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RATIONAL CURVES ON K3 SURFACES IN $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

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Abstract. We discuss Manin and Batyrev's notion of the arithmetic stratification of a variety, and, for an irreducible surface V embedded in $\mathbb{P}^m$, compare it with the spectrum of degrees of rational curves on V. We study this spectrum for the class of K3 surfaces generated by smooth (2,2,2) forms in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Introduction

In this paper, we study the spectrum of degrees of rational curves on varieties in the class of K3 surfaces generated by smooth (2,2,2) forms in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We show in Theorem 2.1 that generic and non-generic varieties of this type are distinguished by this spectrum. On a generic variety of this type, there is an action of a group of automorphisms $\mathcal{A} \cong \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ generated by σ_1, σ_2 and σ_3. We define a vector height $\mathbf{h}$ on rational curves in V with the property $\deg C = (1,1,1) \cdot \mathbf{h}(C)$, and show in Theorem 3.1 that there exist linear maps T_i such that $\mathbf{h}(\sigma_i C) = T_i \mathbf{h}(C)$. Thus, if we can find a rational curve C in V, and can measure $\mathbf{h}(C)$, then this can be used to find a large subset of the spectrum of degrees of rational curves of V. In particular, if C is a rational fiber with respect to a projection onto one of the components of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, then the degree of C is either 2 or 4, and in Theorem 4.1 we show that the number of rational curves on V in the $\mathcal{A}$-orbit of C with degree less than T is $(2/\pi\sqrt{3} \deg C)T + O(\sqrt{T} \log T)$.

1. The arithmetic stratification of a variety

We begin with a discussion of Batyrev and Manin's *arithmetic stratification* of a variety [B-M].

Let $V = V(K)$ be a variety, defined over a number field K, and, for simplicity, embedded in a projective space $\mathbb{P}^m$. Let H be the usual universal exponential height on $\mathbb{P}^m$. Let U be a subset of V, and define

$$N_U(T) = \#\{P \in U : H(P) < T\}.$$

Associated to U, we can define a zeta function

$$Z_U(s) = \sum_{P \in U} (H(P))^{-s}.$$

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There exists an $\alpha_U \in \mathbb{R}$ such that $Z_U(s)$ converges for all s with $\mathrm{Re}(s) > \alpha_U$ and diverges if $\mathrm{Re}(s) < \alpha_U$. We call α_U the *exponent* of $N_U(T)$, since it is related to $N_U(T)$ by the well known result (for divergent $N_U(T)$)

$$\limsup_{T\to\infty} \frac{\log N_U(T)}{\log T} = \alpha_U.$$

Note that α_U depends on the number field K. For example, if $V = K$, then $\alpha_V = 2[K:\mathbb{Q}]$. Let us normalize this exponent by setting $\beta_U = \alpha_U/[K:\mathbb{Q}]$. This is the exponent of $N_U(T)$ with respect to the non-universal height $\hat{H} = H^{[K:\mathbb{Q}]}$. It may still be the case that β_U depends on K (for example, let V be an elliptic curve with rank 0 over $\mathbb{Q}$ and positive rank over K), but the hope is that β_U is constant for all number fields over some sufficiently large K.

We define the *arithmetic stratification* S_V of V to be:

$$\begin{aligned} S_V &= \{\beta_U : U \text{ is a non-empty Zariski open subset of } V\} \\ &\subset \mathbb{R}^+ \cup \{0, -\infty\}. \end{aligned}$$

Let

$$\hat{\beta}_V = \inf S_V.$$

Then for any $\epsilon > 0$, there exists a Zariski closed subset $W(\epsilon)$ of V so that

$$N_V(T) = N_{W(\epsilon)}(T) + a(T)T^{\hat{\beta}_V},$$

where

$$a(T) = \begin{cases} O(T^{\epsilon}), \\ \Omega(T^{-\epsilon}). \end{cases}$$

The constants implied by the big O and Ω depend on ϵ. If $S_V \neq \{\hat{\beta}_V\}$, then the first term dominates the asymptotic behaviour of $N_V(T)$ for ϵ sufficiently small. Since $W(\epsilon)$ is a proper subvariety of V, we often dismiss this term as a characterization of $W(\epsilon)$ and not V. The arithmetic stratification of V is a crude measure of the possibilities for $W(\epsilon)$, but is more accurately a reflection of the characteristics of V.

We can completely characterize S_V for curves. We note that V is a finite union of irreducible curves, so let us first assume V is irreducible. If V has genus $g \geq 2$, then it has a finite number of rational points, so $S_V = \{-\infty\}$. If V has genus $g = 1$, then $S_V = \{0\}$ if the rank of V is positive, and $S_V = \{-\infty\}$ otherwise. If V is an irreducible rational curve embedded in $\mathbb{P}^m$, then there exist coprime polynomials $X_i(t) \in K[t]$ so that the map

$$t \mapsto (X_0(t), ..., X_m(t))$$

is a map of K onto all but finitely many points in V. The degree of V is the maximum of the degrees of the polynomials $X_i(t)$, and $S_V = \{\beta_V\} = \{2/\deg V\}$. Returning now to V an arbitrary curve, we see that S_V is always finite.

If the dimension of V is at least 2, then it may be possible that S_V is infinite. In the following sections we exhibit such an example, provided Manin and Batyrev's conjecture [B-M, Conjecture A] is true (that is, provided $\hat{\beta}_V \leq 0$).

Suppose β_U is greater than $\beta_{U'}$ and both are in S_V. Then there exists a Zariski closed subset W in V (equal to the closure of $U \cup U' \setminus U \cap U'$) such that $\beta_W = \beta_U$. Thus, we can instead consider the spectrum

$$S_V^* = \{\beta_W : W \text{ is a Zariski closed proper subset of } V\}.$$

Note that either $\beta_V \notin S_V^*$, in which case $S_V = \{\beta_V\}$; or $\beta_V \in S_V^*$ and $S_V \subset S_V^*$. In the latter case, we also have either

$$S_V = \{\beta \in S_V^* : \beta > \hat{\beta}_V\}$$

or

$$S_V = \{\beta \in S_V^* : \beta \geq \hat{\beta}_V\}.$$

Note also that since W is a finite union of proper irreducible subvarieties of V, there exists an irreducible subvariety W' in W so that $\beta_{W'} = \beta_W$. Thus we may characterize S_V^* as

$$S_V^* = \{\beta_W : W \text{ is a proper irreducible subvariety of } V\}.$$

Let us now suppose V is an irreducible variety of dimension two. If W is a non-empty proper irreducible subvariety of V, then W is a curve or point, and as we saw earlier, if $\beta_W > 0$, then W is a *rational* curve. Thus, for an irreducible surface V,

$$S_V^* = \{2/\deg W : W \text{ is an irreducible rational curve in } V\} \cup S_0,$$

where S_0 is either $\{0, -\infty\}$ or $\{-\infty\}$. In particular,

$$S_V \subset \{2/n : n \in \mathbb{Z}\} \cup \{0, -\infty\}.$$

Thus, if V is a surface and $\hat{\beta}_V > 0$, then S_V is finite. This gives us some information about three dimensional hypersurfaces: If V is a variety of dimension three, then S_V has at most one cluster point and that point is its infimum $\hat{\beta}_V$.

In this vein, Batyrev and Manin wonder whether S_V is finite for every Fano variety [B-M, Question 1.10]. More generally, one might wonder whether it is ever possible for a cluster point of S_V to have a value other than zero.

For a variety V with trivial canonical divisor, Batyrev and Manin conjecture [B-M, Conjecture A]

$$\hat{\beta}_V = \inf S_V \leq 0.$$

In the following, we study a class of surfaces with trivial canonical divisor.

2. A family of K3 surfaces

Let V be a smooth variety generated by a $(2,2,2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then V is the zero locus of a non-singular polynomial over K in three projective variables which is quadratic in each variable. That is to say, if we write $(X, Y, Z) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where $X = (X_0 : X_1)$, etc., then V is the zero locus of a non-singular polynomial F of the form

$$F(X,Y,Z) = \sum_{i_k + j_k = 2} a_{i_1 j_1 i_2 j_2 i_3 j_3} X_0^{i_1} X_1^{j_1} Y_0^{i_2} Y_1^{j_2} Z_0^{i_3} Z_1^{j_3}$$

with coefficients in K. This variety is a K3 surface, and hence its canonical divisor is trivial.

We can write

$$F(X,Y,Z) = X_0^2 F_0(Y,Z) + X_0 X_1 F_1(Y,Z) + X_1^2 F_2(Y,Z). \tag{1}$$

The equations $F_i = 0$ describe three curves in $\mathbb{P}^1 \times \mathbb{P}^1$, which generically do not intersect (over $\bar{\mathbb{Q}}$.) If there is no common point of intersection, then for all $P = (P_1, P_2, P_3) \in V$, the equation

$$F(X, P_2, P_3) = 0$$

is quadratic in X, and since $P_1 \in \mathbb{P}^1(K)$ is a solution, the other root P_1' is in $\mathbb{P}^1(K)$. Thus, we can define a rational automorphism of V,

$$\sigma_1 : (P_1, P_2, P_3) \mapsto (P_1', P_2, P_3).$$

In a similar fashion, there exist three curves associated to each of the Y and Z components. We call V *generic* within this family of K3 surfaces if none of these three sets of three curves have a common point of intersection over $\bar{\mathbb{Q}}$.

Since we have defined the notion of degree for rational curves in terms of an embedding into a projective space $\mathbb{P}^m$, let us embed $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^7$ via the Segre embedding:

$$\begin{aligned}\phi : (X, Y, Z) \quad \mapsto \quad & (X_0Y_0Z_0 : X_0Y_0Z_1 : X_0Y_1Z_0 : X_1Y_0Z_0 \\ & \qquad : X_0Y_1Z_1 : X_1Y_0Z_1 : X_1Y_1Z_0 : X_1Y_1Z_1).\end{aligned}$$

We define the degree of C to be $\deg(\phi(C))$.

Now suppose C is an irreducible rational curve in V. Then there exist polynomials $X_i(t),\ Y_i(t),\ Z_i(t) \in K[t]$ so that $X_0(t)$ and $X_1(t)$ are coprime, etc., and the map

$$\mathbf{X}(t) = ((X_0(t) : X_1(t)), (Y_0(t) : Y_1(t)), (Z_0(t) : Z_1(t)))$$

is a map from K onto all but finitely many points of C. Define the height

$$h_X(C) = \max\{\deg X_0(t), \deg X_1(t)\}$$

and define h_Y and h_Z similarly. Note that the height $h_X(C)$ is well defined, even though the polynomials $X_0(t)$ and $X_1(t)$ are not necessarily uniquely defined. Now define the vector height

$$\mathbf{h}(C) = (h_X(C), h_Y(C), h_Z(C)).$$

Then

$$\deg(C) = h_X(C) + h_Y(C) + h_Z(C) = (1, 1, 1) \cdot \mathbf{h}(C).$$

Theorem 2.1. *Let V be in the family of K3 surfaces generated by smooth $(2, 2, 2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. For such a V, the following are equivalent:*
1) V is generic within this family of K3 surfaces.
2) $2 \notin S_V^*$ for any number field L over K.
3) $\phi(V)$ contains no lines over $\bar{\mathbb{Q}}$.

Proof. Suppose V is not generic within this family of K3 surfaces. Then there exists a common point of intersection in $\bar{\mathbb{Q}}$ for one of the three sets of curves described above. For ease, let us assume that point is $(P_2, P_3) \in \mathbb{P}^1 \times \mathbb{P}^1$ over a number field L and that $F_i(P_2, P_3) = 0$ for $i = 1,\ 2$ and 3. Then $F(X, P_2, P_3) = 0$ for all $X \in \mathbb{P}^1$, which describes a curve C in V with $\mathbf{h}(C) = (1, 0, 0)$. Thus, $2 \in S_V^*$, and since $\deg C = 1$, $\phi(C)$ is a line in $\phi(V)$.

Note that there exists a rational curve C over L with $\beta_C = 2$ if and only if $\phi(C)$ is a line. Suppose such a C exists. Then $\deg C = 1$, so $\mathbf{h}(C) = (1, 0, 0)$, $(0, 1, 0)$ or

$(0,0,1)$. Let us assume without loss of generality that $\mathbf{h}(C) = (1,0,0)$. Then there exists a map

$$t \mapsto (X(t), Y, Z)$$

from L onto almost all of C with Y and Z constant. This is possible only if $F_i(Y,Z) = 0$ for all i. That is, such a C exists only if V is not generic. □

3. Generic surfaces in this family

If V is generic within this family, then associated to each of the components X, Y and Z, there is a rational automorphism σ_1, σ_2 and σ_3. The group $\mathcal{A}$ of automorphisms generated by σ_1, σ_2 and σ_3 is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ [W2] (see also [B].)

Theorem 3.1. *Suppose C is an irreducible rational curve on V. Then $\sigma_i C$ is an irreducible rational curve on V and*

$$\mathbf{h}(\sigma_i C) = T_i \mathbf{h}(C),$$

where

$$T_1 = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix}.$$

Proof. We show this for σ_1. Note that σ_1 fixes Y and Z, so $h_Y(\sigma_1 C) = h_Y(C)$ and $h_Z(\sigma_1 C) = h_Z(C)$. Let

$$\sigma_1(X(t), Y(t), Z(t)) = (X'(t), Y(t), Z(t)).$$

Then, by considering Eq. (1) affinely, we have for $F_0 X_1 X_1' \neq 0$

$$\frac{X_0'(t) X_0(t)}{X_1'(t) X_1(t)} = \frac{F_2(Y(t), Z(t))}{F_0(Y(t), Z(t))}.$$

Recall that X_0 and X_1 are coprime, so X_0 divides F_2 and X_1 divides F_0 (see (1)). Thus, there exists a polynomial $m(t) \in K[t]$ so that

$$mX_0' = \frac{F_2}{X_0} \qquad \text{and} \qquad mX_1' = \frac{F_0}{X_1}.$$

If m is not a constant, then there exists a $t_0 \in \bar{\mathbb{Q}}$ so that $m(t_0) = 0$. Then $F_0(Y(t_0), Z(t_0)) = F_0(t_0) = 0$ and $F_2(t_0) = 0$. If $X_0(t_0) \neq 0$ and $X_1(t_0) \neq 0$, then we must have $F_1(t_0) = 0$, which means V is not generic. Thus, we may assume $X_0(t_0) = 0$ and $X_1(t_0) \neq 0$. Let $r \geq 1$ be the multiplicity of the root t_0 for X_0. Then $(t - t_0)^{r+1}$ divides F_2, and hence it also divides $-X_0^2 F_0 - X_1^2 F_2 = X_0 X_1 F_1$. Thus, $F_1(t_0) = 0$, and again we get V is not generic. Thus $m(t)$ is a constant. Hence,

$$\begin{aligned} \deg X_0' &= \deg F_2 - \deg X_0, \\ \deg X_1' &= \deg F_0 - \deg X_1. \end{aligned}$$

Let us write $\mathbf{h}(C) = (n, m, l)$, and

$$\begin{aligned} X_0(t) &= a_0 t^n + \text{ terms of lower degree}, \\ X_1(t) &= a_1 t^n + \ldots. \end{aligned}$$

Since $h_X(C) = n$, at least one of a_0 and a_1 is non-zero. That is, $a = (a_0 : a_1) \in \mathbb{P}^1$. Let us also write

$$\begin{array}{rclcrcl} Y_0(t) & = & b_0 t^m + ... & \text{and} & Y_1(t) & = & b_1 t^m + ..., \\ Z_0(t) & = & c_0 t^l + ... & \text{and} & Z_1(t) & = & c_1 t^l + ... \end{array}$$

Since $F_0(Y, Z)$ is a quadratic polynomial in Y and Z, we expect $\deg F_0(t) = 2m+2l$. If not, then the largest powers must cancel. That is, $F_0(b, c) = 0$.

So suppose first that $\deg X_0 = \deg X_1 = n$. Then $h_X(\sigma_1 C) \neq 2m + 2l - n$ only if $F_0(b, c) = F_2(b, c) = 0$. But then $\deg(X_0^2 F_0) < 2m + 2l + 2n$ and $\deg(X_1^2 F_2) < 2m + 2l + 2n$, so $\deg(X_0 X_1 F_1) < 2m + 2l + 2n$. That is, $\deg F_1 \neq 2m + 2l$, so $F_1(b, c) = 0$, and V is not generic.

Now suppose $n = \deg X_0 > \deg X_1$. Let us write $\deg X_1 = n - \epsilon$ with $\epsilon > 0$. If $\deg F_0 = 2m + 2l$, then $\deg(X_0^2 F_0) = 2m + 2l + 2n$, while $\deg(X_0 X_1 F_1)$ and $\deg(X_1^2 F_2)$ are strictly smaller. Thus we cannot have $\deg F_0 = 2m + 2l$, so $F_0(b, c) = 0$. Let us write $\deg F_0 = 2m + 2l - \delta_1$. Now suppose $\deg F_1 = 2m + 2l$. Then

$$\begin{aligned} \deg(X_0^2 F_0) &= 2n + 2m + 2l - \delta_1, \\ \deg(X_0 X_1 F_1) &= 2n + 2m + 2l - \epsilon, \\ \deg(X_1^2 F_2) &= 2n + 2m + 2l - 2\epsilon - \delta_2, \end{aligned}$$

where $\delta_2 \geq 0$. Since these three terms add to zero, and the degree of the last is less than the degrees of the other two, those degrees must be equal. That is, $\delta_1 = \epsilon$. Hence,

$$\begin{aligned} \deg X_0' &= 2m + 2l - \delta_2 - n \leq 2m + 2l - n, \\ \deg X_1' &= 2m + 2l - \epsilon - (n - \epsilon) = 2m + 2l - n, \end{aligned}$$

as desired.

Now suppose $F_1(b, c) = 0$. Since we also have $F_0(b, c) = 0$, we must have $F_2(b, c) \neq 0$. That is, $\deg F_2 = 2m + 2l$. Thus,

$$\begin{aligned} \deg X_0' &= 2m + 2l - n, \\ \deg X_1' &= 2m + 2l - \delta_1 - (n - \epsilon), \end{aligned}$$

so we must show that $\delta_1 \geq \epsilon$. Suppose otherwise – that is, $\delta_1 < \epsilon$. Then

$$\begin{aligned} \deg(X_0^2 F_0) &= 2n + 2m + 2l - \delta_1 > 2n + 2m + 2l - \epsilon, \\ \deg(X_0 X_1 F_1) &< 2n - \epsilon + 2m + 2l, \\ \deg(X_1^2 F_2) &= 2n - 2\epsilon + 2m + 2l < 2n + 2m + 2l - \epsilon. \end{aligned}$$

Thus, these terms cannot cancel to give zero, so we must have $\delta_1 \geq \epsilon$ as desired.

We leave to the reader the cases not covered by taking an affine perspective of Eq. (1). □

The matrices T_i are not surprisingly the same as were found in [B]. However, when compared to results found for rational points, the striking feature of this theorem is the absence of an error term.

4. Orbits of rational fibers

If we can find a rational curve C in V, and are able to calculate $\mathbf{h}(C)$, then we can use Theorem 2.1 to find an infinite subset of S_V^*.

Let C be a fiber on V with respect to the projection onto one of the variables. We know that in $\bar{\mathbb{Q}}$, there are a finite positive number of such fibers which are singular. Such a rational fiber is the zero locus of a singular (2,2) form in $\mathbb{P}^1 \times \mathbb{P}^1$. One can parameterize such a curve with polynomials $Y_i(t)$ and $Z_i(t)$ of degree at most two. Thus, for any generic V, there exists an L over K so that one can find a rational curve C in V over L with $\mathbf{h}(C) = (0, m, n)$ and $m,\ n \leq 2$. But since there are no curves on a generic V with $\deg C = 1$, we know $\mathbf{h}(C) = (0, 1, 1)$ or $(0, 2, 2)$ (note that if $\mathbf{h}(C) = (0, 1, 2)$, then $\mathbf{h}(\sigma_3 C) = (0, 1, 0)$). Thus, over L, S_V^* contains one of the two following sets:

$$\{1, 1/3, 1/7, 1/13, 1/19, 1/21, 1/31, 1/37, 1/39, 1/43, 1/49, ...\},$$
$$\{1/2, 1/6, 1/14, 1/26, 1/38, 1/42, 1/62, 1/74, 1/78, 1/86, 1/98, ...\}.$$

Counting the number of elements in the orbit of a vector under the action of the group generated by $T_1,\ T_2$ and T_3 is usually equivalent to a difficult lattice point problem in hyperbolic space. We are fortunate that the orbit of a vector of the form $(0, c, c)$ is an exception, and can be handled using geometric methods.

Theorem 4.1. *Suppose C is a rational curve on V and $\mathbf{h}(C) = (0, 1, 1)$ or $(0, 2, 2)$. Then the number of rational curves on V in the $\mathcal{A}$-orbit of C with degree less than T is*

$$\frac{2}{\pi\sqrt{3}\deg C} T + O(\sqrt{T}\log T).$$

Proof. Note that

$$\begin{aligned} T_1(u^2, v^2, (u+v)^2) &= ((2u+v)^2, v^2, (u+v)^2), \\ T_2(u^2, v^2, (u+v)^2) &= (u^2, (u+2v)^2, (u+v)^2). \end{aligned}$$

Thus, we are counting the number of coprime positive pairs (u, v) which satisfy

$$u^2 + v^2 + uv < \frac{T}{\deg C}.$$

By Theorem 3.5 of [B] this is

$$\frac{6}{\pi^2}|R| + O(|\partial R| \log T),$$

where R is the portion of the above ellipse in the first quadrant, and $|R|$ and $|\partial R|$ are its area and perimeter. The area of this region is

$$\frac{T}{\sqrt{3}\deg C} \arctan(\sqrt{3}) = \frac{\pi T}{3\sqrt{3}\deg C},$$

and the perimeter is $O(\sqrt{T})$, giving us the result. □

As we noted before, if V is generic in this family of K3 surfaces, then there exists a sufficiently large field K so that there exists on $V(K)$ a rational curve of degree 2 or 4. On the other hand, there is never a curve of degree 1. Thus one might find the following quantity of interest:

$$S_{V(\bar{\mathbb{Q}})}^* = \bigcup_K S_{V(K)}^*,$$

where the union is taken over all number fields K. It seems conceivable that there might be a sufficiently large K so that $S^*_{V(\bar{\mathbb{Q}})} = S^*_{V(K)}$.

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