diam} A' \geq 1/2$. Then, $A' = u + v \cdot A$ where $u, v \in [0,1]$ with $v \geq 1/2$ and as $\sup A = 1$, $u + v \leq 1$. We show that $A' \cap G_\varepsilon \neq \emptyset$. Put $x_k = \max(1, [n_k u])$ and $y_k = \max([n_k/2], [n_k v] - 1)$. Then $x_k, y_k, x_k + y_k \in \mathbb{N}_{n_k}$, and $y_k \geq [n_k/2]$. Thus, by the choice of B_{n_k}, we have

$$A_{x_k,y_k} \cap B_{n_k} \neq \emptyset.$$

If $m \in A_{x_k,y_k} \cap B_{n_k}$, then there is an $a \in A$ such that $m = x_k + [ay_k]$, and it follows that

$$\begin{aligned} n_k u + a n_k v - 4 &\leq (n_k u - 1) + a(n_k v - 2) - 1 < m \\ &\leq (n_k u + 1) + a n_k v < n_k u + a n_k v + 4. \end{aligned}$$

Hence, $(m-4)/n_k < u + av < (m+4)/n_k$. However, if $m \in B_{n_k}$ then $((m-4)/n_k, (m+4)/n_k) \subset G_\varepsilon$ and thus $a + uv \in A' \cap G_\varepsilon$. This completes the construction of G_ε.

Fix a number $c \in (1/2, 1)$, and let $\delta_n = 2c^n - c^{n-1}$. For each $n = 1, 2, \ldots$ define

$$G_\varepsilon^n = [0,1] \cap \bigcup_{k=0}^{\infty} \left((2c^n \cdot G_\varepsilon) + k\delta_n\right).$$

Then G_ε^n intersects every set $A' \subset [0,1]$ with $A' \approx A$ and $c^n \leq \operatorname{diam} A' \leq c^{n-1}$. Indeed, if $u = \inf A'$, $v = \sup A'$ and k is the largest integer such that $k\delta_n \leq u$, then $[u,v] \subset 2c^n \cdot [0,1] + k\delta_n$ since

$$v = (v-u) + u \leq c^{n-1} + (k+1) \cdot \delta_n = 2c^n + k\delta_n.$$

Let $A'' = (A' - k\delta_n)/2c^n$. Then $A'' \subset [0,1]$ and $\operatorname{diam} A'' \geq 1/2$; therefore $A'' \cap G_\varepsilon \neq \emptyset$. Then it follows that A' intersects $2c^n \cdot G_\varepsilon + k\delta_n$ and hence, A' intersects G_ε^n.

Choose $\varepsilon_n > 0$ such that $\lambda(G^n_{\varepsilon_n}) < 2^{-n-1}$ and set

$$G = \bigcup_{n=1}^{\infty} G^n_{\varepsilon_n}.$$

Then, $\lambda(G) < 1/2$ and if $A \approx A' \subset [0,1]$ then $A' \cap G \neq \emptyset$. The proof is completed by letting F be $[0,1] \setminus G$. □

For $A \subset \mathbb{R}$ and $u < v$ we shall denote by $A(u,v)$ the length of the longest component of $(u,v) \setminus A$.

Theorem 2. *Let A be a bounded set such that*

$$\inf_{u<v} \frac{A(u,v)}{v-u} = 0.$$

Then there is a measurable set of positive measure which contains no subset similar to A.

Proof. We may assume $\inf A = 0$ and $\sup A = 1$. Let $0 < \varepsilon < 1$ be given. There exist $u, v \in A$ such that $u < v$ and $|I| < \varepsilon(v-u)$ whenever $I \subset (u,v) \cap A^c$. Let $n = [6/\varepsilon(v-u)]$ and $B = \{kb : k = 1, 2, \dots\} \cap \mathbb{N}_n$ where $b = [1/\varepsilon]$. We claim that each set $A_{x,y}$ contains a sequence of at least b consecutive integers and hence, intersects B. Indeed, $A_{x,y} \supset \{x + [ay] : a \in [u,v] \cap A\} = A^*$, and using the fact that $y \geq n/2$,

$$\operatorname{diam} A^* = [vy] - [uy] \geq vy - uy - 1 \geq (v-u)\frac{n}{3} > b.$$

Since each component of $(u,v) \setminus A$ is shorter than $\frac{\varepsilon}{4}(v-u) < 1/n \leq 1/y$ it is clear that A^* and hence $A_{x,y}$ contains every integer between $x + [uy]$ and $x + [vy]$. Hence, $\liminf_{n\to\infty} \frac{\Lambda_n}{n} < \varepsilon$ and as $\varepsilon > 0$ is arbitrary, $\liminf_{n\to\infty} \frac{\Lambda_n}{n} = 0$ and Theorem 1 can be applied. □

The following result was proved independently by Eigen [2] and Falconer [7], and is an immediate corollary of Theorem 2.

Corollary 1 (Eigen [2] and Falconer [7])**.** *Suppose $\{a_n\}$ is a decreasing sequence of positive numbers converging to 0 with*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

Then there is a measurable set of positive measure which contains no subset similar to $\{a_n\}$.

Our next theorem extends this result.

Corollary 2. *Suppose $\{a_n\}$ is a decreasing sequence of positive numbers converging to 0 with $\{a_n - a_{n+1}\}$ monotonic, and $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Then there is a measurable set of positive measure which contains no subset similar to $\{a_n\}$.*

Proof. If $\frac{a_{n+1}}{a_n} > 1 - \varepsilon$ then the length of any component of $[0, a_n] \setminus \{a_i\}$ is less than εa_n and so Theorem 2 can be applied. □

It is easy to see that there are sequences $\{a_n\}$ which satisfy the hypothesis of Corollary 2 but for which $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$.

We have included an expanded list of references.

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