SIMPLE QUOTIENTS OF HYPERBOLIC 3-MANIFOLD GROUPS

D. D. LONG AND A. W. REID

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Abstract. We show that hyperbolic 3-manifolds have residually simple fundamental group.

1. Introduction

Let $G$ be a finitely generated group and $X$ a property of groups, e.g. finite, simple, $p$-group. $G$ is said to be residually $X$, if for any element $g \neq 1$, there is a group $H$ with property $X$ and a surjective homomorphism $\phi : G \to H$ such that $\phi(g) \neq 1$.

Of interest to us are residual properties of groups $\pi_1(M)$ where $M$ is a compact orientable 3-manifold with infinite fundamental group. Now it is well-known that if $M$ is a hyperbolic 3-manifold, that is the quotient of hyperbolic 3-space by a torsion-free Kleinian group, then $\pi_1(M)$ is residually finite. In this note we prove a much stronger result which seems to have been unnoticed previously. First we make a definition.

Definition 1.1. Let $M$ be a compact orientable 3-manifold with infinite fundamental group and $\rho : \pi_1(M) \to SL(2, \mathbb{C})$ a faithful representation whose image lies in $SL(2, \overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. In this case we define $\rho$ to be an algebraic representation.

With this we can state:

Theorem 1.2. Let $M$ be a compact orientable 3-manifold such that $\pi_1(M)$ admits an algebraic representation. Then $\pi_1(M)$ is residually simple.

A particular case of this is:

Corollary 1.3. Let $M$ be a finite volume hyperbolic 3-manifold. Then $\pi_1(M)$ is residually simple.

In fact we shall show more; the simple groups will all be of the type $PSL(2,F)$ for finite fields $F$ of prime cardinality. A corollary of Theorem 1.2 is a new proof of a result originally observed by Magnus, which follows by taking $M$ to be a handlebody:

Corollary 1.4. Nonabelian free groups are residually simple.
2. General remarks

We start with some general remarks about algebraic representations of 3-manifold groups, and groups $PSL(2, \mathbb{F})$ which do not necessarily surject. This is just a reformulation of classical notions about linear groups; cf. [6]. For details on number fields and their completions, see [2] for instance.

Let $M$ be a compact orientable 3-manifold, and $\rho$ an algebraic representation of $\pi_1(M)$ into $SL(2, \mathbb{C})$. Denote the image of the group $\pi_1(M)$ under $\rho$ by $\Gamma$. Let $k = \mathbb{Q}(tr \gamma : \gamma \in \Gamma)$ denote the trace-field of $\Gamma$. Since $\rho$ is an algebraic representation, $k$ is a finite extension of $\mathbb{Q}$. Let $A \Gamma$ be the algebra

$$\{ \sum a_i \gamma_i : a_i \in k, \gamma_i \in \Gamma \}.$$

This is a quaternion algebra over $k$ as follows from [1]. We remark that for finite volume hyperbolic 3-manifolds $k$ is always a finite extension of $\mathbb{Q}$; cf. [4], Proposition 6.7.4.

By the classification theorem for quaternion algebras $A \Gamma$ is unramified at all but a finite number of places of $k$, [5]. In particular, for all but a finite number of prime ideals $\mathfrak{p}$ of $k$, $A \otimes_k k_{\mathfrak{p}} \cong M(2, k_{\mathfrak{p}})$. Thus on specifying an isomorphism between $A \otimes_k k_{\mathfrak{p}}$ and $M(2, k_{\mathfrak{p}})$, we induce a representation of $\Gamma$ into $SL(2, k_{\mathfrak{p}})$.

Since $M$ is compact, $\Gamma$ is finitely generated and finitely presented, therefore for all but finitely many prime ideals, we actually induce a representation of $\Gamma$ into $SL(2, O_{\mathfrak{p}})$ where $O_{\mathfrak{p}}$ are the $\mathfrak{p}$-adic integers in $k_{\mathfrak{p}}$, since only finitely many $k$-primes can divide denominators of elements of $\Gamma$.

Denote by $\pi_{\mathfrak{p}}$ a local uniformizing parameter for $O_{\mathfrak{p}}$. The unique maximal ideal of $O_{\mathfrak{p}}$ is $\pi O_{\mathfrak{p}}$, and $O_{\mathfrak{p}}/\pi O_{\mathfrak{p}}$ is a finite field with $p^n$ elements where $\mathfrak{p}$ divides $p$ for a rational prime $p$ and $n$ is the inertial degree of $\mathfrak{p}$. Therefore reduction induces a homomorphism (so far, not necessarily surjective) of $\Gamma$ into $SL(2, F_{p^n})$ where $F_{p^n}$ is the finite field with $p^n$ elements. By composing the map $\pi_1(M) \rightarrow \Gamma$, with the above, and then projectivising, we get a homomorphism $\phi_{\mathfrak{p}}$ of $\pi_1(M)$ into $PSL(2, F_{p^n})$, for infinitely many finite fields $F$.

**Lemma 2.1.** There are infinitely many $k$-primes $\mathfrak{p}$ such that the homomorphisms $\phi_{\mathfrak{p}}$ constructed above are nontrivial and map $\pi_1(M)$ into $PSL(2, F)$ where $F$ has prime cardinality.

**Proof.** It is a well-known consequence of how prime ideals behave in finite extensions of $\mathbb{Q}$, that there are infinitely many many rational primes that split completely in the finite extension $k/\mathbb{Q}$; see [2] Theorem 4.12 for example. Now a rational prime $p$ splits completely if and only if the inertial degree of the $k$-prime divisors of $p$ are all equal to 1. In particular we deduce that there are infinitely many rational primes $p$ with $\mathfrak{p}|p$ such that $\phi_{\mathfrak{p}}$ maps $\pi_1(M)$ into $PSL(2, F_{p^n})$ where $F_{p^n}$ has $p$ elements.

It is also easy to see that infinitely many of these homomorphisms are non-trivial. For if $\gamma \in \Gamma$ is given, reduction of, for example, the $(1, 2)$-entry of $\gamma$ will be non-zero for all but a finite number of $k$-primes—since entries of $\gamma$ will be $\mathfrak{p}$-adic units for all but a finite number of $\mathfrak{p}$. Hence, the number of $k$-primes such that $\phi_{\mathfrak{p}}(\gamma)$ is trivial is finite.

In particular Lemma 2.1 implies that we may construct infinitely many non-trivial representations of $\pi_1(M)$ into groups $PSL(2, F)$ where the cardinalities of $F$ are distinct primes.
3. Proof of Theorem 1.2

To prove Theorem 1.2 we shall make use of the description of subgroups of the groups $PSL(2, F)$ where $|F|$ is of odd prime cardinality. The following is deduced from [3], Theorem 6.25, together with the observation that all abelian subgroups of such $PSL(2, F)$ are cyclic.

**Theorem 3.1.** Let $p$ be an odd rational prime; then a complete list of subgroups of $PSL(2, F)$ where $|F| = p$ is

1. Cyclic groups of order $p$ and order $n$ where $n$ divides $(p^2 - 1)/2$.
2. Dihedral groups of order $n$ where $n$ is as in 1.
3. Semi-direct products of cyclic groups of order $p$ with cyclic groups of order $(p - 1)/2$.
4. $A_4$, $S_4$ or $A_5$.

We now show that the homomorphisms $\phi_\wp$ constructed above actually surject infinitely many groups $PSL(2, F)$ as in the statement of Lemma 2.1.

The group $\Gamma$ is never soluble of any finite degree. This follows for example by the fact that they contain free non-abelian groups, as they are non-elementary subgroups of $SL(2, \mathbb{C})$. Thus for the remainder of the proof, we fix some nontrivial element $\alpha$ which lies deep in the solubility series of $\pi_1(M)$.

Now suppose then that we are given some element $\gamma \in \Gamma$. As observed above, we can find (infinitely many) $k$-primes $\wp$ so that $\phi_\wp(\gamma) \neq 1$ and $\phi_\wp(\alpha) \neq 1$.

Since the homomorphic image of a term in the solubility series for a group lies in the same term of the solubility series of the image, the fact that the element $\alpha$ maps nontrivially means that the image of the group $\pi_1(M)$ cannot be of type 1, 2, 3 nor $A_4$ or $S_4$ in the list provided by Theorem 3.1 since these are all soluble of small fixed degree.

Thus the map $\phi_\wp$ will be shown to have been a surjection if we show that $\phi_\wp(\Gamma) \neq A_5$ for infinitely many $\wp$. Now if infinitely many of the homomorphisms constructed surject $A_5$, then since there are only finitely many normal subgroups in $\Gamma$ of index 60, it follows that for infinitely many of these homomorphisms the kernels coincide. However this is impossible. These homomorphisms were constructed by reducing $\Gamma$ modulo $\pi_\wp O_{\wp}$, hence if infinitely many homomorphisms had the same kernel this would mean that the elements in this matrix group were congruent to the identity modulo infinitely many $\pi_\wp O_{\wp}$, which is clearly false.

Thus we have $\phi_\wp$ that surjects $\pi_1(M)$ onto some $PSL(2, F)$ and which maps $\gamma$ non-trivially.

In fact we may conclude that under the homomorphisms $\phi_\wp$ constructed above, $\pi_1(M)$ surjects infinitely many of the simple groups $PSL(2, F)$, with $|F|$ of odd prime cardinality. 

4. Application

A motivation for this result arises from trying to show that covers of hyperbolic 3-manifolds have positive first Betti number. With this in mind, an application of this result stems from the following question raised by D. Cooper. Here $inj(M)$ denotes the injectivity radius of $M$, which is simply half the length of the shortest closed geodesic in $M$:

**Question.** Is there a number $K > 0$ so that if $M$ is a closed hyperbolic 3-manifold and $inj(M) > K$, then $rank(H_1(M; \mathbb{Q})) > 0$?
An affirmative answer to this question, taken with the fact that $\pi_1(M)$ is residually finite, implies in particular that every closed hyperbolic 3-manifold has a finite sheeted covering with positive first Betti number. However our main theorem shows that actually more would be true.

**Corollary 4.1.** If the above question has an affirmative answer, then every rational hyperbolic homology 3-sphere has infinite virtual Betti number.

**Proof.** We recall that $M$ is said to have infinite virtual Betti number if given any integer $N$, one can find a finite sheeted covering of $M$ whose first Betti number is larger than $N$. Equivalently, $M$ has infinite virtual Betti number if the rank of $H_2(\tilde{M};\mathbb{Z})$ is unbounded as $\tilde{M}$ ranges over all finite covers of $M$.

Since, given any constant $C$, $M$ has only finitely many geodesics of length at most $C$, our main theorem implies that one can find regular covers of $M$ having arbitrarily large injectivity radius for which the group of covering transformations has the form $PSL(2,\mathbb{F})$. An affirmative answer to the question implies that we may assume that these manifolds all have $H_2(M;\mathbb{Z})$ having rank at least one. The action of the covering group gives a series of representations

$$\alpha_F : PSL(2,\mathbb{F}) \to GL(H_2(M;\mathbb{Z}))$$

Suppose to the contrary that the ranks of the groups $H_2(M;\mathbb{Z})$ were bounded, by $P$ say. Then since there is a bound on the size of the finite subgroups in $GL(P,\mathbb{Z})$, and the sizes of the groups $PSL(2,\mathbb{F})$ are going to infinity, we would eventually see that some $\alpha_F$ is nonfaithful, hence trivial. It follows that the fixed homology $H_1(M;\mathbb{Q})^{PSL(2,\mathbb{F})} \cong H_1(M;\mathbb{Q})$. However using the transfer map we see that the left hand side of this isomorphism is $H_1(M;\mathbb{Q})$, a contradiction, since we assumed that $M$ was a rational homology sphere. \qed

**References**


Department of Mathematics, University of California, Santa Barbara, California 93106

E-mail address: long@math.ucsb.edu

Department of Mathematics, University of Texas, Austin, Texas 78712

E-mail address: areid@math.utexas.edu