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ENVELOPING SEMIGROUPS AND MAPPINGS ONTO THE TWO-SHIFT

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Abstract. Enveloping semigroups of topological actions of semigroups G on compact spaces are studied. For zero dimensional spaces, and under modest conditions on G, the enveloping semigroup is shown to be the Stone-Čech compactification if and only if some Cartesian product has the two-shift as a factor. Examples are discussed showing that, unlike in the measure theory case, positive entropy does not imply the existence of such a factor even if the Cartesian product has large entropy.

Let G be a discrete multiplicative semigroup with at least one element g_0 such that $g \to g_0 g$ is injective. Let X be a compact Hausdorff space. We say $\sigma : G \times X \to X$, written $(g, x) \to \sigma_g(x)$, is an action of G on X, and we say (X, G, σ) is a G-flow, if $\sigma_{gh} = \sigma_g \circ \sigma_h$ for all $g, h \in G$. We sometimes refer to the flow X when G and σ are understood. We will assume that if $g_1 \neq g_2$ then there is at least one $x \in X$ such that $\sigma_{g_1}(x) \neq \sigma_{g_2}(x)$. Given a G-flow X let $E = E_X$ be the closure of $\{\, \sigma_g : g \in G \,\}$ in X^X with the product topology. Clearly E is compact and it can easily be shown that E is a subsemigroup of X^X where $(\xi\eta)(x) = \xi(\eta(x))$ for $\xi, \eta \in X^X$ and $x \in X$. E is called the enveloping semigroup of X (the enveloping semigroup was first introduced by Ellis in [3]. Other sources in the literature dealing with enveloping semigroups include [1], [4] and [5]). Let $G_E = \{\, \sigma_g : g \in G \,\}$. Observe $g \to \sigma_g$ is an injection of G onto the dense subset G_E of E.

Let C be the set of all continuous real valued functions with domain X and let B be the set of all bounded real valued functions with domain G. For $x \in X$ and $f \in C$ let $f_x \in B$ be defined by $f_x(g) = f(\sigma_g(x))$. Let $A = A(X) \subseteq B$ be the smallest uniformly closed algebra containing $\{\, f_x : x \in X,\ f \in C \,\}$. We slightly abuse notation to use the same symbol f_x for the corresponding function defined on G_E. If $x \in X$, and $f \in C$ define $\overline{f_x} : E \to \mathbf{R}$ by $\overline{f_x}(\eta) = f(\eta(x))$. Observe that each $\overline{f_x}$ is in $C_{\mathbf{R}}(E)$, the space of continuous real valued functions on E. Clearly $\overline{f_x}$ is the continuous extension of f_x from G_E to E. It follows that every $\phi \in A$ has a (unique) continuous extension $\overline{\phi} : E \to \mathbf{R}$.

Proposition 1. $\{\, \overline{\phi} : \phi \in A \,\} = C_{\mathbf{R}}(E)$. *In particular, $A = B$ if and only if the embedding $g \to \sigma_g$ is a realization of the Stone-Čech compactification β of G.*

Proof. It is clear that $\{\, \overline{\phi} : \phi \in A \,\} \subseteq C_{\mathbf{R}}(E)$ is a uniformly closed subalgebra of $C_{\mathbf{R}}(E)$. If ξ and η are distinct elements of E then there is some $x \in X$ with

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$\xi(x) \neq \eta(x)$. Choose $f \in C$ so that $f(\xi(x)) \neq f(\eta(x))$, and thus $\overline{f_x}(\xi) \neq \overline{f_x}(\eta)$. This shows that $\{\, \overline{\phi} : \phi \in A \,\}$ separates points of E and so, by the Stone-Weierstrass theorem, $\{\, \overline{\phi} : \phi \in A \,\} = C_{\mathbf{R}}(E)$.

As is well-known, a realization of the Stone-Čech compactification of the discrete space G consists of a compact space E and an injective mapping $g \to \sigma_g$ of G onto a dense subspace G_E of E such that:

(1) The set G_E is discrete as a subspace of E.
(2) Every bounded function on G_E extends to a continuous function on E.

In fact (1) follows from (2) and (2) is simply the assertion that $A = B$. We have assumed $g \to \sigma_g$ is injective, and G_E is dense by the definition of E. This shows that E is the Stone-Čech compactification when $A = B$. The converse assertion is an immediate consequence of (2). □

Remark. A similar result holds if G has a completely regular topology and the action σ of G on X is jointly continuous.

Proposition 2. *Let I be any non-void index set. Let X^I be the product space with the product topology and let σ^I be the product action on X^I, defined by $\sigma^I_g(\overline{x})_i = \sigma_g(\overline{x}_i)$ for $\overline{x} \in X^I$ and $i \in I$. Then $A(X^I) = A(X)$.*

Proof. Clearly $A(X) \subseteq A(X^I)$, so we establish the reverse inclusion. Let $\pi_i(\overline{x}) = \overline{x}_i$ for $\overline{x} \in X^I$ and $i \in I$. Clearly $\{\, f \circ \pi_i : f \in C,\ i \in I \,\}$ separates points of X^I. Given $f \in C_{\mathbf{R}}(X^I)$ and $\epsilon > 0$, it follows from the Stone-Weierstrass theorem that there exist a positive integer k, functions $f^1, \ldots, f^k \in C$, indices $i_1, \ldots, i_k \in I$, and a real polynomial p of k variables such that $|f(\overline{x}) - p(f^1(\pi_{i_1}(\overline{x})), \ldots, f^k(\pi_{i_k}(\overline{x})))| < \epsilon$ for all $\overline{x} \in X^I$. Examination of the σ^I action shows that

$$|f_{\overline{x}}(g) - p({f^1}_{x_1}(g), \ldots, {f^k}_{x_k}(g))| < \epsilon.$$

Since $A(X)$ consists of the uniform closure of just such polynomial expressions, this inequality shows $f \in A(X)$. □

We will apply the above results to actions on a totally disconnected space. We begin with a general definition of a shift space on a finite set.

Definition. Let α be a finite set with the discrete topology. Let $X = \alpha^G$ be the product space with the product topology. Let σ be defined by $(\sigma_g(x))_h = x_{hg}$. We call (X, G, σ) the G-shift on α.

Remark. The mapping σ defined above is an action, and $g \neq h \implies \sigma_g \neq \sigma_h$.

Proof. First observe $(\sigma_{gh}(x))_t = x_{tgh} = (\sigma_h(x))_{tg} = (\sigma_g(\sigma_h(x)))_t\ \forall g, h, t \in G$ and $x \in X$. This shows $\sigma_{gh} = \sigma_g \circ \sigma_h$. Because X has the product topology, evaluation $x \to x_{tg}$ is continuous for fixed t and g. But continuity of $\sigma_g : X \to X$ is equivalent to continuity of $x \to (\sigma_g(x))_t = x_{tg}$ at every $x \in X$ and for every $t \in G$. This shows each σ_g is continuous. Finally, we have assumed the existence of an element $g_0 \in G$ such that $g \to g_0 g$ is injective. Given distinct g and h in G let $x \in X$ be such that $x_{g_0 g} \neq x_{g_0 h}$. Then $\sigma_g(x) \neq \sigma_h(x)$ since $(\sigma_g(x))_{g_0} \neq (\sigma_h(x))_{g_0}$. □

Proposition 3. *Let (X, G, σ) be a G-shift on some finite set α of cardinality greater than 1. Then $A_X = B$, and so E_X is the Stone-Čech compactification of G.*

Proof. Clearly it is sufficient to show that for any subset $U \subseteq G$, the characteristic function $\chi = \chi_U$ of U is in A. We can map α bijectively to $\{0, 1, \ldots, k-1\}$ for some $k > 1$ and for notational simplicity we assume that α in fact equals $\{0, 1, \ldots, k-1\} \subset \mathbf{R}$. Choose $g_0 \in G$ so $g \to g_0 g$ is injective, and let $V = g_0 U$. Define $\hat{x} \in X$ as the characteristic function of V. Define $f : X \to \mathbf{R}$ by $f(x) = x_{g_0}$. Observe $f_{\hat{x}}(g) = f(\sigma_g(\hat{x})) = (\sigma_g(\hat{x}))_{g_0} = \hat{x}_{g_0 g} = \chi(g)$. □

Definition. If (X, G, σ) and (Y, G, τ) are flows then a continuous mapping $\Pi : X \to Y$ is called a flow mapping if $\tau_g(\Pi(x)) = \Pi(\sigma_g(x))$ for all $x \in X$ and $g \in G$. If also Π is surjective then we call Π a factor mapping of X onto Y and we call Y a factor of X.

Theorem 1. *Let (X, G, σ) be an action of the discrete semigroup G on the compact totally disconnected Hausdorff space X, satisfying $g \neq h \implies \sigma_g \neq \sigma_h$. Assume that there is an element $g_0 \in G$ such that $g \to g_0 g$ is injective. Let $A(X)$, B and E_X be as above. Let (Y, G, τ) be the G-shift on the two symbol set $\alpha = \{0, 1\}$. Assume that there is at least one $\hat{y} \in Y$ such that $\{\, \tau_g(\hat{y}) : g \in G \,\}$ is dense in Y (i.e., (Y, G, τ) is transitive). Then the following four statements are equivalent:*

(1) *For some finite non-void index set I, the shift action (Y, G, τ) is a factor of the product action (X^I, G, σ^I).*
(2) *For some non-void index set I, the shift action (Y, G, τ) is a factor of the product action (X^I, G, σ^I).*
(3) *$A = B$.*
(4) *E_X, together with the mapping $g \to \sigma_g$, is a realization of the Stone-Čech compactification of G.*

Proof. Clearly (1) $\implies$ (2) and we have seen from Proposition 1 that (3) $\iff$ (4). Also (2) $\implies$ $B = A(Y)$ by Proposition 3, but $A(Y) \subseteq A(X^I) = A(X)$ by Proposition 2, and $A(X) \subseteq B$. So (2) $\implies$ (3).

We complete the proof by showing (3) $\implies$ (1).

Assume (3) holds. Select $\hat{y}$ such that $\{\, \sigma_g(\hat{y}) : g \in G \,\}$ is dense in Y. Since $\hat{y}$ has domain G and values in $\{0, 1\}$ it is an element of B. From (3) and by the Stone-Weierstrass theorem, there exist a positive integer k, functions $f^1, \ldots, f^k$ in C, points $\hat{x}_1, \ldots, \hat{x}_k$ in X, and a polynomial p in k variables so that

$$|p(f^1_{\hat{x}_1}(g), \ldots, f^k_{\hat{x}_k}(g)) - \hat{y}_g| < \tfrac{1}{4}$$

for all $g \in G$. Since X is totally disconnected, any two points of X can be distinguished by some continuous function that assumes only two values. Again using the Stone-Weierstrass theorem, we may rechoose k, p, and the f^i, and renumber the x_i so that, in addition to the above inequality, the range T of $p(f^1_{x_1}(g), \ldots, f^k_{x_k}(g))$ is a finite set. Let $\gamma \in (\frac{1}{4}, \frac{3}{4}) \setminus T$ and let χ be the characteristic function of $[\gamma, \infty)$. Define $\Pi : X^k \to Y$ by $\Pi(x_1, \ldots, x_k)(g) = \chi(p(f^1_{x_1}(g), \ldots, f^k_{x_k}(g)))$. The map Π is continuous on X since χ is continuous on T. Calculate

$$\begin{aligned}\tau_h(\Pi(x_1, \ldots, x_k))(g) &= \Pi(x_1, \ldots, x_k)(gh) = \chi(p(f^1_{x_1}(gh), \ldots, f^k_{x_k}(gh)))\\ &= \chi(p(f^1(\sigma_{gh}(x_1), \ldots, f^k(\sigma_{gh}(x_k)))))\\ &= \chi(p(f^1(\sigma_g(\sigma_h(x_1))), \ldots, f^k(\sigma_g(\sigma_h(x_k))))) = \Pi(\sigma^I_h(x_1, \ldots, x_k))(g),\end{aligned}$$

so Π is a flow mapping. Finally, $\Pi : X^I \to Y$ is surjective since $\Pi(\hat{x}_1, \ldots, \hat{x}_k) = \hat{y}$. □

The hypothesis in Theorem 1 requiring Y to be transitive is impractical as it may not be easily verifiable. In contrast, the transitivity of Y follows from the following modest condition on G, when the cardinality of G is infinite, which is easy to check:

Definition. G is said to satisfy the Separation Condition if the cardinality $|G|$ of G is infinite and if for all $h, k \in G$, $|\{g \in G : hg = k\}| < |G|$.

Theorem 2. *If G satisfies the Separation Condition, then the shift space (Y, G, τ) is transitive.*

We first need the lemma:

Lemma. *Let G be an infinite semigroup and let G_0 and G_1 be a partition of G. Let y^c be the characteristic function of G_1. Then y^c has dense τ-orbit if and only if whenever F_0 and F_1 are disjoint finite subsets of G, there is an element $g \in G$ such that $F_0 g \subseteq G_0$ and $F_1 g \subseteq G_1$.*

Proof (of Lemma). "if": Pick $y \in Y$ and let F be a finite subset of G. Let $F_i = \{g \in G : y_g = i\}, i = 0, 1$. Choose $g \in G$ such that $F_i g \subseteq G_i, i = 0, 1$. Pick $h_i \in F_i$. Then $(\sigma_g y^c)_{h_i} = y^c_{h_i g} = i = y_{h_i}$. This shows that given y and F, there is a g such that $\sigma_g y^c$ agrees with y on F. That is, the orbit of y^c is dense in Y.

The converse is proven similarly. □

Proof (of Theorem 2). We construct sets G_i^ζ, $i = 0, 1$, inductively, using transfinite induction to account for uncountable G. It will be clear from the construction that $G_i^\eta \subseteq G_i^\nu$ if $\eta < \nu$.

For any ordinal η, we let $\overline{\eta}$ be the set of strict predecessors of η. Now let ξ be the least ordinal such that $|\overline{\xi}| = |G|$. Let $\mathcal{F}$ consist of all ordered pairs of disjoint finite subsets of G. Let $\zeta \to (F_0^\zeta, F_1^\zeta)$ be a bijective mapping of $\overline{\xi}$ onto $\mathcal{F}$.

Set $G_0^0 = G_1^0 = \emptyset$.

Suppose that for some ordinal ζ with $|\overline{\zeta}| < |G|$ the sets G_0^ζ and G_1^ζ have been defined such that $G_0^\zeta \cap G_1^\zeta = \emptyset$ and $|G_i^\zeta| < |G|, i = 0, 1$. Choose $g_\zeta \in G$ such that the four sets $G_0^\zeta, G_1^\zeta, F_0^\zeta g_\zeta, F_1^\zeta g_\zeta$ are pairwise disjoint. Such a g_ζ exists since $|G_0^\zeta \cup G_1^\zeta \cup F_0^\zeta \cup F_1^\zeta| < |G|$ and since G satisfies the Separation Condition. Set $G_i^{\zeta+1} = G_i^\zeta \cup F_i^\zeta g_\zeta$. Then $G_0^{\zeta+1}$ and $G_1^{\zeta+1}$ have been defined, are disjoint, and have cardinality strictly less than that of G.

If ζ is a limit ordinal with $|\overline{\zeta}| < |G|$ and if $G_i^\eta, i = 0, 1$, for all $\eta < \zeta$, then set $H_i^\zeta = \cup_{\eta<\zeta} G_i^\eta$ and choose g_ζ so that $H_0^\zeta, H_1^\zeta, F_0^\zeta g_\zeta, F_1^\zeta g_\zeta$ are pairwise disjoint. Set $G_i^\zeta = H_i^\zeta \cup F_i^\zeta$.

Finally, set $G_0 = \cup_{\eta<\zeta} G_0^\eta$ and $G_1 = G \backslash G_0$. Let y^c be the characteristic function of G_1. It is now easy to check that if $(F_0^\zeta, F_1^\zeta) \in \mathcal{F}$, then g_ζ satisfies the condition of the lemma. □

We now consider the case $G = (\mathbf{Z}, +)$, the integers under addition. We let $\sigma = \sigma_1$ and then $\sigma^n = \sigma_n$.

It is well-known that a mixing subshift of finite type has a Cartesian product with the full 2-shift as a factor (see [2]) and so the enveloping semigroup E is the Stone-Čech compactification β of $\mathbf{Z}$. The first theorem in [6] implies that no flow "constructed from minimal flows" can have $E = \beta$. Let us be more precise: A bounded function $F : \mathbf{Z} \to \mathbf{R}$ is called minimal if for every finite $F \subset \mathbf{Z}$ and every

$\epsilon > 0$ the set $\{ n \in \mathbf{Z} : |F(n+i) - F(i)| < \epsilon \ \forall i \in F \}$ is syndetic in $\mathbf{Z}$. Let $\mathcal{U}$ be the smallest uniformly closed algebra containing every minimal function. It is shown in [6] (Theorem 1) that $\mathcal{U}$ is properly contained in B, the algebra of bounded real valued functions defined on $\mathbf{Z}$. A $\mathbf{Z}$-flow X is minimal if there are no closed invariant non-void proper subsets. This condition is equivalent to: For every $x \in X$ and open set U containing x, the set $\{ n \in \mathbf{Z} : \sigma^n(x) \in U \}$ is syndetic. From this it is easy to see that for a minimal flow X every f_x is a minimal function. It follows that every minimal flow X has $A \neq B$. Moreover, if we take Cartesian products, subflows, or factors of flows satisfying $A \subseteq \mathcal{U}$ then the resulting flow will also have $A \subseteq \mathcal{U}$ and so $E \neq \beta$. We know, both from specific construction (see for instance [7]) and from the Jewitt-Krieger theorem, that there are many minimal subshifts of positive entropy.

Before giving the next example it is convenient to introduce some terminology. Let (X, d) be a compact metric space and let σ be a homeomorphism of X. Say that a point $x \in X$ has an *arithmetically clustered orbit* if for every $\epsilon > 0$ there is an arithmetic subset $S = a + L\mathbf{Z} \subseteq \mathbf{Z}$ $(a, L \in \mathbf{Z})$ such that if $n, m \in S$ then $d(\sigma^n x, \sigma^m x) < \epsilon$. We say (X, σ) *clusters arithmetically*, or is of class $\mathcal{A}$, if every point of X has an arithmetically clustered orbit. The property of arithmetically clustered is unchanged if d is replaced by some other topologically equivalent (hence uniformly equivalent, since X is compact) metric.

Proposition 4. (1) *If (X, σ) and (Y, τ) are in $\mathcal{A}$ then $(X \times Y, \sigma \times \tau)$ is in $\mathcal{A}$.*
(2) *If (X, σ) is in $\mathcal{S}$ and if (Y, τ) is a factor of (X, σ) then (Y, τ) is in $\mathcal{S}$.*
(3) *The 2-shift does not cluster arithmetically.*

Proof. (1) Let $(x, y) \in X \times Y$ and let $\epsilon > 0$. We use the same symbol d to denote the metric on each space. It suffices to find an arithmetic set S such that $n, m \in S$ implies $d(\sigma^n x, \sigma^m x) < \epsilon$ and $d(\tau^n x, \tau^m x) < \epsilon$. Let $S_x = a + L\mathbf{Z}$ be the appropriate arithmetic set for x and ϵ. Choose $\delta > 0$ so that if y_1, y_2 in Y and $d(y_1, y_2) < \delta$ then $d(\tau^k y_1, \tau^k y_2) < \epsilon$ for $0 \leq k \leq L - 1$. Now choose $S_Y = b + M\mathbf{Z}$ so that $m, n \in S_Y$ implies $d(\tau^m y, \tau^n y) < \delta$. Choose $u \in \mathbf{Z}$ and $0 \leq k \leq L - 1$ so that $a + uL = b + k$. Let $S = a + uL + LM\mathbf{Z}$. Observe that $S \subseteq S_x$ so $n, m \in S$ implies $d(\sigma^n x, \sigma^m x) < \epsilon$. Write S as $b + k + LM\mathbf{Z}$. If $n, m \in S$ then $n - k, m - k \in S_Y$ so $d(\tau^{n-k} y, \tau^{m-k} y) < \delta$ yielding $d(\tau^n y, \tau^m y) < \epsilon$.
(2) This is a direct consequence of uniform continuity.
(3) The point $0, 1, 1, 0, 0, 0, 1, 1, 1, 1, \ldots$ does not have an arithmetically clustered orbit.

□

Remark. Theorem 1 implies that no zero dimensional $\mathcal{A}$ flow can have its enveloping semigroup equal to β. The dimensional assumption is superfluous. Suppose (X, σ) clusters arithmetically and suppose the function f is in the corresponding algebra $A(X)$. It is easy to show that for every $\epsilon > 0$ there is an arithmetic set S such that if $n, m \in S$ then $|f(n) - f(m)| < \epsilon$. It follows that $A \neq B$ and so $E \neq \beta$.

The next example, a mild variant of one provided by B. Weiss at the 1995 CBMS conference in Bakersfield, is a non-minimal subshift (of the 2-shift) of positive entropy. It belongs to the class $\mathcal{A}$ and so $E \neq \beta$.

Let $m = m_1, m_2, \cdots \in \{0, 1, 2, 3, \ldots, 9\}^{\mathbf{N}} = M$. With each $m \in M$ we associate $I_m \subseteq \mathbf{Z}$ defined by $i \in I_m$ if and only if there exist $n \in \mathbf{N}$ and $k \in \mathbf{Z}$ such that

$|i - k10^n - \sum_{j=1}^n m_j 10^{j-1}| < n$. For example, if $m = 3, 2, 5 \dots$ then, for any k, $3 + 10k \in I_m$, $\{22, 23, 24\} + 100k \subset I_m$, $[521, 525] + 1000k \subset I_m \dots$. We let $x \in X \subset \{0,1\}^{\mathbf{Z}}$ if there is some $m \in M$ such that $x_i = 0$ for each $i \in I_m$. We will say such an x and m are paired. In general neither x nor m determines the other. We let σ be the shift: $\sigma(x)_i = x_{i+1}$.

Proposition 5. *X is a closed shift invariant set and the flow (X, σ) has positive topological entropy.*

Proof. If $x \in X$ and $m \in M$ are paired in the above manner then $m - 1$ and $m + 1$ are paired with $\sigma(x)$ and $\sigma^{-1}(x)$ respectively, where $m + 1$ is obtained by changing the leading string of nines (if there is one) to zeroes and then increasing the first non-nine digit (if there is one) by 1, and $m - 1$ is defined analogously so that $(m + 1) - 1 = m$.

Now suppose $x^{(\nu)}$ is a sequence of points in X converging to a point x in $\{0,1\}^{\mathbf{Z}}$. We will show $x \in X$. Let $m^\nu \in M$ be paired with x^ν. Since M is compact, we can assume (passing to a subsequence if necessary) that m^ν converges to some $m \in M$. It is easily seen that x and m are paired. This shows X is closed.

Next let X_1 be the set of x paired with $\mathbf{1} = 1, 1, 1, \dots$. If we look at the entries in the interval $[1, 10^k]$ we see that ones are specified at $I_{\mathbf{1}} \cap [1, 10^k]$ and any pattern of zeroes and ones may appear at the remaining entries. The set I consists of $\{1, 11, 21, \dots\} \cup \{10, 12, 110, 112, \dots\} \cup \{109, 113, 1109, 1113, \dots\} \cup \dots$. Thus

$$\begin{aligned}
[\mathrm{card}(I \cap [1, 10^k]) &< 10^k \left(\frac{1}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \dots\right) \\
&= 10^k \left(\frac{1}{10} + \frac{2}{10^2} \frac{1}{1 - \frac{1}{10}}\right) = \frac{11}{90} \times 10^k.
\end{aligned}$$

A finite block $x_1, \dots, x_n$ in $\{0,1\}^n$ is called an admissible n-block if there is some two-side continuation $\dots, x_1, \dots, x_n \dots$ to an element of X. We have just shown that for $n = 2^k$ the number of admissible n-blocks is at least $2^{\frac{79}{90} n}$. This, as is well-known, implies that σ has positive entropy. ☐

It is obvious from the construction that (X, σ) belongs to $\mathcal{A}$.

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