

RADIATION CONDITIONS AND UNIQUENESS FOR STATIONARY OSCILLATIONS IN ELASTIC PLATES

CHRISTIAN CONSTANDA

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ABSTRACT. Sommerfeld-type radiation conditions are indicated for the solutions of the system governing the small stationary oscillations in plates with transverse shear deformation, and a uniqueness theorem is proved in the case of the corresponding exterior Dirichlet and Neumann problems.

1. PRELIMINARIES

In what follows Greek and Latin suffixes are assumed to take the values 1,2 and 1,2,3, respectively, and the superscript T denotes matrix transposition. If L is a scalar operator and Ψ a vector function, then $L\Psi$ means that L is applied to each component of Ψ . Also, if X is a space of scalar functions, $\Psi \in X$ means that each component of Ψ belongs to X .

Let Ω be a domain in \mathbb{R}^3 bounded by a closed surface $\partial\Omega$, which is occupied by an isotropic and homogeneous elastic material of Lamé coefficients λ and μ and of density ρ . Also, let $x = (x_1, x_2, x_3)^T$ be the position vector of a generic point in Ω . The equations governing the small dynamic deformations of this body can be written in the form [1, Ch. I, §11.1]

$$\mu\Delta\Phi + (\lambda + \mu)\operatorname{grad}\operatorname{div}\Phi + \rho F = \rho\ddot{\Phi},$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T$ is the displacement vector and two superposed dots denote the second order time-derivative. If the body force F is of the form

$$F(x, t) = \operatorname{Re}[f(x)e^{-i\omega t}],$$

where f is a complex-valued vector function, and if the boundary conditions (prescribed on $\partial\Omega$) have similar expressions, then the body performs stationary oscillations of frequency ω and its expected displacements are of the form

$$\Phi(x, t) = \operatorname{Re}[\varphi(x)e^{-i\omega t}],$$

with $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$ also complex-valued and satisfying

$$(1) \quad \mu\Delta\varphi + (\lambda + \mu)\operatorname{grad}\operatorname{div}\varphi + \rho\omega^2\varphi + \rho f = 0.$$

In the case of an elastic plate we have $\Omega = S \times (-h_0/2, h_0/2)$, where S is a domain in \mathbb{R}^2 bounded by a closed contour ∂S and $h_0 = \operatorname{const} \ll \operatorname{diam} S$. Owing to this

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special geometry, the study of bending of plates makes use of certain simplifying assumptions. Thus, when transverse shear deformation is taken into account, the fundamental (kinematic) assumption is that the displacement field is of the form

$$\varphi_\beta = x_3 u_\beta(x_1, x_2), \quad \varphi_3 = u_3(x_1, x_2).$$

Let $x = (x_1, x_2)^T$ be the position vector of a generic point in S , and let $w = (u_1, u_2, 0)^T$. If the body forces and moments, and the forces and moments acting on the faces $x_3 = \pm h_0/2$, are negligible, then the usual procedure of averaging (1) over the thickness of the plate [2, §2.1] leads to the system of equations

$$(2) \quad \begin{aligned} h^2(\lambda + \mu) \operatorname{grad} \operatorname{div} w + h^2 \mu \Delta w + \mu \theta w - \mu \operatorname{grad} u_3 &= 0, \\ (\Delta + k^2) u_3 + \operatorname{div} w &= 0, \end{aligned}$$

where $h^2 = h_0^2/12$, $k^2 = \rho \omega^2 / \mu$ and $\theta = k^2 h^2 - 1$.

A solution $u = w + (0, 0, u_3)^T$ of (2) is called regular if $u \in C^2(S) \cap C^1(\bar{S})$. From now on we assume that

$$(3) \quad \lambda + \mu > 0, \quad \mu > 0, \quad \omega > \frac{1}{h} \sqrt{\frac{\mu}{\rho}}.$$

The first two conditions ensure that (2) is elliptic; the last one is equivalent to $k^2 h^2 > 1$.

We denote by S^+ the bounded domain enclosed by ∂S , write $S^- = \mathbb{R}^2 \setminus (S^+ \cup \partial S)$, and assume that ∂S has a continuous normal of unit outward (with respect to S^+) vector $\nu = (\nu_1, \nu_2)^T$.

The exterior Dirichlet and Neumann problems (D^-) and (N^-) consist in finding a solution of (2) in S^- which takes prescribed values on ∂S , or for which the bending and shear stress vector Tu [3] takes prescribed values on ∂S , respectively.

In [3] radiation conditions were indicated under which (D^-) and (N^-) have at most one regular solution. Unfortunately, these conditions are not of the form one might expect for stationary oscillations. In what follows we intend to sharpen considerably the results of the analysis in [3] and give full Sommerfeld-type conditions that imply uniqueness for the solutions of these boundary value problems.

System (2) describes the small stationary flexural oscillations of a plate, and its solutions play an important role in the study of scattering of elastic waves that occur, for example, in the process of non-destructive testing of plate-like structures.

2. RADIATION CONDITIONS

Setting

$$\begin{aligned} \mu \theta w^{(1)} &= -h^2(\lambda + 2\mu) \operatorname{grad} \operatorname{div} w + \mu \operatorname{grad} u_3, \\ \theta w^{(2)} &= h^2 \operatorname{curl} \operatorname{curl} w, \end{aligned}$$

we arrive at the representation [3]

$$(4) \quad w = w^{(1)} + w^{(2)},$$

where

$$(5) \quad \begin{aligned} (\Delta + k_1^2)(\Delta + k_2^2)w^{(1)} &= 0, \\ (\Delta + k_3^2)w^{(2)} &= 0 \end{aligned}$$

and

$$(6) \quad \begin{aligned} \operatorname{curl} w^{(1)} &= 0, \\ \operatorname{div} w^{(2)} &= 0. \end{aligned}$$

Also, u_3 satisfies the equation

$$(7) \quad (\Delta + k_1^2)(\Delta + k_2^2)u_3 = 0.$$

Here k_1^2, k_2^2 and k_3^2 are defined by

$$(8) \quad \begin{aligned} k_1^2 + k_2^2 &= \frac{k^2(\lambda + 3\mu)}{\lambda + 2\mu}, & k_1^2 k_2^2 &= \frac{\mu k^2(k^2 h^2 - 1)}{h^2(\lambda + 2\mu)}, \\ k_3^2 &= k^2 - \frac{1}{h^2}. \end{aligned}$$

In view of (3), all of k_1^2, k_2^2 and k_3^2 are strictly positive and distinct.

From $(2)_2$ and $(6)_2$ it follows that

$$(9) \quad (\Delta + k^2)u_3 + \operatorname{div} w^{(1)} = 0.$$

The fact that $w^{(1)}$ and u_3 do not satisfy Helmholtz equations is highly unsatisfactory. However, this drawback is easily eliminated.

Theorem 1. *Any regular solution of (2) can be written in the form*

$$(10) \quad u = U^{(1)} + U^{(2)} + U^{(3)} + v^{(1)} + v^{(2)},$$

where $U_3^{(j)} = 0, v_\gamma^{(\beta)} = 0$ and

$$(11) \quad (\Delta + k_j^2)U^{(j)} = 0, \quad \operatorname{curl} U^{(\beta)} = 0, \quad \operatorname{div} U^{(3)} = 0,$$

$$(12) \quad (\Delta + k_\beta^2)v^{(\beta)} = 0.$$

Proof. Setting

$$U^{(1)} = -\frac{(\Delta + k_2^2)w^{(1)}}{k_1^2 - k_2^2}, \quad U^{(2)} = \frac{(\Delta + k_1^2)w^{(1)}}{k_1^2 - k_2^2}, \quad U^{(3)} = w^{(2)}$$

and

$$v^{(\beta)} = (0, 0, u_3^{(\beta)})^T,$$

where

$$u_3^{(1)} = -\frac{(\Delta + k_2^2)u_3}{k_1^2 - k_2^2}, \quad u_3^{(2)} = \frac{(\Delta + k_1^2)u_3}{k_1^2 - k_2^2},$$

we easily see that

$$(13) \quad w^{(1)} = U^{(1)} + U^{(2)}, \quad u_3 = u_3^{(1)} + u_3^{(2)}$$

and that, by (5)–(7) and (4), equalities (11) and (12) are satisfied. □

Consider a circular disk K_R with the centre at the origin (chosen in S^+) and radius R sufficiently large so that ∂S lies strictly inside the circumference ∂K_R .

In view of the radiation conditions for the Helmholtz equation in the plane [4, §228], equations (11)₁ and (12) now suggest that, as $R \rightarrow \infty$, the radiation conditions that are appropriate for a regular solution of (2) in S^- are

$$(14) \quad U^{(j)} = O(R^{-1/2}), \quad \frac{\partial U^{(j)}}{\partial R} - ik_j U^{(j)} = O(R^{-3/2}),$$

$$(15) \quad u_3^{(\beta)} = O(R^{-1/2}), \quad \frac{\partial u_3^{(\beta)}}{\partial R} - ik_\beta u_3^{(\beta)} = O(R^{-3/2}).$$

3. ASYMPTOTIC ESTIMATES

In what follows we assume that $x \in \partial K_R$, $x = (x_1, x_2, 0)^T$, and write $R_0 = x/R$.

Theorem 2. *For R large, a regular solution u of (2) in S^- which satisfies the radiation conditions (14) and (15) also satisfies the estimates*

$$(16) \quad \begin{aligned} R_0 \times U^{(\beta)} &= O(R^{-3/2}), \\ \operatorname{div} U^{(\beta)} - ik_\beta R_0 \cdot U^{(\beta)} &= O(R^{-3/2}), \\ R_0 \cdot U^{(3)} &= O(R^{-3/2}), \\ \operatorname{curl} U^{(3)} - ik_3 R_0 \times U^{(3)} &= O(R^{-3/2}), \\ U^{(\beta)} &= ik_\beta^{-1}(k^2 - k_\beta^2)u_3^{(\beta)} R_0 + O(R^{-3/2}). \end{aligned}$$

Proof. The first four asymptotic equalities (16) are established just as in three-dimensional elasticity [1, Ch. III, §2.3] by means of (14) and (15). To prove (16)₅, we replace (13)₂ and $\Delta u_3^{(\beta)} = -k_\beta^2 u_3^{(\beta)}$ (from (12)₂) in (9), which becomes

$$(k^2 - k_1^2)u_3^{(1)} + (k^2 - k_2^2)u_3^{(2)} + \operatorname{div} w^{(1)} = 0.$$

We now apply grad to both sides above and use the equality $\operatorname{grad} \operatorname{div} = \operatorname{curl} \operatorname{curl} + \Delta$, (6)₁, (13)₁, and the substitutions $\Delta U^{(\beta)} = -k_\beta^2 U^{(\beta)}$ (from (11)₁) to find that

$$(k^2 - k_1^2) \operatorname{grad} u_3^{(1)} + (k^2 - k_2^2) \operatorname{grad} u_3^{(2)} - k_1^2 U^{(1)} - k_2^2 U^{(2)} = 0.$$

Taking the scalar product of R_0 with both sides above and then using (15)₂ yields

$$(17) \quad ik_1(k^2 - k_1^2)u_3^{(1)} + ik_2(k^2 - k_2^2)u_3^{(2)} = k_1^2 R_0 \cdot U^{(1)} + k_2^2 R_0 \cdot U^{(2)} + O(R^{-3/2}).$$

On the other hand, in view of (13)₁, (16)₂ and (12), from (9) it follows that

$$(k^2 - k_1^2)u_3^{(1)} + (k^2 - k_2^2)u_3^{(2)} + ik_1 R_0 \cdot U^{(1)} + ik_2 R_0 \cdot U^{(2)} = O(R^{-3/2}).$$

We adjoin this equality to (17) multiplied by i to form a system of simultaneous linear equations for $R_0 \cdot U^{(\beta)}$:

$$\begin{aligned} ik_1 R_0 \cdot U^{(1)} + ik_2 R_0 \cdot U^{(2)} &= -(k^2 - k_1^2)u_3^{(1)} - (k^2 - k_2^2)u_3^{(2)} + O(R^{-3/2}), \\ ik_1^2 R_0 \cdot U^{(1)} + ik_2^2 R_0 \cdot U^{(2)} &= -k_1(k^2 - k_1^2)u_3^{(1)} - k_2(k^2 - k_2^2)u_3^{(2)} + O(R^{-3/2}). \end{aligned}$$

Since, by (3), $k_1 \neq k_2$ and $k_1 k_2 \neq 0$, this system has the unique solution

$$(18) \quad R_0 \cdot U^{(\beta)} = ik_\beta^{-1}(k^2 - k_\beta^2)u_3^{(\beta)} + O(R^{-3/2}).$$

Now, by (16)₁, (18), and the properties of the triple vector product,

$$\begin{aligned} R_0 \times (R_0 \times U^{(\beta)}) &= O(R^{-3/2}) = (R_0 \cdot U^{(\beta)})R_0 - U^{(\beta)} \\ &= ik_\beta^{-1}(k^2 - k_\beta^2)u_3^{(\beta)} R_0 - U^{(\beta)} + O(R^{-3/2}), \end{aligned}$$

from which we obtain (16)₅. □

For $x \in \partial K_R$, the components of the boundary moment and shear stress vector Tu are (see [3], with ν regarded as the vector $(\nu_1, \nu_2, 0)^T$)

$$(19) \quad \begin{aligned} (Tu)_\alpha &= h^2 \left[\lambda(\operatorname{div} w)R_0 + 2\mu \frac{\partial w}{\partial R} + \mu R_0 \times \operatorname{curl} w \right]_\alpha, \\ (Tu)_3 &= \mu R_0 \cdot w + \frac{\partial u_3}{\partial R}. \end{aligned}$$

Theorem 3. *If $x \in \partial K_R$ and u is a regular solution of (2) in S^- satisfying the radiation conditions (14) and (15), then for R large Tu satisfies the estimates*

$$(20) \quad \begin{aligned} [TU^{(\beta)} - ih^2(\lambda + 2\mu)k_\beta U^{(\beta)}]_\alpha &= O(R^{-3/2}), \\ [TU^{(3)} - ih^2\mu k_3 U^{(3)}]_\alpha &= O(R^{-3/2}), \\ U^{(\beta)} \cdot U^{(3)} &= O(R^{-2}). \end{aligned}$$

Proof. In view of (19)₁, (11)₂, (16)₂ and (14)₂,

$$\begin{aligned} (TU^{(\beta)})_\alpha &= h^2 \left[\lambda(\operatorname{div} U^{(\beta)})R_0 + 2\mu \frac{\partial U^{(\beta)}}{\partial R} \right]_\alpha \\ &= ih^2 k_\beta [\lambda(R_0 \cdot U^{(\beta)})R_0 + 2\mu U^{(\beta)}]_\alpha + O(R^{-3/2}). \end{aligned}$$

Using the triple product $R_0 \times (R_0 \times U^{(\beta)})$ and taking (16)₁ into account, we establish the first estimate (20). The second estimate is derived similarly, by means of (19)₁, (11)₃, (14)₂, and (16)_{2,3}. Finally, from the equality

$$U^{(3)} \times (R_0 \times U^{(\beta)}) = (U^{(3)} \cdot U^{(\beta)})R_0 - (R_0 \cdot U^{(3)})U^{(\beta)},$$

(14)₁ and (16)_{1,3} we deduce (20)₃. □

4. UNIQUENESS OF THE SOLUTION

The preceding results can now be combined to answer the question of uniqueness for the solutions of the exterior Dirichlet and Neumann problems.

Theorem 4. *If conditions (3) and the radiation conditions (14) and (15) hold, then each of (D^-) , (N^-) has at most one regular solution.*

Proof. The assertion is proved if we show that for $u = 0$ or $Tu = 0$ on ∂S the system (2) has only the trivial solution $u = 0$ in S^- .

The procedure used to establish the reciprocity relation for bending of plates with transverse shear deformation [2, §2.1], applied to a regular solution u of the homogeneous problem (D^-) or (N^-) in the bounded domain $S^- \cap K_R$, leads to the equality [3]

$$(21) \quad \int_{\partial K_R} (\bar{u} \cdot Tu - u \cdot T\bar{u}) ds = 0.$$

First we evaluate all the terms occurring in the expression for $\bar{u} \cdot Tu$ on ∂K_R arising from (10) and (19) for R large. Thus, by (14)₁ and (20)₁,

$$(22) \quad \bar{U}^{(\beta)} \cdot TU^{(\beta)} = ih^2(\lambda + 2\mu)k_\beta |U^{(\beta)}|^2 + O(R^{-2}).$$

Similarly,

$$(23) \quad \bar{U}^{(1)} \cdot TU^{(2)} + \bar{U}^{(2)} \cdot TU^{(1)} = ih^2(\lambda + 2\mu)(k_2 \bar{U}^{(1)} \cdot U^{(2)} + k_1 \bar{U}^{(2)} \cdot U^{(1)}) + O(R^{-2}).$$

By (14)₁ and (20),

$$(24) \quad \bar{U}^{(\beta)} \cdot TU^{(3)} = O(R^{-2}), \quad \bar{U}^{(3)} \cdot TU^{(\beta)} = O(R^{-2}).$$

By (14)₁ and (20)₂,

$$(25) \quad \bar{U}^{(3)} \cdot TU^{(3)} = ih^2\mu k_3 |U^{(3)}|^2 + O(R^{-2}).$$

By (19)₂, (15)₁ and (16)₃,

$$(26) \quad \bar{v}^{(\beta)} \cdot TU^{(3)} = \mu \bar{u}_3^{(\beta)} R_0 \cdot U^{(3)} = O(R^{-2}).$$

By (19)₂ and (15),

$$(27) \quad \begin{aligned} & (\bar{v}^{(1)} + \bar{v}^{(2)}) \cdot (Tv^{(1)} + Tv^{(2)}) \\ &= \mu(\bar{u}_3^{(1)} + \bar{u}_3^{(2)}) \left(\frac{\partial u_3^{(1)}}{\partial R} + \frac{\partial u_3^{(2)}}{\partial R} \right) \\ &= \mu(ik_1 |u_3^{(1)}|^2 + ik_2 |u_3^{(2)}|^2 + ik_2 \bar{u}_3^{(1)} u_3^{(2)} + ik_1 \bar{u}_3^{(2)} u_3^{(1)}) + O(R^{-2}). \end{aligned}$$

By (19)₂ and (13)₁, the remaining terms are

$$(28) \quad (\bar{v}^{(1)} + \bar{v}^{(2)}) \cdot (TU^{(1)} + TU^{(2)}) = \mu(\bar{u}_3^{(1)} + \bar{u}_3^{(2)}) R_0 \cdot (U^{(1)} + U^{(2)}).$$

Combining (22)–(28) with their complex conjugates, we compute the expression of the integrand in (21):

$$(29) \quad \begin{aligned} & \bar{u} \cdot Tu - u \cdot T\bar{u} \\ &= 2i\{h^2(\lambda + 2\mu)(k_1 |U^{(1)}|^2 + k_2 |U^{(2)}|^2) + h^2\mu k_3 |U^{(3)}|^2 \\ & \quad + \mu(k_1 |u_3^{(1)}|^2 + k_2 |u_3^{(2)}|^2) + h^2(\lambda + 2\mu)(k_1 + k_2) \operatorname{Re}(U^{(1)} \cdot \bar{U}^{(2)}) \\ & \quad + \mu(k_1 + k_2) \operatorname{Re}(u_3^{(1)} \bar{u}_3^{(2)}) + \mu \operatorname{Im}[(\bar{u}_3^{(1)} + \bar{u}_3^{(2)}) R_0 \cdot (U^{(1)} + U^{(2)})]\} + O(R^{-2}). \end{aligned}$$

We need to estimate the terms without a modulus in (29).

By (16)₅ and (15)₁,

$$(30) \quad \operatorname{Re}(U^{(1)} \cdot \bar{U}^{(2)}) = (k_1 k_2)^{-1} (k^2 - k_1^2)(k^2 - k_2^2) \operatorname{Re}(u_3^{(1)} \bar{u}_3^{(2)}) + O(R^{-2}).$$

From (18),

$$(31) \quad \begin{aligned} R_0 \cdot U^{(1)} &= ik_1^{-1} (k^2 - k_1^2) u_3^{(1)} + O(R^{-3/2}), \\ R_0 \cdot U^{(2)} &= ik_2^{-1} (k^2 - k_2^2) u_3^{(2)} + O(R^{-3/2}). \end{aligned}$$

If from the combination $[(31)_1 \bar{u}_3^{(2)} + (31)_2 \bar{u}_3^{(1)}] + [(31)_1 \bar{u}_3^{(1)} + (31)_2 \bar{u}_3^{(2)}]$ we now subtract its complex conjugate expression, we arrive at

$$(32) \quad \begin{aligned} \operatorname{Im}[(\bar{u}_3^{(1)} + \bar{u}_3^{(2)})R_0 \cdot (U^{(1)} + U^{(2)})] \\ = [k_1^{-1}(k^2 - k_1^2) + k_2^{-1}(k^2 - k_2^2)] \operatorname{Re}(u_3^{(1)} \bar{u}_3^{(2)}) \\ + k_1^{-1}(k^2 - k_1^2)|u_3^{(1)}|^2 + k_2^{-1}(k^2 - k_2^2)|u_3^{(2)}|^2 + O(R^{-2}). \end{aligned}$$

Hence, by (32), (30) and (8),

$$\begin{aligned} h^2(\lambda + 2\mu)(k_1 + k_2) \operatorname{Re}(U^{(1)} \cdot \bar{U}^{(2)}) \\ + \mu(k_1 + k_2) \operatorname{Re}(u_3^{(1)} \bar{u}_3^{(2)}) + \mu \operatorname{Im}[(\bar{u}_3^{(1)} + \bar{u}_3^{(2)})R_0 \cdot (U^{(1)} + U^{(2)})] \\ = \mu[k_1^{-1}(k^2 - k_1^2)|u_3^{(1)}|^2 + k_2^{-1}(k^2 - k_2^2)|u_3^{(2)}|^2] + O(R^{-2}). \end{aligned}$$

Consequently, from (29) we obtain

$$\begin{aligned} \bar{u} \cdot Tu - u \cdot T\bar{u} \\ = 2i[h^2(\lambda + 2\mu)(k_1|U^{(1)}|^2 + k_2|U^{(2)}|^2) + h^2\mu k_3|U^{(3)}|^2 \\ + \mu(k_1 k_2)^{-1}k^2(k_2|u_3^{(1)}|^2 + k_1|u_3^{(2)}|^2)] + O(R^{-2}). \end{aligned}$$

Replacing this in (21) and taking the limit as $R \rightarrow \infty$, in view of conditions (3) we deduce that

$$(33) \quad \lim_{R \rightarrow \infty} \int_{\partial K_R} |U^{(j)}|^2 ds = \lim_{R \rightarrow \infty} \int_{\partial K_R} |u_3^{(\beta)}|^2 ds = 0.$$

To complete the proof, we recall that the $U^{(j)}$ and $u_3^{(\beta)}$ satisfy the Helmholtz equations (11) or (12) and the corresponding radiation conditions (14) or (15), so, by the classical argument based on (33) [4, §229], we conclude that

$$U^{(j)} = u_3^{(\beta)} = 0 \quad \text{in } S^-.$$

This means that $u = 0$ in S^- , as required. □

Remarks. (i) According to (10), (11)₁ and (12), the displacement field appears to have a spectrum of 5 ‘basic’ time-harmonic oscillations. By (11)₂, (14)₁ and (16)₁, the $U^{(\alpha)}$ are potential vector-functions whose component tangent to the wave front decays faster than the normal one away from the origin. Consequently, these vectors seem to describe asymptotically longitudinal oscillations. By (11)₃, (14)₁ and (16)₃, $U^{(3)}$ is a solenoidal vector whose component normal to the wave front decays faster than the tangential one at infinity, so $U^{(3)}$ may be regarded as describing asymptotically transverse oscillations. The $v^{(\beta)}$ could be said to represent ‘purely flexural’ oscillations, with no equivalent in three-dimensional elasticity. Postulating the Sommerfeld radiation conditions (14) and (15) is equivalent to assuming that there are no sources at infinity for any of these basic spectral components.

(ii) The constants k_i occurring in the expression of u coincide with the wave numbers of the small-amplitude monochromatic waves propagating in the plate [5]. Under conditions (3) there are three distinct such waves.

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