

A NOTE ON THE DENSITY OF s -DIMENSIONAL SETS

JAMES FORAN

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ABSTRACT. Sets in Euclidean spaces which are measurable with respect to Hausdorff s -dimensional measure with $0 < s < 1$ are shown to have an at most countable set of points where the exact s -density exists and is finite and non-zero.

The purpose of this note is to give a slight improvement to the result of Marstrand [2] which states that for $0 < s < 1$, the set of points of an s -set at which the exact density exists and is finite is of s -measure 0. This result was given a simpler proof by Falconer [1, p. 55–56] and the theorem below uses his notation and definitions. Indeed, this stronger result is patterned on his proof.

Definitions. $B_r(y) = \{x : d(x, y) \leq r\}$ is the *closed ball* about x of radius r .

$H_\delta^s(E) = \inf \sum_{i=1}^{\infty} |U_i|$ where the inf is over all countable covers of E with $\text{diam}(U_i) \leq \delta$.

$H^s(E) = \lim_{\delta \rightarrow 0} H_\delta^s(E)$ is the *Hausdorff outer measure* of E .

$D^s(E, x) = \lim_{r \rightarrow 0} H^s(E \cap B_r(x)) / (2r)^s$ is the *density* of E at x provided the limit exists.

The *upper density* and *lower density* are defined using the lim sup and lim inf respectively.

Theorem. *Suppose E is an s -measurable set in Euclidean n -dimensional space with $0 < s < 1$. Then the set of points x where the density $D^s(E, x)$ exists and is finite and non-zero is an at most countable set of points.*

Proof. Given a natural number k and $0 < s < 1$, suppose there is a set E in a Euclidean space for which the set of points A where $D^s(E, x)$ is finite and non-zero is an uncountable set. Let A_j be the set of all $x \in A$ for which $r < 1/j$ implies $(2r)^s/j < H^s(E \cap B_r(x)) < (2r)^s \cdot j$. Since $A = \bigcup A_j$ there is a natural number N so that A_N is uncountable and thus there is a point $y \in A_N$ which is an accumulation point of A_N . Let $0 < \eta < 1$ and let $x \in A_N$ with $d(x, y) = r$ and $r(1 + \eta) < 1/N$. Let $A_{r,\eta}(y) = B_{r(1+\eta)}(y) \setminus B_{r(1-\eta)}(y)$, the annulus centered at y . Note (as in [1]) that

$$\begin{aligned} (2r)^{-s} H^s(E \cap A_{r,\eta}(y)) &= (2r)^{-s} H^s(E \cap B_{r(1+\eta)}(y)) - (2r)^{-s} H^s(E \cap B_{r(1-\eta)}(y)) \\ &\rightarrow D^s(E, y)((1 + \eta)^s - (1 - \eta)^s). \end{aligned}$$

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Then, since $B_{r\eta/2}(x) \subset A_{r,\eta}(y)$, it follows that

$$r^s \eta^s / N < H^s(E \cap B_{r\eta/2}(x)) \leq H^s(E \cap A_{r,\eta}(y))$$

and on dividing the terms of this inequality by $(2r)^s$ and letting r approach 0, it follows that

$$\eta^s / (N \cdot 2^s) \leq D^s(E, y)((1 + \eta)^s - (1 - \eta)^s) = D^s(E, y)(2s\eta + O(\eta^2))$$

which is impossible for sufficiently small η . This contradiction shows that each A_j consists of isolated points and from this it also follows that A is an at most countable set. □

To complete this picture, suppose that $\{x_i\}$ is a sequence of points in a Euclidean space and $\{d_i\}$ is a sequence of extended real numbers with $0 < d_i \leq \infty$. The following example shows that it is possible for a set to have exact s -density d_i at each point x_i .

Example. Given points x_i in Euclidean k -dimensional space, numbers d_i with $0 < d_i \leq \infty$ and $s \in (0, 1)$, there is a set E of finite s -measure so that $D^s(E, x_i) = d_i$.

Construction. The set will be constructed on a sequence of line segments parallel to one of the axes. First, let $0 < d < \infty$ and $s \in (0, 1)$ be given. Place a closed set X_N of s -measure $2^s \cdot d(N^{-s} - (N + 1)^{-s})$ in each interval $((N + 1)^{-1}, N^{-1})$. Then $E_d = \bigcup X_i \cup \{0\}$ is a closed set. Since $2^s \cdot d \cdot \sum_{n=N}^{\infty} (1/n^s - 1/(n + 1)^s) / (2r)^s = d \cdot N^{-s} / r^s$, if $1/(N + 1) \leq r < 1/N$, it follows that

$$d \cdot (N + 1)^{-s} / r^s \leq H^s(E_d \cap (-r, r)) / (2r)^s \leq d \cdot N^{-s} / r^s.$$

Since the opposite sides of this inequality approach d as r approaches 0, the density of E_d is d at 0. If it is desired to have a set E_∞ of finite measure and s -density ∞ at the origin, let $d_n \uparrow \infty$ and construct closed sets $X_N \subset ((N + 1)^{-1}, N^{-1})$ of s -measure $2^s \cdot d_n \cdot (N^{-s} - (N + 1)^{-s})$. Then the same calculations show that the set $E_\infty = \bigcup X_i \cup \{0\}$ has density ∞ at 0. If the d_n are chosen so that

$$\sum d_n(n^{-s} - (n + 1)^{-s}) < \infty,$$

then the s -measure of the set E_∞ will be finite. Note that E_d can be considered to be a subset of E^k of points on the first coordinate axis with s -density d at the origin. Suppose $\{x_i\}$ and $\{d_i\}$ are given. Let $E_1 = F_1 = E_{d_1} + x_1$. Let $r_1 = 1$ and let $r_n = \min\{2^{-n}, d(x_i, x_j) : i < j \leq n\}$. Suppose E_{n-1} and F_{n-1} have been constructed. Let $\varepsilon_n > 0$ with $\varepsilon_n < r_n / 2^n$ so that $H^s(B_{\varepsilon_n}(x_n) \cap E_{n-1}) < 2^{-n} \cdot r_{n+1}^s$ and $H^s(B_{\varepsilon_n}(0) \cap E_{d_n}) < 2^{-n} \cdot r_{n+1}^s$. Let $F_n = (E_{d_n} \cap B_{\varepsilon_n}(0)) + x_n$ and let $E_n = (E_{n-1} \setminus B_{\varepsilon_n}(x_n)) \cup F_n$. Then $E = \limsup E_n$ is the required set. It clearly has finite s -measure and if $r > 0$ is given with $r_{n+1}(1 - 2^{-n}) \leq r < r_n(1 - 2^{-n})$, then $H^s(F_i \cap B_r(x_i)) - 2^{-n+1} \cdot r_{n+1}^s \leq H^s(E \cap B_r(x_i)) \leq H^s(F_i \cap B_r(x_i)) + 2^{-n+1} \cdot r_{n+1}^s$. Dividing the terms in this inequality by $(2r)^s$ and letting $r \rightarrow 0$, the two sides of the resulting inequality approach d_i . It follows that the s -density of E at x_i is d_i .

Some natural questions which arise are:

1. Is it possible that the set in the example above be constructed so that it has the exact preassigned s -density only at the preassigned points?
2. What kind of structure does the set of points where the s -density is infinite have?
3. Is there a generalization for $s > 1$; for example, is it possible that the set of points is always of σ -finite $n - 1$ measure where $n - 1 < s < n$?

REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-KANSAS CITY, KANSAS CITY, MISSOURI 64110

E-mail address: `jforan@cctr.umkc.edu`