

ON THE EXTENDED HILBERT'S INEQUALITY

GAO MINGZHE AND YANG BICHEN

(Communicated by J. Marshall Ash)

ABSTRACT. In this paper, it is shown that the extended Hilbert's inequality for double series can be refined by the aid of the Euler-Maclaurin summation formula. The extreme cases $p \rightarrow 1^+$ and $q \rightarrow +\infty$ are discussed.

1. INTRODUCTION

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\sum_{n=1}^{\infty} a_n^p < +\infty$ and $\sum_{n=1}^{\infty} b_n^q < +\infty$, then an extended Hilbert's inequality may be written in the form

$$(1) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \left(\pi / \sin \frac{\pi}{p} \right) \left(\sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}.$$

As is well known, the constant factor $\pi / \sin \frac{\pi}{p}$ contained in (1) is best possible. In other words, $\pi / \sin \frac{\pi}{p}$ cannot be replaced by any positive number smaller than it (cf. [1], [2]). But we may move the factor $\pi / \sin \frac{\pi}{p}$ of the right-hand side of (1) to the inside of the summation and write it in the following form:

$$(2) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \left\{ \sum_{n=1}^{\infty} (\pi / \sin \frac{\pi}{p} - \alpha_n(q)) a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} (\pi / \sin \frac{\pi}{p} - \alpha_n(p)) b_n^q \right\}^{\frac{1}{q}}$$

where $\alpha_n(r) \downarrow 0$ ($r = p, q$). Clearly, it will offer a refined form of (1). In this paper it will be shown that we can take $\alpha_n(r) = \lambda / n^{1-\frac{1}{r}}$, where λ is a positive real number that is independent of r . Furthermore, we prove also that $\lambda = 1 - \gamma$, where γ is the Euler constant.

Before proving our results we need to define some functions. Throughout this paper we assume that $x \in [1, +\infty)$ and $r \in (1, +\infty)$.

Let us define the following functions:

$$(3) \quad u(x) = x^{1-\frac{1}{r}} I(x),$$

where $I(x)$ is defined by

$$(4) \quad I(x) = \int_0^{\frac{1}{x}} \frac{1}{1+t} \left(\frac{1}{t} \right)^{\frac{1}{r}} dt,$$

Received by the editors December 6, 1995 and, in revised form, August 29, 1996.

1991 *Mathematics Subject Classification*. Primary 26D15.

Key words and phrases. Double series, infimum, Euler-Maclaurin summation formula.

and

$$(5) \quad v(x) = \int_1^{\infty} \rho(t)F(x, t) dt,$$

where $\rho(t)$ and $F(x, t)$ are defined respectively by

$$\rho(t) = t - [t] - \frac{1}{2} \quad \text{and} \quad F(x, t) = \frac{(r+1)xt + x^2}{r(x+t)^2 t^{1+1/r}}.$$

For convenience we define

$$(6) \quad \lambda_r(x) = u(x) + v(x) - \frac{x}{2(x+1)}$$

where $u(x)$ and $v(x)$ are defined respectively by (3) and (5). Particularly, in the case $x = 1$, $\lambda_r(1)$ is denoted by $\lambda(r)$. We know from (6) that

$$(7) \quad \lambda(r) = u(1) + v(1) - \frac{1}{4}.$$

We will show that $\lambda(r)$ can be written in the form

$$(8) \quad \lambda(r) = J(r) + R(r),$$

where J and R are defined respectively by

$$J(r) = \int_0^1 \frac{1}{1+t} \left(\frac{1}{t}\right)^{\frac{1}{r}} dt - \frac{13r+2}{48r}$$

and

$$R(r) = \frac{\theta}{5760} \left(3 + \frac{20}{r} + \frac{18}{r^2} + \frac{4}{r^3}\right) \quad (0 < \theta < 1).$$

2. LEMMAS

The aim of the section is to prove the following inequalities are valid:

$$\lambda_r(x) \geq \lambda(r) > \lambda$$

where λ is an infimum of $\lambda(r)$.

Lemma 1. *Let $I(x)$ be the function defined by (4). Then*

$$(9) \quad I(x) \geq \frac{r(2r-1)x^{\frac{1}{r}}}{(r-1)((2r-1)x+r-1)}.$$

Proof. Using integration by parts we obtain

$$(10) \quad I(x) = \frac{rx^{\frac{1}{r}}}{(r-1)(x+1)} + \frac{r}{r-1}K(x),$$

where $K(x)$ is defined by

$$K(x) = \int_0^{\frac{1}{x}} \frac{t^{1-\frac{1}{r}}}{(1+t)^2} dt.$$

Define the functions f and g respectively by

$$f(t) = \frac{1}{(1+t)^2} \left(\frac{1}{x}\right)^{1-\frac{1}{r}} \quad \text{and} \quad g(t) = (xt)^{1-\frac{1}{r}}, \quad t \in \left[0, \frac{1}{x}\right].$$

Evidently $f(t)$ is nonnegative and monotone decreasing in $[0, \frac{1}{x}]$ and $g(t)$ satisfies the constraint $0 \leq g(t) \leq 1$. According to Steffensen's inequality we have

$$(11) \quad \int_{\frac{1}{x}-c}^{\frac{1}{x}} f(t) dt \leq \int_0^{\frac{1}{x}} f(t)g(t) dt = K(x) \leq \int_0^c f(t) dt,$$

where $c = \int_0^{\frac{1}{x}} g(t) dt = \int_0^{\frac{1}{x}} (xt)^{1-\frac{1}{r}} dt = \frac{r}{x(2r-1)}$. Hence

$$K(x) \geq \int_{\frac{1}{x}-c}^{\frac{1}{x}} \frac{1}{(1+t)^2} \left(\frac{1}{x}\right)^{1-\frac{1}{r}} dt = -\frac{x^{\frac{1}{r}}}{x+1} + \frac{x^{\frac{1}{r}}(2r-1)}{x(2r-1) + (r-1)}.$$

Substituting it in the second term of the right-hand side of (10) we obtain after simplification that (9) is valid. \square

Lemma 2. *Let $\lambda_r(x)$ be the function defined by (6). Then*

$$(12) \quad \lambda_r(n) \geq \lambda(r), \quad n \in N,$$

where $\lambda(r)$ is defined by (7).

Proof. At first, consider the function $u(x)$ defined by (3). Taking derivatives and after simplification we have

$$u'(x) = \frac{r-1}{r}x^{-\frac{1}{r}}I(x) - \frac{x^{1-\frac{1}{r}}}{1+x}.$$

By Lemma 1 we obtain easily the following inequality:

$$(13) \quad u'(x) \geq r/(x+1)((2r-1)x + (r-1)).$$

Define the functions F_1 and F_2 by

$$F_1(t) = \frac{r+1}{r(x+t)^2t^{1/r}} \quad \text{and} \quad F_2(t) = \frac{2x}{(x+t)^3t^{1/r}}, \quad t \in [1, +\infty).$$

Obviously $F_i(t) \downarrow 0$ ($t \rightarrow +\infty$) and after calculations $F_i''(t) > 0$. In the paper [4] it has been proved that

$$-\frac{1}{8}F_i(1) < \int_1^\infty \rho(t)F_i(t) dt < -\frac{1}{12}F_i\left(\frac{3}{2}\right), \quad i = 1, 2.$$

Hence we obtain from (5) that

$$(14) \quad \begin{aligned} v'(x) &= \int_1^\infty \rho(t) \left(\frac{x+rt+t-xr}{r(x+t)^3t^{1/r}}\right) dt \\ &= \int_1^\infty \rho(t)F_1(t) dt - \int_1^\infty \rho(t)F_2(t) dt \\ &> -\frac{1}{8}F(1) + \frac{1}{12}F_2\left(\frac{3}{2}\right) \\ &= -\frac{r+1}{8r(x+1)^2} + \frac{4x}{3(2x+3)^3} \left(\frac{2}{3}\right)^{\frac{1}{r}}. \end{aligned}$$

We can obtain from (13) and (14) that

$$\begin{aligned} \lambda_r'(x) &= u'(x) + v'(x) - \frac{1}{2(x+1)^2} \\ &> \frac{(-2r^2 + 3r + 1)x + (3r^2 + 4r + 1)}{8r(x+1)^2((2r-1)x + (r-1))} + \frac{4x}{3(2x+3)^3} \left(\frac{2}{3}\right)^{\frac{1}{r}}. \end{aligned}$$

By direct computations we have the following conclusions (see Notes at the end of this paper):

When $r \geq 4$, $(\frac{2}{3})^{\frac{1}{r}} > \frac{9}{10}$, $\lambda'_r(x) > 0$ is true. And when $1 < r < 4$, $(\frac{2}{3})^{\frac{1}{r}} > \frac{2}{3}$, $\lambda'_r(x) > 0$ is also true. This implies that $\lambda_r(x)$ is monotone increasing. Whence (12) is valid. \square

Lemma 3. *Let $\lambda(r)$ be the function defined by (8). Then if $\lambda = \inf\{\lambda(r)\}$ we have $\lim_{r \rightarrow \infty} \lambda(r) = \lambda$.*

Proof. Evidently the function $J(r)$ is continuously differentiable in $(1, +\infty)$. Hence

$$(15) \quad J'(r) = \frac{1}{r^2} \int_0^1 \frac{\ln t}{1+t} \left(\frac{1}{t}\right)^{\frac{1}{r}} dt + \frac{1}{24r^2}.$$

Substituting $t = e^{-y}$ in (15) we obtain easily that

$$\begin{aligned} J'(r) &= -\frac{1}{r^2} \int_0^{+\infty} \frac{ye^{-\alpha y}}{1+e^{-y}} dy + \frac{1}{24r^2} \\ &< -\frac{1}{2r^2} \int_0^{+\infty} ye^{-\alpha y} dy = -\frac{1}{2(r-1)^2} + \frac{1}{24r^2} < 0, \end{aligned}$$

where $\alpha = 1 - \frac{1}{r}$. Hence the function $J(r)$ is monotone decreasing. Clearly $R(r)$ is also monotone decreasing. Thus

$$\lambda = \inf\{\lambda(r)\} = \lim_{r \rightarrow \infty} \lambda(r). \quad \square$$

By Lemma 3, we obtain at once the following results:

$$(16) \quad \lambda(r) > \lambda = \ln 2 - \frac{13}{48} + \frac{\theta}{1920} \quad (0 < \theta < 1).$$

3. MAIN RESULTS

Theorem 1. *Let $q \geq p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $0 < \sum_{n=1}^{\infty} a_n^p < +\infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < +\infty$, then*

$$(17) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{ \sum_{n=1}^{\infty} \left(\frac{\pi}{\sin \frac{\pi}{p}} - \lambda/n^{\frac{1}{p}} \right) a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left(\frac{\pi}{\sin \frac{\pi}{q}} - \lambda/n^{\frac{1}{q}} \right) b_n^q \right\}^{\frac{1}{q}},$$

where $\lambda = 1 - \gamma$ and γ is the Euler constant. λ is the largest constant that keeps (17) valid and is independent of r ($r = p, q$).

Proof. We may apply Hölder's inequality to estimate the left-hand side of (17) as follows:

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m}{(m+n)^{\frac{1}{p}}} \left(\frac{m}{n}\right)^{\frac{1}{pq}} \cdot \frac{b_n}{(m+n)^{\frac{1}{q}}} \left(\frac{n}{m}\right)^{\frac{1}{pq}} \\ &\leq \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m^p}{m+n} \left(\frac{m}{n}\right)^{\frac{1}{q}} \right\}^{\frac{1}{p}} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_n^q}{m+n} \left(\frac{n}{m}\right)^{\frac{1}{p}} \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{\frac{1}{q}} \right) a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{\frac{1}{p}} \right) b_n^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=1}^{\infty} \omega_q(n) a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \omega_p(n) b_n^q \right\}^{\frac{1}{q}}, \end{aligned}$$

where $\omega_r(n)$ ($r = p, q$) is defined by

$$\omega_r(n) = \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{\frac{1}{r}}.$$

Applying the Euler-Maclaurin summation formula to $\omega_r(n)$ and using the relation $\sin \frac{\pi}{p} = \sin \frac{\pi}{q}$, we obtain

$$\begin{aligned} \omega_r(n) &= \int_1^{\infty} g(t) dt + \frac{1}{2}g(1) + \int_1^{\infty} \rho(t)g'(t) dt \\ &= \int_0^{\infty} g(t) dt - \int_0^1 g(t) dt + \frac{1}{2}g(1) + \int_1^{\infty} \rho(t)g'(t) dt, \end{aligned}$$

where the function g is defined by

$$g(t) = \frac{1}{n+t} \left(\frac{n}{t}\right)^{\frac{1}{r}}, \quad t \in (0, +\infty).$$

Note that

$$\int_0^{\infty} g(t) dt = \pi / \sin \frac{\pi}{p}, \quad \int_0^1 g(t) dt = \int_0^{\frac{1}{n}} \frac{1}{1+t} \left(\frac{1}{t}\right)^{\frac{1}{r}} dt = u(n)/n^{1-\frac{1}{r}}$$

and

$$\int_1^{\infty} \rho(t)g'(t) dt = v(n)/n^{1-\frac{1}{r}},$$

where $u(x)$ and $v(x)$ are the functions defined by (3) and (5) respectively. Hence

$$\begin{aligned} \omega_r(n) &= \pi / \sin \frac{\pi}{p} - \left(u(n) + v(n) - \frac{n}{2(n+1)} \right) / n^{1-\frac{1}{r}} \\ (18) \quad &= \pi / \sin \frac{\pi}{p} - \lambda_r(n) / n^{1-\frac{1}{r}}, \end{aligned}$$

where $\lambda_r(n)$ is the function defined by (6).

In view of (12) we have

$$(19) \quad \omega_r(n) \leq \pi / \sin \frac{\pi}{p} - \lambda(r) / n^{1-\frac{1}{r}}.$$

When $n = 1$ it follows from (18) that

$$\lambda(r) = \lambda_r(1) = \pi / \sin \frac{\pi}{p} - \omega_r(1).$$

Applying the Euler-Maclaurin summation formula to $\omega_r(1)$ we have

$$\begin{aligned} \omega_r(1) &= \sum_{m=1}^{\infty} \frac{1}{1+m} \left(\frac{1}{m}\right)^{\frac{1}{r}} = \int_1^{\infty} f(t) dt + \frac{1}{2}f(1) + \sum_{k=1}^{s-1} -\rho_s \\ &= \int_0^{\infty} f(t) dt - \int_0^1 f(t) dt + \frac{1}{2}f(1) + \sum_{k=1}^{s-1} -\rho_s \\ &= \pi / \sin \frac{\pi}{p} - \int_0^1 f(t) dt + \frac{1}{4} + \sum_{k=1}^{s-1} -\rho_s. \end{aligned}$$

Hence the term $\lambda(r)$ that appears in (19) can be written in the form

$$(20) \quad \lambda(r) = \int_0^1 f(t) dt - \frac{1}{4} - \sum_{k=1}^{s-1} +\rho_s,$$

where $f(t) = \frac{1}{1+t} \left(\frac{1}{t}\right)^{\frac{1}{r}}$, $\sum_{k=1}^{s-1} = \sum_{k=1}^{s-1} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1)$ and the B_j 's are the Bernoulli numbers, viz. $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ etc., and ρ_s is the remainder of the form

$$\rho_s = \frac{B_{2s}\theta}{(2s)!} f^{(2s-1)}(1) \quad (0 < \theta < 1).$$

For $s = 2$ we obtain (8) from (20). By virtue of Lemma 3 and (19) we get that

$$(21) \quad \omega_r(n) < \pi / \sin \frac{\pi}{p} - \lambda / n^{1-\frac{1}{r}}.$$

It remains to show that $\lambda = 1 - \gamma$, where γ is the Euler constant. For $n = 1$, using the Euler-Maclaurin summation formula we obtain from (18) that

$$\begin{aligned} (22) \quad \lambda(r) &= \pi / \sin \frac{\pi}{p} - \left\{ \sum_{m=1}^{k-1} \frac{1}{1+m} \left(\frac{1}{m}\right)^{\frac{1}{r}} + \sum_{m=k}^{\infty} \frac{1}{1+m} \left(\frac{1}{m}\right)^{\frac{1}{r}} \right\} \\ &= \pi / \sin \frac{\pi}{p} - \left\{ \sum_{m=1}^{k-1} \frac{1}{1+m} \left(\frac{1}{m}\right)^{\frac{1}{r}} + \int_k^{\infty} f(t) dt + \frac{1}{2}f(k) - \frac{\theta}{12}f'(k) \right\} \\ &= \int_0^k f(t) dt - \sum_{m=1}^{k-1} \frac{1}{1+m} \left(\frac{1}{m}\right)^{\frac{1}{r}} - \frac{1}{2}f(k) + \frac{\theta}{12}f'(k) \quad (0 < \theta < 1), \end{aligned}$$

where $f(t) = \frac{1}{1+t} \left(\frac{1}{t}\right)^{\frac{1}{r}}$. In accordance with the definition of the Euler constant γ , i.e.

$$\sum_{m=0}^{k-1} \frac{1}{1+m} = \gamma + \ln(k-1) + \varepsilon_{k-1} \quad (\varepsilon_{k-1} \rightarrow 0, \text{ if } k \rightarrow +\infty)$$

and by Lemma 3 we obtain from (22)

$$\begin{aligned} \lambda &= \lim_{r \rightarrow \infty} \lambda(r) = \int_0^k \frac{1}{1+t} dt - \sum_{m=1}^{k-1} \frac{1}{1+m} - \frac{1}{2(1+k)} - \frac{\theta}{12(1+k)^2} \\ &= 1 - \gamma + \Delta R, \end{aligned}$$

where ΔR is the error of the form

$$\Delta R = \ln \frac{k+1}{k-1} - \varepsilon_{k-1} - \frac{1}{2(1+k)} - \frac{\theta}{12(1+k)^2} \quad (0 < \theta < 1).$$

This implies that the bigger we take the value of k , the smaller the value of $|\Delta R|$. Let $k \rightarrow +\infty$. Then we obtain that $\lambda = 1 - \gamma$.

Based on Lemma 3 and (21), it follows that λ is the largest constant that keeps (17) valid and is independent of r ($r = p, q$).

Thus we have completed the proof of the theorem. □

The value of λ is given numerically as follows:

$$\lambda = 0.422784335098467 \dots$$

In particular, in the case $r = 2$, it follows from (8) that

$$\lambda(2) = J(2) + R(2) = \frac{\pi}{2} - \frac{7}{24} + \frac{\theta}{320}.$$

In view of (19) we have

$$\omega_2(n) \leq \pi - \lambda(2)/\sqrt{n}.$$

Therefore we obtain a sharp result of Hilbert's inequality.

Theorem 2. *If $0 < \sum_{n=1}^{\infty} a_n^2 < +\infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < +\infty$, then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{ \sum_{n=1}^{\infty} (\pi - \alpha/\sqrt{n}) a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} (\pi - \alpha/\sqrt{n}) b_n^2 \right\}^{\frac{1}{2}},$$

where $\alpha = \frac{\pi}{2} - \frac{7}{24} + \frac{\theta}{320}$ ($0 < \theta < 1$).

Finally, the extreme cases $p \rightarrow 1^+$ and $q \rightarrow +\infty$ are discussed. Note that $\frac{1}{p} + \frac{1}{q} = 1$ and $q \geq p > 1$. In the paper [3] it has been proved that $\lambda(p) > \frac{1}{p-1}$, where $\lambda(p)$ is defined by (20). Now we may prove that $\lambda(p) \sim \frac{1}{p-1}$ when $p \rightarrow 1^+$. In fact, for $r = p$ and $x = 1$ we consider the function defined by (4) and denote it by $h(p)$. We have

$$h(p) = \int_0^1 \frac{1}{1+t} \left(\frac{1}{t}\right)^{\frac{1}{p}} dt.$$

From (10) we obtain

$$h(p) = \frac{p}{2(p-1)} + \frac{p}{p-1} k(1), \quad \text{where } k(1) = \int_0^1 \frac{t^{1-\frac{1}{p}}}{(1+t)^2} dt.$$

Use (11) to estimate $k(1)$. When $x = 1$ we have

$$c = \int_0^1 g(t) dt = \int_0^1 t^{1-\frac{1}{p}} dt = \frac{p}{2p-1},$$

$$\int_{1-c}^1 f(t) dt = \int_{1-c}^1 \frac{dt}{(1+t)^2} = \frac{p}{2(3p-2)}$$

and $\int_0^c f(t) dt = \frac{p}{3p-1}$. Hence

$$\frac{p}{2(3p-2)} \leq k(1) \leq \frac{p}{3p-1}.$$

Since $\lim_{p \rightarrow 1^+} \frac{p}{2(3p-2)} = \lim_{p \rightarrow 1^+} \frac{p}{3p-1} = \frac{1}{2}$, it follows that $\lim_{p \rightarrow 1^+} k(1) = \frac{1}{2}$. Whence $\lim_{p \rightarrow 1^+} (p-1)h(p) = 1$. It follows from (8) that $\lim_{p \rightarrow 1^+} \lambda(p)/\frac{1}{p-1} = 1$.

ACKNOWLEDGMENTS

The authors express their gratitude to the referee, whose thoughtful suggestions have improved the exposition of this work. The authors wish also to express their sincerest thanks to Professor J. Marshall Ash and Professor L. C. Hsu for their assistance and encouragements.

NOTES

$\lambda'_r(x) > g(x)$, where

$$g(x) = \frac{(-2r^2 + 3r + 1)x + (3r^2 + 4r + 1)}{8r(x+1)^2((2r-1)x + (r-1))} + \frac{4x}{3(2x+3)^3} \left(\frac{2}{3}\right)^{\frac{1}{r}}, \quad r > 1, x \geq 1.$$

When $r \geq 4$, $(\frac{2}{3})^{\frac{1}{r}} > \frac{9}{10}$. Hence

$$\begin{aligned} g(x) &> \frac{(-2r^2 + 3r + 1)x + (3r^2 + 4r + 1)}{8r(x+1)^2((2r-1)x + (r-1))} + \frac{6x}{5(2x+3)^3} \\ &= \frac{A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5}{40r(x+1)^2(2x+3)^3((2r-1)x + (r-1))} > 0, \end{aligned}$$

where

$$\begin{aligned} A_1 &= 16r^2 + 72r + 40, & A_2 &= 556r + 220, \\ A_3 &= 192r^2 + 1386r + 450, & A_4 &= 588r^2 + 1437r + 405, \\ A_5 &= 13r(3r^2 + 4r + 1) \end{aligned}$$

when $1 < r < 4$, $(\frac{2}{3})^{\frac{1}{r}} > \frac{2}{3}$. Hence

$$\begin{aligned} g(x) &> \frac{-2r^2 + 3r + 1}{8r(x+1)^2((2r-1)x + (r-1))} + \frac{8x}{9(2x+3)^3} \\ &= \frac{B_1x^4 + B_2x^3 + B_3x^2 + B_4x + B_5}{72r(x+1)^2(2x+3)^3((2r-1)x + (r-1))} > 0, \end{aligned}$$

where

$$\begin{aligned} B_1 &= -16r^2 + 152r + 72, & B_2 &= -112r^2 + 1068r + 396, \\ B_3 &= 256r^2 + 2562r + 810, & B_4 &= 1036r^2 + 2609r + 729, \\ B_5 &= 243(3r^2 + 4r + 1). \end{aligned}$$

Consequently, we have $\lambda'(x) > g(x) > 0$.

REFERENCES

1. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge Univ. Press, 1952. MR **13**:727e
2. D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, 1970. MR **43**:448

3. Gao Mingzhe, *An improvement of Hardy-Riesz's Extension of the Hilbert Inequality*, J. Math. Res. Exposition, Vol. 14, No. 2 (1994). MR **95m**:26040
4. Zhao Dejun, *On a Refinement of Hilbert's Double Series Theorem*, Math. In Practice and Theory, Beijing, China.

DEPARTMENT OF MATHEMATICS, XIANGXI EDUCATION COLLEGE FOR NATIONALITIES, JISHOU,
HUNAN 416000, PEOPLE'S REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS, GUANGDONG COLLEGE OF EDUCATION, GUANGZHOU 510303,
PEOPLE'S REPUBLIC OF CHINA